Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 560586, 14 pages
doi:10.1155/2012/560586

Research Article

On Integral Inequalities of Hermite-Hadamard Type for s-Geometrically Convex Functions

Tian-Yu Zhang,[1] Ai-Ping Ji,[1] and Feng Qi[2]

[1] *College of Mathematics, Inner Mongolia University for Nationalities, Inner Mongolia Autonomous Region, Tongliao City 028043, China*
[2] *School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China*

Correspondence should be addressed to Feng Qi, qifeng618@gmail.com

Received 12 May 2012; Accepted 19 May 2012

Academic Editor: Yonghong Yao

The authors introduce the concept of the s-geometrically convex functions. By the well-known Hölder inequality, they establish some integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions and apply these inequalities to special means.

1. Introduction

We firstly list several definitions and some known results.

Definition 1.1. A function $f : I \subset \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 (see [1]). A function $f : I \subset \mathbb{R}_0 = [0, +\infty) \to \mathbb{R}_0$ is said to be s-convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for some $s \in (0,1]$, where $x, y \in I$, and $\lambda \in [0,1]$.

If $s = 1$, the s-convex function becomes a convex function on $\mathbb{R}_0$.

Theorem 1.3 ([2], Theorem 2.2). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$, $a < b$.*

(i) *If $|f'(x)|$ is a convex function on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)}{8}. \tag{1.3}$$

(ii) *If $|f'(x)|^{p/(p-1)}$ is a convex function on $[a, b]$, for $p > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{(1/p)}}\left(\frac{\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.4}$$

Theorem 1.4 ([3], Theorems 2.3 and 2.4). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$, $a < b$. If $|f'(x)|^p$ is convex on $[a, b]$, for $p > 1$, then*

$$\begin{gathered}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left(\left|f'(a)\right| + \left|f'(b)\right|\right), \\ \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ \times\left[\left(\left|f'(a)\right|^{p/(p-1)} + 3\left|f'(b)\right|^{p/(p-1)}\right)^{(p-1)/p} \right. \\ \left. + \left(3\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}\right)^{(p-1)/p}\right].\end{gathered} \tag{1.5}$$

Theorem 1.5 ([4], Theorem 1–4). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$, $a < b$. If $|f'(x)|^q$ is s-convex on $[a, b]$, for $q > 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\quad\le \frac{b-a}{2}\left(\frac{1}{2}\right)^{(q-1)/q}\left(\frac{2+(1/2)^s}{(s+1)(s+2)}\right)^{1/q}\left(\left|f'(a)\right|^s + \left|f'(b)\right|^s\right)^{1/q}.\end{aligned} \tag{1.6}$$

Theorem 1.6 ([5], Theorem 4). *Let $f : I \to \mathbb{R}_0$ be differentiable on I°, $a, b \in I$, $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some $s \in (0,1]$, and $p, q \geq 1$, such that $(1/q)+(1/p) = 1$ then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq 2^{-1/p}\frac{\left(|f'(a)|^q + (s+1)|f'((a+b)/2)|^q\right)^{1/q}}{\{(s+1)(s+2)\}^{1/q}}$$
$$+ 2^{-1/p}\frac{\left(|f'(b)|^q + (s+1)|f'((a+b)/2)|^q\right)^{1/q}}{\{(s+1)(s+2)\}^{1/q}}$$
$$= 2^{-1/p}\left[\left(\beta(s+1,2)|f'(a)|^q + \beta(s+2,1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right.$$
$$\left. + \left(\beta(s+1,2)|f'(b)|^q + \beta(s+2,1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}\right]. \tag{1.7}$$

Theorem 1.7 ([6], Theorems 2.2–2.4). *Let $f : I \to \mathbb{R}_0$ be differentiable on I°, $a, b \in I$, $a < b$, and $f' \in L([a,b])$.*

(i) *If $|f'(x)|$ is s-convex on $[a,b]$ for some $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4(s+1)(s+2)}\left[|f'(a)| + 2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|\right]$$
$$\leq \frac{(2^{2-s}+1)(b-a)}{4(s+1)(s+2)}\left[|f'(a)| + |f'(b)|\right]. \tag{1.8}$$

(ii) *If $|f'(x)|^{p/(p-1)}$ $(p > 1)$ is a s-convex function on $[a,b]$ for some $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{2/q}\left[\left(\left(2^{1-s}+s+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q}\right.$$
$$\left. + \left(2^{1-s}|f'(a)|^q|f'(a)|^{p/(p-1)} + \left(2^{1-s}+s+1\right)|f'(b)|^q\right)^{1/q}\right], \tag{1.9}$$

where $1/p + 1/q = 1$.

(iii) *If* $|f'(x)|^q$ $(q \geq 1)$ *is s-convex on* $[a,b]$ *for some* $s \in (0,1]$, *then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{8}\left(\frac{2}{(s+1)(s+2)}\right)^{1/q}\left[\left(\left(2^{1-s}+1\right)|f'(a)|^q + 2^{1-s}|f'(b)|^q\right)^{1/q} + \left(2^{1-s}|f'(a)|^q|f'(a)|^q + \left(2^{1-s}+1\right)|f'(b)|^q\right)^{1/q}\right]. \quad (1.10)$$

Now we introduce the definition of the s-geometrically convex function.

Definition 1.8. A function $f : I \subset \mathbb{R}_+ = (0,+\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda} \quad (1.11)$$

for $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.9. A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq [f(x)]^{\lambda^s}[f(y)]^{(1-\lambda)^s} \quad (1.12)$$

for some $s \in (0,1]$, where $x, y \in I$ and $\lambda \in [0,1]$.

If $s = 1$, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions and then apply these inequalities to special means.

2. A Lemma

In order to prove our results, we need the following lemma.

Lemma 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I°, *and* $a, b \in I$, *with* $a < b$. *If* $f' \in L([a,b])$, *then*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{4}\int_0^1 \left[tf'\left((1-t)a + t\frac{a+b}{2}\right) + (t-1)f'\left((1-t)\frac{a+b}{2} + tb\right)\right]dt,$$

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{b-a}{4}\int_0^1\left[(t-1)f'\left((1-t)a+t\frac{a+b}{2}\right)+tf'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt. \tag{2.1}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned}
&\int_0^1\left[tf'\left((1-t)a+t\frac{a+b}{2}\right)+(t-1)f'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt\\
&\quad=\frac{2}{b-a}\left[tf\left((1-t)a+t\frac{a+b}{2}\right)\Big|_0^1-\int_0^1 f\left((1-t)a+t\frac{a+b}{2}\right)dt\right]\\
&\qquad+\frac{2}{b-a}\left[(t-1)f\left((1-t)\frac{a+b}{2}+tb\right)\Big|_0^1-\int_0^1 f\left((1-t)\frac{a+b}{2}+tb\right)dt\right]\\
&\quad=\frac{4}{b-a}\left(f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right),\\
&\int_0^1\left[(t-1)f'\left((1-t)a+t\frac{a+b}{2}\right)+tf'\left((1-t)\frac{a+b}{2}+tb\right)\right]dt\\
&\quad=\frac{2}{b-a}\left[(t-1)f\left((1-t)a+t\frac{a+b}{2}\right)\Big|_0^1-\int_0^1 f\left((1-t)a+t\frac{a+b}{2}\right)dt\right]\\
&\qquad+\frac{2}{b-a}\left[tf\left((1-t)\frac{a+b}{2}+tb\right)\Big|_0^1-\int_0^1 f\left((1-t)\frac{a+b}{2}+tb\right)dt\right]\\
&\quad=\frac{4}{b-a}\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right).
\end{aligned} \tag{2.2}$$

This completes the proof of Lemma 2.1. □

3. Main Results

Theorem 3.1. *Let $f : I\subset\mathbb{R}_+\to\mathbb{R}_+$ be differentiable on I°, $a,b\in I$, with $a<b$, and $f'\in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $q\geq 1$ and $s\in(0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}G_1(s,q;g_1(\alpha),g_2(\alpha)) \tag{3.1}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}G_1(s,q;g_2(\alpha),g_1(\alpha)), \tag{3.2}$$

where

$$g_1(\alpha) = \begin{cases} \dfrac{1}{2}, & \alpha = 1, \\ \dfrac{\alpha \ln \alpha - \alpha + 1}{[\ln \alpha]^2}, & \alpha \neq 1, \end{cases} \qquad g_2(\alpha) = \begin{cases} \dfrac{1}{2}, & \alpha = 1, \\ \dfrac{\alpha - \ln \alpha - 1}{[\ln \alpha]^2} & \alpha \neq 1, \end{cases} \tag{3.3}$$

$$\alpha(u, v) = \left|f'(a)\right|^{-u}\left|f'(b)\right|^{v}, u, v > 0, \tag{3.4}$$

$$\begin{aligned} & G_1(s, q; g_1(\alpha), g_2(\alpha)) \\ &= \begin{cases} \left|f'(a)\right|^{s}\left[g_1\left(\alpha\left(\dfrac{sq}{2}, \dfrac{sq}{2}\right)\right)\right]^{1/q} + \left|f'(a)f'(b)\right|^{s/2}\left[g_2\left(\alpha\left(\dfrac{sq}{2}, \dfrac{sq}{2}\right)\right)\right]^{1/q}, \\ \qquad\qquad \left|f'(a)\right| \leq 1, \\ \left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\dfrac{q}{2s}, \dfrac{sq}{2}\right)\right)\right]^{1/q} + \left|f'(a)\right|^{1/2s}\left|f'(b)\right|^{s/2}\left[g_2\left(\alpha\left(\dfrac{q}{2s}, \dfrac{sq}{2}\right)\right)\right]^{1/q}, \\ \qquad\qquad \left|f'(b)\right| \leq 1 \leq \left|f'(a)\right|, \\ \left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\dfrac{q}{2s}, \dfrac{q}{2s}\right)\right)\right]^{1/q} + \left|f'(a)f'(b)\right|^{1/2s}\left[g_2\left(\alpha\left(\dfrac{q}{2s}, \dfrac{q}{2s}\right)\right)\right]^{1/q}, \\ \qquad\qquad 1 \leq \left|f'(b)\right|. \end{cases} \end{aligned} \tag{3.5}$$

Proof. (1) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ & \leq \frac{b-a}{4}\int_0^1 \left[t\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right| + |t-1|\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right| \right] dt \\ & \leq \frac{b-a}{4}\left\{ \left(\int_0^1 t dt\right)^{1-1/q}\left[\int_0^1 t\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right|^q dt\right]^{1/q} \right. \\ & \qquad \left. + \left(\int_0^1 (1-t)dt\right)^{1-1/q}\left[\int_0^1 (1-t)\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|^q dt\right]^{1/q} \right\} \\ & \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{ \left[\int_0^1 t\left|f'\left(a^{(2-t)/2}b^{t/2}\right)\right|^q dt\right]^{1/q} \right. \\ & \qquad \left. + \left[\int_0^1 (1-t)\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|^q dt\right]^{1/q} \right\} \end{aligned}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 t\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt\right]^{1/q} + \left[\int_0^1 (1-t)\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt\right]^{1/q}\right\}. \tag{3.6}$$

If $0<\mu\leq 1\leq \eta, 0<\alpha, s\leq 1$, then

$$\mu^{\alpha^s}\leq \mu^{\alpha s}, \quad \eta^{\alpha^s}\leq \eta^{\alpha/s}. \tag{3.7}$$

(i) If $|f'(a)|\leq 1$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \\
&\quad\leq \int_0^1 t\left(\left|f'(a)\right|^{(sq/2)(2-t)}\left|f'(b)\right|^{(sqt/2)}\right)dt = \left|f'(a)\right|^{sq}g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \\
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \\
&\quad\leq \int_0^1 (1-t)\left(\left|f'(a)\right|^{(sq/2)(1-t)}\left|f'(b)\right|^{(sq/2)(1+t)}\right)dt = \left|f'(a)f'(b)\right|^{sq/2}g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.8}$$

(ii) If $|f'(b)|\leq 1\leq |f'(a)|$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \\
&\quad\leq \int_0^1 t\left(\left|f'(a)\right|^{(q/2s)(2-t)}\left|f'(b)\right|^{(sq/2)t}\right)dt = \left|f'(a)\right|^{q/s}g_1\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right), \\
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \\
&\quad\leq \int_0^1 (1-t)\left(\left|f'(a)\right|^{(q/2s)(1-t)}\left|f'(b)\right|^{(sq/2)(1+t)}\right)dt = \left|f'(a)\right|^{q/2s}\left|f'(b)\right|^{sq/2}g_2\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.9}$$

(iii) If $1\leq |f'(b)|$, by (3.7), we obtain that

$$\begin{aligned}
&\int_0^1 t\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \\
&\quad\leq \int_0^1 t\left(\left|f'(a)\right|^{(q/2s)(2-t)}\left|f'(b)\right|^{(q/2s)t}\right)dt = \left|f'(a)\right|^{q/s}g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right),
\end{aligned}$$

$$\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt$$
$$\leq \int_0^1 (1-t)\left(\left|f'(a)\right|^{(q/2s)(1-t)}\left|f'(b)\right|^{(q/2s)(1+t)}\right)dt = \left|f'(a)f'(b)\right|^{q/2s} g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{3.10}$$

From (3.6) to (3.10), (3.1) holds.

(2) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\left(\int_0^1 (1-t)dt\right)^{1-1/q}\left(\int_0^1 (1-t)\left|f'\left((1-t)a+t\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\
&\qquad \left.+\left(\int_0^1 t dt\right)^{1-1/q}\left(\int_0^1 t\left|f'\left((1-t)\frac{a+b}{2}+tb\right)\right|^q dt\right)^{1/q}\right] \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[\int_0^1 (1-t)\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt\right]^{1/q}\right. \\
&\qquad \left.+\left[\int_0^1 t\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt\right]^{1/q}\right\}.
\end{aligned} \tag{3.11}$$

(i) If $|f'(a)| \leq 1$, by (3.7), we have

$$\begin{aligned}
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \leq \left|f'(a)\right|^{sq} g_2\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \\
&\int_0^1 t\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \leq \left|f'(a)f'(b)\right|^{sq/2} g_1\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.12}$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (3.7), we have

$$\begin{aligned}
&\int_0^1 (1-t)\left(\left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}\right)dt \leq \left|f'(a)\right|^{q/s} g_2\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right), \\
&\int_0^1 t\left(\left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}\right)dt \leq \left|f'(a)\right|^{q/2s}\left|f'(b)\right|^{sq/2} g_1\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right).
\end{aligned} \tag{3.13}$$

(iii) If $1 \le |f'(b)|$, by (3.7), we have

$$\int_0^1 (1-t)\left(|f'(a)|^{q((2-t)/2)^s}|f'(b)|^{q(t/2)^s}\right)dt \le |f'(a)|^{q/s} g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right),$$
$$\int_0^1 t\left(|f'(a)|^{q((1-t)/2)^s}|f'(b)|^{q((1+t)/2)^s}\right)dt \le |f'(a)f'(b)|^{q/2s} g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right). \tag{3.14}$$

From (3.11) to (3.14), (3.2) holds. This completes the required proof. □

Applying Theorem 3.1 to $q = 1, s = 1$, respectively, results in the following corollary.

Corollary 3.2. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$ for $s \in (0,1]$, then*

(i) *when $q = 1$, one has*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4} G_1(s,1;g_1(\alpha),g_2(\alpha)),$$
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4} G_1(s,1;g_2(\alpha),g_1(\alpha)). \tag{3.15}$$

(ii) *when $s = 1$, one has*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1(1,q;g_1(\alpha),g_2(\alpha)),$$
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q} G_1(1,q;g_2(\alpha),g_1(\alpha)), \tag{3.16}$$

where $g_1(\alpha)$, $g_2(\alpha)$, $\alpha(u,v)$, $G_1(s,q;g_2(\alpha),g_1(\alpha))$ are same with (3.3)–(3.5).

Theorem 3.3. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, a, $b \in I$, with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, for $q > 1$ and $s \in (0,1]$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s,q;g_3(\alpha)), \tag{3.17}$$

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(s,q;g_3(\alpha)), \tag{3.18}$$

where

$$G_2(s,q;g_3(\alpha)) =$$
$$\begin{cases} \left(\left|f'(a)\right|^{s} + \left|f'(a)f'(b)\right|^{s/2}\right)\left[g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right)\right]^{1/q}, & \left|f'(a)\right| \le 1, \\ \left(\left|f'(a)\right|^{1/s} + \left|f'(a)\right|^{1/2s}\left|f'(b)\right|^{s/2}\right)\left[g_3\left(\alpha\left(\frac{q}{2s},\frac{sq}{2}\right)\right)\right]^{1/q}, & \left|f'(b)\right| \le 1 \le \left|f'(a)\right|, \\ \left(\left|f'(a)\right|^{1/s} + \left|f'(a)f'(b)\right|^{1/2s}\right)\left[g_3\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q}, & 1 \le \left|f'(b)\right|, \end{cases} \tag{3.19}$$

$$g_3(\alpha) = \begin{cases} 1, & \alpha = 1, \\ \dfrac{\alpha - 1}{\ln \alpha}, & \alpha \ne 1, \end{cases}$$

and $\alpha(u,v)$ *is the same as in* (3.4).

Proof. (1) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a,b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{b-a}{4}\int_0^1 \left[t\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right| + |t-1|\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|\right]dt \\ &\le \frac{b-a}{4}\left[\left(\int_0^1 t^{q/(q-1)}dt\right)^{1-1/q}\left(\int_0^1 \left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\right. \\ &\quad \left.+\left(\int_0^1 (1-t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 \left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|^q dt\right)^{1/q}\right] \\ &\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left(\int_0^1 \left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt\right)^{1/q}\right. \\ &\qquad \left.+\left(\int_0^1 \left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt\right)^{1/q}\right]. \end{aligned} \tag{3.20}$$

(i) If $|f'(a)| \le 1$, we have

$$\begin{aligned} \int_0^1 \left|f'(a)\right|^{q((2-t)/2)^s}\left|f'(b)\right|^{q(t/2)^s}dt &\le \left|f'(a)\right|^{sq} g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \\ \int_0^1 \left|f'(a)\right|^{q((1-t)/2)^s}\left|f'(b)\right|^{q((1+t)/2)^s}dt &\le \left|f'(a)f'(b)\right|^{sq/2} g_3\left(\alpha\left(\frac{sq}{2},\frac{sq}{2}\right)\right), \end{aligned} \tag{3.21}$$

(ii) If $|f'(b)| \leq 1 \leq |f'(a)|$, we have

$$\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right),$$
$$\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \leq |f'(a)|^{q/2s} |f'(b)|^{sq/2} g_3\left(\alpha\left(\frac{q}{2s}, \frac{sq}{2}\right)\right). \tag{3.22}$$

(iii) If $1 \leq |f'(b)|$, we have

$$\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \leq |f'(a)|^{q/s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right),$$
$$\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \leq |f'(a) f'(b)|^{q/2s} g_3\left(\alpha\left(\frac{q}{2s}, \frac{q}{2s}\right)\right). \tag{3.23}$$

From (3.20) to (3.23), (3.17) holds.

(2) Since $|f'(x)|^q$ is s-geometrically convex and monotonically decreasing on $[a, b]$, from Lemma 2.1 and Hölder inequality, we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \left[\left(\int_0^1 |f'(a)|^{q((2-t)/2)^s} |f'(b)|^{q(t/2)^s} dt \right)^{1/q} \right. \\ &\qquad \left. + \left(\int_0^1 |f'(a)|^{q((1-t)/2)^s} |f'(b)|^{q((1+t)/2)^s} dt \right)^{1/q} \right]. \end{aligned} \tag{3.24}$$

From (3.24) and (3.21) to (3.23), (3.18) holds. This completes the proof. □

If taking $s = 1$ in Theorem 3.3, we can derive the following corollary.

Corollary 3.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a, b \in I$, with $a < b$, and $f' \in L([a, b])$. If $|f'(x)|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q > 1$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(1, q; g_3(\alpha)),$$
$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} G_2(1, q; g_3(\alpha)), \tag{3.25}$$

where $\alpha(u, v)$, $G_2(s, q; g_3(\alpha))$, and $g_3(\alpha)$ are the same as in Theorem 3.3.

4. Application to Special Means

Let

$$A(a,b)=\frac{a+b}{2},\qquad L(a,b)=\frac{b-a}{\ln b-\ln a}\quad (a\neq b),$$
$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p},\quad a\neq b,\ p\in R,\ p\neq 0,\ -1 \tag{4.1}$$

be the arithmetic, logarithmic, generalized logarithmic means for $a,b>0$ respectively.

Let $f(x)=x^s/s,\ x\in(0,1],\ 0<s<1$, $q\geq 1$, and then the function

$$\left|f'(x)\right|^q=x^{(s-1)q} \tag{4.2}$$

is monotonically decreasing on $(0,1]$. For $\lambda\in[0,1]$, we have

$$(s-1)q(\lambda^s-\lambda)\leq 0,\qquad (s-1)q\left((1-\lambda)^s-(1-\lambda)\right)\leq 0. \tag{4.3}$$

Hence, $|f'(x)|^q$ is s-geometrically convex on $(0,1]$ for $0<s<1$.

Theorem 4.1. *Let* $0<a<b\leq 1$, $0<s<1$, *and* $q\geq 1$. *Then*

$$\begin{aligned}
\left|[A(a,b)]^s-[L_s(a,b)]^s\right|&\leq\frac{(b-a)^{1-1/q}s}{8}\left(\frac{4s}{(1-s)q}L(a,b)\right)^{1/q}\\
&\quad\times\left[a^{(s-1)/(2s)}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)-b^{(s-1)q/(2s)}\right\}^{1/q}\right.\\
&\quad\left.+\,b^{(s-1)/(2s)}\left\{a^{(s-1)q/(2s)}-\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q}\right]\\
\left|A(a^s,b^s)-[L_s(a,b)]^s\right|&\\
\leq\frac{(b-a)^{1-1/q}s}{8}&\left(\frac{4s}{(1-s)q}L(a,b)\right)^{1/q}\\
&\quad\times\left[b^{(s-1)/(2s)}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)-b^{(s-1)q/(2s)}\right\}^{1/q}\right.\\
&\quad\left.+\,a^{(s-1)/(2s)}\left\{a^{(s-1)q/(2s)}-\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q}\right].
\end{aligned}\tag{4.4}$$

In particular, if $q = 1$*, one has*

$$\left|[A(a,b)]^s - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{4}\left[L_{(s-1)/(2s)-1}(a,b)\right]^{(s-1)/s-2}[L(a,b)]^2$$
$$\left|A(a^s,b^s) - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{4}L(a,b)\left\{2\left[L_{(s-1)/s-1}(a,b)\right]^{(s-1)/s-1} - \left[L_{(s-1)/(2s)-1}(a,b)\right]^{(s-1)/s-2}L(a,b)\right\}. \tag{4.5}$$

Proof. Let $f(x) = x^s/s,\ x \in (0,1],\ 0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \ge 1$ and

$$\left|f'(a)\right|^{1/s}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}a^{(s-1)/2s}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q},$$
$$\left|f'(a)f'(b)\right|^{1/(2s)}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}b^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{a^{(s-1)q/(2s)} - \left[L_{(s-1)q/((2s)-1)}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right\}^{1/q},$$
$$\left|f'(a)f'(b)\right|^{1/(2s)}\left[g_1\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}b^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left\{\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b) - b^{(s-1)q/(2s)}\right\}^{1/q},$$
$$\left|f'(a)\right|^{1/s}\left[g_2\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)\right]^{1/q} = (b-a)^{-1/q}a^{(s-1)/(2s)}\left(\frac{2sL(a,b)}{(1-s)q}\right)^{1/q}\left[a^{(s-1)q/(2s)} - \left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b)\right]^{1/q}. \tag{4.6}$$

By Theorem 3.1, Theorem 4.1 is thus proved. □

Theorem 4.2. *Let* $0 < a < b \le 1$*,* $s \in (0,1)$*, and* $q > 1$*. Then one has*

$$\left|[A(a,b)]^s - [L_s(a,b)]^s\right| \le \frac{(b-a)s}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}A\left(a^{(s-1)/(2s)},b^{(s-1)/(2s)}\right)$$
$$\times\left[\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/2s-1}L(a,b)\right]^{1/q}$$

$$\left|A(a^s,b^s)-[L_s(a,b)]^s\right| \le \frac{(b-a)s}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q} A\left(a^{(s-1)/(2s)},b^{(s-1)/(2s)}\right) \times \left[\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1} L(a,b)\right]^{1/q}. \tag{4.7}$$

Proof. Let $f(x)=x^s/s$, $x\in(0,1]$, $0<s<1$. Then $|f'(a)|=a^{s-1}>b^{s-1}=|f'(b)|\ge 1$ and

$$g_3\left(\alpha\left(\frac{q}{2s},\frac{q}{2s}\right)\right)=a^{-(s-1)q/(2s)}\left[L_{(s-1)q/(2s)-1}(a,b)\right]^{(s-1)q/(2s)-1}L(a,b). \tag{4.8}$$

Using Theorem 3.3, Theorem 4.2 is thus proved. □

Acknowledgments

The research was supported by Science Research Funding of Inner Mongolia University for Nationalities (Project no. NMD1103).

References

[1] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions," *Aequationes Mathematicae*, vol. 48, no. 1, pp. 100–111, 1994.

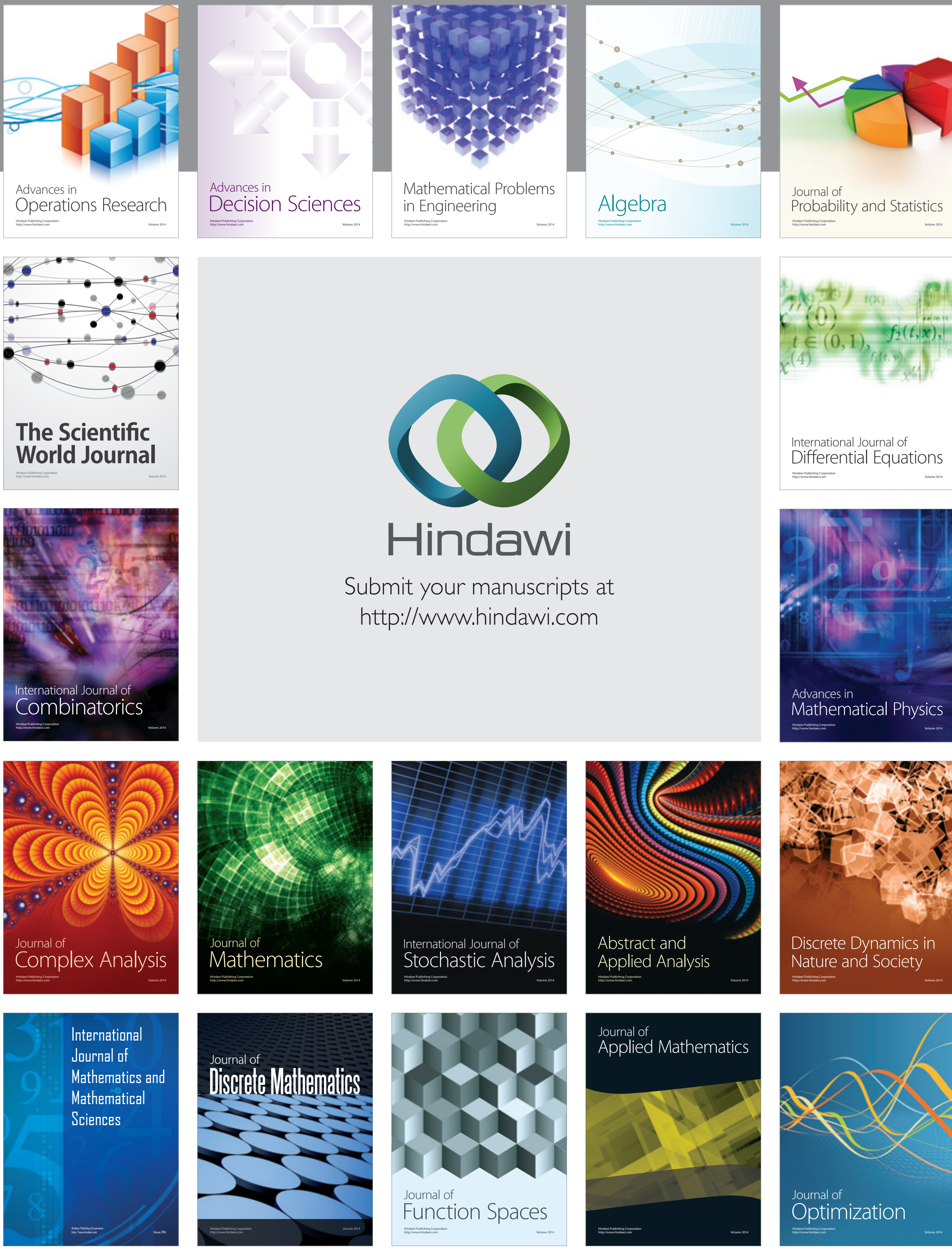
[2] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[3] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.

[4] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard-type inequalities for s-convex functions," *Applied Mathematics and Computation*, vol. 193, no. 1, pp. 26–35, 2007.

[5] S. Hussain, M. I. Bhatti, and M. Iqbal, "Hadamard-type inequalities for s-convex functions. I," *Punjab University. Journal of Mathematics*, vol. 41, pp. 51–60, 2009.

[6] M. W. Alomari, M. Darus, and U. S. Kirmaci, "Some inequalities of Hermite-Hadamard type for s-convex functions," *Acta Mathematica Scientia B*, vol. 31, no. 4, pp. 1643–1652, 2011.



Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization