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Research Article

A New Approach to the Approximation of Common Fixed Points of an Infinite Family of Relatively Quasinonexpansive Mappings with Applications

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By using a specific way of choosing the indexes, we propose an iteration algorithm generated by the monotone CQ method for approximating common fixed points of an infinite family of relatively quasinonexpansive mappings. A strong convergence theorem without the stronger assumptions of the AKTT condition and the *AKTT condition imposed on the involved mappings is established in the framework of Banach space. As application, an iterative solution to a system of equilibrium problems is studied. The result is more applicable than those of other authors with related interest.

1. Introduction

Let C be a nonempty and closed convex subset of a real Banach space E. A mapping $T : C \to E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping T is said to be quasi-nonexpansive if $F(T) := \{x \in C : x = Tx\} \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C,\ p \in F(T). \tag{1.2}$$

It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. There are many methods for approximating fixed points of quasi-nonexpansive mappings. In 1953, Mann [1] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.3}$$

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$. Approximation of fixed points of nonexpansive mappings via Mann's algorithm has extensively been investigated. One of the fundamental convergence results was proved by Reich [2]. In infinite-dimensional Hilbert spaces, Mann iteration can yield only weak convergence (see [3, 4]).

Attempts to modify the Mann iteration method (1.3) for strong convergence have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.3) for a nonexpansive mapping T from C into itself in a Hilbert space: from an arbitrary $x_0 \in C$,

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{aligned} \tag{1.4}$$

where P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H. They proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

Recently, Su and Qin [6] introduced a monotone CQ method for nonexpansive mapping, defined as follows: from an arbitrary $x_0 \in C$,

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_0 &= \{z \in C : \|y_0 - z\| \le \|x_0 - z\|\}, \qquad Q_0 = C, \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{aligned} \tag{1.5}$$

and it proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

We now recall some definitions concerning relatively quasi-nonexpansive mappings. Let E be a real smooth Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. The normalized duality mapping J from E to E^* is defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E, \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between E and E^*. Readers are directed to [7] (and its review [8]), where the properties of the duality mapping and several related topics are presented. The function $\phi : E \times E \to \mathrm{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.7}$$

Let T be a mapping from C into E. A point p in C is said to be an *asymptotic fixed point* [9] of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

We say that the mapping T is *relatively nonexpansive* (see [10]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;

(R2) $\phi(p, Tx) \leq \phi(p, x), \forall p \in F(T)$;

(R3) $F(T) = \widehat{F}(T)$.

If T satisfies (R1) and (R2), then T is called *relatively quasi-nonexpansive*.

Several articles have provided methods for approximating fixed points of relatively quasi-nonexpansive mappings [11–16]. Employing the ideas of Su and Qin [6], and of Aoyama et al. [17], in 2008, Nilsrakoo and Saejung [18] used the following iterations to obtain strong convergence theorems for common fixed points of a countable family of relatively quasi-nonexpansive mappings in a Banach space

$$\begin{gathered}
x_0 \in C, \quad C_{-1} = Q_{-1} = C; \\
y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_n x_n), \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_n x_0, \quad \forall n \geq 0.
\end{gathered} \tag{1.8}$$

However, the results were obtained under two stronger assumption conditions, namely, the $AKTT$-condition and the ${}^*AKTT$-condition imposed on the involved mappings.

Inspired and motivated by those studies mentioned above, in this paper, we use a modified type of the iteration scheme (1.8) for approximating common fixed points of an infinite family of relatively quasi-nonexpansive mappings; without stronger assumptions imposed on the involved mappings, a strong convergence theorem in Banach spaces is obtained for solving a system of equilibrium problems. The results improve those of other authors with related interest.

2. Preliminaries

Throughout the paper, let E be a real Banach space. We say that E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \qquad x \neq y \Longrightarrow \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geq \epsilon \Longrightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{2.2}$$

It is known that if E is uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$; the limit (2.3) is attained uniformly for $x \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$; the limit (2.3) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

We also know the following properties (see, e.g., [19] for details).

(1) E (E^*, resp.) is uniformly convex $\Leftrightarrow$ E^* (E, resp.) is uniformly smooth.

(2) $Jx \neq \emptyset$ for each $x \in E$.

(3) If E is reflexive, then J is a mapping from E onto E^*.

(4) If E is strictly convex, then $Jx \cap Jy = \emptyset$ as $x \neq y$.

(5) If E is smooth, then J is single-valued.

(6) If E has a Fréchet differentiable norm, then J is norm-to-norm continuous.

(7) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

(8) If E is a Hilbert space, then J is the identity operator.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathrm{R}^+$ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2. \tag{2.4}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{2.5}$$

Moreover, we know the following results.

Lemma 2.1 (see [13]). *Let E be a strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.2 (see [11]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x\| - \|y\|) \le \phi(x, y) \tag{2.6}$$

for all $x, y \in B_r := \{z \in E : PzP \le r\}$.

Let C be a nonempty and closed convex subset of E. Suppose that E is reflexive, strictly convex, and smooth. It is known in [20] that for any $x \in E$, there exists a unique point $x^* \in C$ such that

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x). \tag{2.7}$$

Following Alber [21], we denote such an x^* by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C. What follows are the well-known facts concerning the generalized projection.

Lemma 2.3 (see [20]). *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then*

$$x^* = \Pi_C x \Longleftrightarrow \langle x^* - y, Jx - Jx^* \rangle \ge 0, \quad \forall y \in C. \tag{2.8}$$

Lemma 2.4 (see [20]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C. \tag{2.9}$$

Dealing with the generalized projection from E onto the fixed point set of a relatively quasi-nonexpansive mapping, we have the following result.

Lemma 2.5 (see [18]). *Let E be a strictly convex and smooth Banach space, let C be a nonempty and closed convex subset of E, and let T be a relatively quasi-nonexpansive mapping from C into E. Then $F(T)$ is closed and convex.*

Let C be a subset of a Banach space E and let $\{T_n\}$ be a family of mappings from C into E. For a subset B of C, we say that

(i) $(\{T_n\}, B)$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty; \tag{2.10}$$

(ii) $(\{T_n\}, B)$ satisfies *AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{\|JT_{n+1}z - JT_n z\| : z \in B\} < \infty. \tag{2.11}$$

3. Main Results

Recall that an operator T in a Banach space is *closed* if $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, then $Tx = y$.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty and closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty} : C \to E$ be a sequence of closed and relatively quasi-nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Starting from an arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is define by*

$$\begin{gathered} x_1 \in C, \qquad C_0 = Q_0 = C; \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_{i_n} x_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_n x_1, \quad \forall n \geq 1, \end{gathered} \tag{3.1}$$

where $\Pi_n := \Pi_{C_n \bigcap Q_n}$ and $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$; i_n is the solution to the positive integer equation: $n = i + (m-1)m/2$ $(m \geq i, n = 1, 2, \ldots)$, that is, for each $n \geq 1$, there exists a unique i_n such that

$$\begin{gathered} i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \quad i_6 = 3, \quad i_7 = 1, \quad i_8 = 2, \\ i_9 = 3, \quad i_{10} = 4, \quad i_{11} = 1, \ldots. \end{gathered} \tag{3.2}$$

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. We first claim that both C_n and Q_n are closed and convex. This follows from the fact that $\phi(z, y_n) \leq \phi(z, x_n)$ is equivalent to the following:

$$2\langle z, Jx_n - Jy_n \rangle \leq \|x_n\|^2 - \|y_n\|^2. \tag{3.3}$$

It is clear that $F \subset C = C_0 \cap Q_0$. Next, we show that

$$F \subset C_n \cap Q_n, \quad \forall n \geq 1. \tag{3.4}$$

Suppose that $F \subset C_{k-1} \cap Q_{k-1}$ for some $k \geq 2$. Letting $p \in F$, we then have

$$\begin{aligned}
\phi(p, y_k) &= \phi\Big(p, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J T_{i_k} - x_k)\Big) \\
&= \|p\|^2 - 2\langle p, \alpha_k J x_k + (1-\alpha_k) J T_{i_k} x_k \rangle + \|\alpha_k J x_k + (1-\alpha_k) J T_{i_k} x_k\|^2 \\
&\leq \|p\|^2 - 2\alpha_k \langle p, J x_k \rangle - 2(1-\alpha_k)\langle p, J T_{i_k} x_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_{i_k} x_k\|^2 \\
&= \alpha_k \Big(\|p\|^2 - 2\langle p, J x_k \rangle + \|x_k\|^2\Big) + (1-\alpha_k)\Big(\|p\|^2 - 2\langle p, J T_{i_k} x_k \rangle + \|T_{i_k} x_k\|^2\Big) \\
&= \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, T_{i_k} x_k) \\
&\leq \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, x_k) \\
&= \phi(p, x_k).
\end{aligned} \tag{3.5}$$

This implies that $F \subset C_k$. It follows from $x_k = \Pi_{k-1} x_1$ and Lemma 2.3 that

$$\langle x_k - z, J x_1 - J x_k \rangle \geq 0, \quad \forall z \in C_{k-1} \cap Q_{k-1}. \tag{3.6}$$

Particularly,

$$\langle x_k - z, J x_1 - J x_k \rangle \geq 0, \quad \forall p \in F \tag{3.7}$$

and hence $F \subset Q_k$, which yields that

$$F \subset C_k \cap Q_k. \tag{3.8}$$

By induction, (3.4) holds. This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n and Lemma 2.3 that $x_n = \Pi_{Q_n} x_1$. Since $x_{n+1} = \Pi_n x_1 \in Q_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.9}$$

Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing. Using $x_n = \Pi_{Q_n} x_1$ and Lemma 2.4, we have

$$\phi(x_n, x_1) = \phi(\Pi_{Q_n} x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1) \tag{3.10}$$

for all $p \in F$ and for all $n \geq 1$, that is, $\{\phi(x_n, x_1)\}$ is bounded. Then

$$\lim_{n \to \infty} \phi(x_n, x_1) \text{ exists.} \tag{3.11}$$

In particular, by (2.5), the sequence $\{(\|x_n\| - \|x_1\|)^2\}$ is bounded. This implies that $\{x_n\}$ is bounded. Note again that $x_n = \Pi_{Q_n} x_1$ and for any positive integer k, $x_{n+k} \in Q_{n+k-1} \subset Q_n$. By Lemma 2.4,

$$\begin{aligned} \phi(x_{n+k}, x_n) &= \phi\left(x_{n+k}, \Pi_{Q_n} x_1\right) \\ &\leq \phi(x_{n+k}, x_1) - \phi\left(\Pi_{Q_n} x_1, x_1\right) \\ &= \phi(x_{n+k}, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.12}$$

By Lemma 2.2, we have, for any positive integers m, n with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1), \tag{3.13}$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. Then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence in C, so there exists an $x^* \in C$ such that

$$x_n \longrightarrow x^* \quad (n \longrightarrow \infty). \tag{3.14}$$

In view of $x_{n+1} = \Pi_n x_1 \in C_n$ and the definition of C_n, we also have

$$\phi\left(x_{n+1}, y_n\right) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 1. \tag{3.15}$$

This implies that

$$\lim_{n \to \infty} \phi\left(x_{n+1}, y_n\right) = \lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.16}$$

It follows from Lemma 2.2 that

$$\lim_{n \to \infty} \left\|x_{n+1} - y_n\right\| = \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \to \infty} \left\|Jx_{n+1} - Jy_n\right\| = \lim_{n \to \infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.18}$$

On the other hand, we have, for each $n \geq 1$,

$$\begin{aligned} \left\|Jx_{n+1} - Jy_n\right\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1 - \alpha_n) JT_{i_n} x_n)\| \\ &= \|(1 - \alpha_n)(Jx_{n+1} - JT_{i_n} x_n) - \alpha_n (Jx_n - Jx_{n+1})\| \\ &\geq (1 - \alpha_n)\|Jx_{n+1} - JT_{i_n} x_n\| - \alpha_n \|Jx_n - Jx_{n+1}\|, \end{aligned} \tag{3.19}$$

and hence

$$\|Jx_{n+1} - JT_{i_n}x_n\| \leq \frac{1}{1-\alpha_n}\left\|Jx_{n+1} - Jy_n\right\| + \frac{\alpha_n}{1-\alpha_n}\|Jx_n - Jx_{n+1}\|. \tag{3.20}$$

From (3.18) and $\limsup_{n\to\infty}\alpha_n < 1$, we obtain that

$$\lim_{n\to\infty}\|Jx_{n+1} - JT_{i_n}x_n\| = 0. \tag{3.21}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1} - T_{i_n}x_n\| = \lim_{n\to\infty}\left\|J^{-1}(Jx_{n+1}) - J^{-1}(JT_{i_n}x_n)\right\| = 0. \tag{3.22}$$

It follows from (3.17) that, as $n \longrightarrow \infty$,

$$\|x_n - T_{i_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{i_n}x_n\| \longrightarrow 0. \tag{3.23}$$

Now, set $\mathcal{K}_i = \{k \geq 1 : k = i + (m-1)m/2, m \geq i, m \in \mathbb{Z}^+\}$ for each $i \geq 1$. Note that $T_{i_k} = T_i$ whenever $k \in \mathcal{K}_i$. For example, by the definition of $\mathcal{K}_1$, we have $\mathcal{K}_1 = \{1,2,4,7,11,16,\ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Then it follows from (3.23) that

$$\lim_{\mathcal{K}_i \ni k\to\infty}\|T_i x_k - x_k\| = 0, \quad \forall i \geq 1. \tag{3.24}$$

Since $\{x_k\}_{k\in\mathcal{K}_i}$ is a subsequence of $\{x_n\}$, (3.14) implies that $x_k \to x^*$ as $\mathcal{K}_i \ni k \to \infty$. It immediately follows from (3.24) and the closedness of T_i that $x^* \in F(T_i)$ for each $i \geq 1$, and hence $x^* \in F$. Furthermore, by (3.10),

$$\phi(x_*, x_1) = \lim_{n\to\infty}\phi(x_n, x_1) \leq \phi(p, x_1), \quad \forall p \in F. \tag{3.25}$$

This implies that $x^* = \Pi_F x_1$. The proof is completed. □

Remark 3.2. Note that the algorithm (3.1) is based on the projection onto an intersection of two closed and convex sets. An example [22] of how to compute such a projection is given as follows.

Dykstra's Algorithm

Let $\Omega_1, \Omega_2, \ldots, \Omega_p$ be closed and convex subsets of $\mathbb{R}^n$. For any $i = 1, 2, \ldots, p$ and $x^0 \in \mathbb{R}^n$, the sequences $\{x_i^k\}$ are defined by the following recursive formulae:

$$\begin{aligned} x_0^k &= x_p^{k-1}, \\ x_i^k &= P_{\Omega_i}\left(x_{i-1}^k - y_i^{k-1}\right), \quad i = 1, 2, \ldots, p, \\ y_i^k &= x_i^k - \left(x_{i-1}^k - y_i^{k-1}\right), \quad i = 1, 2, \ldots, p, \end{aligned} \tag{3.26}$$

for $k = 1, 2, \ldots$ with initial values $x_p^0 = x^0$ and $y_i^0 = 0$ for $i = 1, 2, \ldots, p$. If $\Omega := \bigcap_{i=1}^p \Omega_i \neq \emptyset$, then $\{x_i^k\}$ converges to $x^* = P_\Omega(x^0)$, where $P_\Omega(x) := \arg\inf_{y \in \Omega} \|y - x\|^2$, for all $x \in \mathbb{R}^n$.

4. Applications

The so-called *convex feasibility problem* for a family of mappings $\{T_i\}_{i=1}^\infty$ is to find a point in the nonempty intersection $\bigcap_{i=1}^\infty F(T_i)$, which exactly illustrates the importance of finding fixed points of infinite families. The following example also clarifies the same thing.

Example 4.1. Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty and closed convex subset of E, and $\{f_i\}_{i=1}^\infty : C \to C$ a countable family of bifunctions satisfying the conditions: for each $i \geq 1$,

(A_1) $f_i(x, x) = 0$;

(A_2) f_i is monotone, that is, $f_i(x, y) + f_i(y, x) \leq 0$;

(A_3) $\limsup_{t \downarrow 0} f_i(x + t(z - x), y) \leq f_i(x, y)$;

(A_4) the mapping $y \mapsto f_i(x, y)$ is convex and lower semicontinuous.

A *system of equilibrium problems* for $\{f_i\}_{i=1}^\infty$ is to find an $x^* \in C$ such that

$$f_i(x^*, y) \geq 0, \quad \forall y \in C, \ i \geq 1, \tag{4.1}$$

whose set of common solutions is denoted by $EP := \bigcap_{i=1}^\infty EP(f_i)$, where $EP(f_i)$ denotes the set of solutions to the equilibrium problem for f_i $(i = 1, 2, \ldots)$. It will be shown in Theorem 4.3 that such a system of problems can be reduced to approximation of some fixed points of a countable family of nonexpansive mappings.

Example 4.2 (see [23]). Let $r > 0$. Define a countable family of mappings $\{T_{r,i}\}_{i=1}^\infty : E \to C$ as follows:

$$T_{r,i}(x) = \left\{ z \in C : f_i(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall i \geq 1. \tag{4.2}$$

Then we have that

(1) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of single-valued mappings;

(2) $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively quasi-nonexpansive mappings;

(3) $F := \bigcap_{i=1}^{\infty} F(T_{r,i}) = EP$.

Now, we have the following result.

Theorem 4.3. *Let C, E, and $\{\alpha_n\}$ be the same as those in Theorem 3.1. Let $\{f_i\}_{i=1}^{\infty} : C \to C$ be a countable family of bifunctions satisfying the conditions (A_1)–(A_4). Let $\{T_{r,i}\}_{i=1}^{\infty} : E \to C$ be a countable family of mappings defined by* (4.2). *Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered}
x_1 \in C, \quad C_0 = Q_0 = C; \\
f_{i_n}(u_n, y) + \frac{1}{r}\langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Ju_n), \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_n x_1, \quad \forall n \geq 1,
\end{gathered} \tag{4.3}$$

where i_n satisfies the positive integer equation: $n = i + (m-1)m/2$ $(m \geq i, n = 1, 2, \ldots)$. If $F := \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \emptyset$, then $\{x_n\}$ strongly converges to $\Pi_F x_1$ which is a common solution of the system of equilibrium problems for $\{f_i\}_{i=1}^{\infty}$.

Proof. Since each $T_{r,i}$ is single-valued, $u_n = T_{r,i_n} x_n$ for all $n \geq 1$. In addition, we have pointed out in Example 4.2 that $F = EP$ and $\{T_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively quasi-nonexpansive mappings. Hence, (4.3) can be rewritten as follows:

$$\begin{gathered}
x_1 \in C, \quad C_0 = Q_0 = C; \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT_{r,i_n} x_n), \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_n x_1, \quad \forall n \geq 1.
\end{gathered} \tag{4.4}$$

Therefore, this conclusion can be obtained immediately from Theorem 3.1. □

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