Research Article

Power Increasing Sequences and Their Some New Applications

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In the work of Bor (2008), we have proved a result dealing with $|\overline{N}, p_n, \theta_n|_k$ summability factors by using a quasi- β -power increasing sequence. In this paper, we prove that result under less and more weaker conditions. Some new results have also been obtained.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). We write $\mathcal{BU}_{\mathcal{O}} = \mathcal{BU} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}} = \{x = (x_k) \in \Omega : \lim_k |x_k| = 0\}$, $\mathcal{BU} = \{x = (x_k) \in \Omega : \sum_k |x_k - x_{k+1}| < \infty\}$ and Ω being the space of all real or complex-valued sequences. A positive sequence $X = (X_n)$ is said to be a quasi- β -power increasing sequence if there exists a constant $K = K(\beta, X) \geq 1$ such that $Kn^{\beta}X_n \geq m^{\beta}X_m$ holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi- β -power increasing sequence for any nonnegative β , but the converse is not true for $\beta > 0$. Moreover, for any positive β there exists a quasi- β -power increasing sequence tending to infinity, but it is not almost increasing (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \ i \ge 1). \tag{1.1}$$

Let (θ_n) be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k, \ k \ge 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |V_n - V_{n-1}|^k < \infty, \tag{1.2}$$

and it is said to be summable $|\overline{N}, p_n, \theta_n|_k$, $k \ge 1$, if (see [4])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |V_n - V_{n-1}|^k < \infty, \tag{1.3}$$

where

$$V_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v. \tag{1.4}$$

If we take $\theta_n = P_n/p_n$, then $|\overline{N}, p_n, \theta_n|_k$ summability reduces to $|\overline{N}, p_n|_k$ summability. Also if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C, 1|_k$ summability (see [5]). Furthermore, if we take $\theta_n = n$, then $|\overline{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ summability (see [6]).

2. Known Result

In [7], we have proved the following theorem dealing with $|\overline{N}, p_n, \theta_n|_k$ summability factors of infinite series.

Theorem 2.1. Let $(\lambda_n) \in \mathcal{BU}_{\mathcal{O}}$, (X_n) be a quasi- β -power increasing sequence for some β $(0 < \beta < 1)$, and let $(\theta_n p_n/P_n)$ be a nonincreasing sequence. Suppose also there exists sequences (λ_n) and (p_n) such that

$$\left| \lambda_m \middle| X_m = O(1) \quad \text{as } m \longrightarrow \infty,$$

$$\sum_{n=1}^m n X_n \middle| \Delta^2 \lambda_n \middle| = O(1),$$

$$\sum_{n=1}^m \frac{P_n}{n} = O(P_m) \quad \text{as } m \longrightarrow \infty.$$
(2.1)

If

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad as \ m \longrightarrow \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m) \quad \text{as } m \longrightarrow \infty,$$
 (2.3)

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k$, $k \ge 1$, where (t_n) is the nth (C, 1) mean of the sequence (na_n) .

Remark 2.2. It should be noticed that, if we take (X_n) as an almost increasing sequence and $\theta_n = P_n/p_n$, then we obtain a theorem of Mazhar (see [8]), in this case the condition " $(\lambda_n) \in \mathcal{BU}_{\mathcal{O}}$ " is not needed.

3. The Main Result

The aim of this paper is to prove Theorem 2.1 under less and more weaker conditions. Now, we prove the following theorem.

Theorem 3.1. Let (X_n) be a quasi- β -power increasing sequence for some β $(0 < \beta < 1)$, and let $(\theta_n p_n/P_n)$ be a nonincreasing sequence. Suppose also there exists sequences (λ_n) and (p_n) such that conditions (2.1) of Theorem 2.1 are satisfied. If

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{3.1}$$

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad \text{as } m \longrightarrow \infty$$
 (3.2)

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n, \theta_n|_k$, $k \ge 1$.

Remark 3.2. It should be noted that conditions (3.1) and (3.2) are the same as conditions (2.2) and (2.3), respectively, when k = 1. When k > 1, conditions (3.1) and (3.2) are weaker than conditions (2.2) and (2.3), respectively. But the converses are not true. In fact, if (2.2) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m).$$
(3.3)

If (3.1) is satisfied, then for k > 1, we obtain that

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = \sum_{n=1}^{m} X_n^{k-1} \frac{|t_n|^k}{n X_n^{k-1}} = O\left(X_m^{k-1}\right) \sum_{n=1}^{m} \frac{|t_n|^k}{n X_n^{k-1}} = O\left(X_m^k\right) \neq O(X_m). \tag{3.4}$$

The similar argument is also valid for the conditions (2.3) and (3.2). Also it should be noted that condition " $(\lambda_n) \in \mathcal{BU}_{\mathcal{O}}$ " has been removed.

We need following lemma for the proof of our theorem.

Lemma 3.3 (see [9]). Under the conditions on the sequences (X_n) and (λ_n) as expressed in the statement of the theorem, one has the following:

$$nX_{n}|\Delta\lambda_{n}| = O(1),$$

$$\sum_{n=1}^{\infty} X_{n}|\Delta\lambda_{n}| < \infty.$$
(3.5)

4. Proof of the Theorem

Let (T_n) denote the (\overline{N}, p_n) mean of the series $\sum a_n \lambda_n$. Then, for $n \ge 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \frac{P_{v-1} \lambda_v}{v} v a_v.$$
(4.1)

By Abel's transformation, we have

$$T_{n} - T_{n-1} = \frac{n+1}{nP_{n}} p_{n} t_{n} \lambda_{n} - \frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} p_{v} t_{v} \lambda_{v} \frac{v+1}{v} + \frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} \Delta \lambda_{v} t_{v} \frac{v+1}{v} + \frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} \Delta \lambda_{v} t_{v} \frac{v+1}{v} + \frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} \Delta \lambda_{v} t_{v} \frac{v+1}{v} + \frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} \Delta \lambda_{v} t_{v} \frac{v+1}{v}$$

$$(4.2)$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$
 (4.3)

Firstly, we have that

$$\sum_{n=1}^{m} \theta_{n}^{k-1} |T_{n,1}|^{k} = \sum_{n=1}^{m} \theta_{n}^{k-1} |\lambda_{n}|^{k-1} |\lambda_{n}| \left(\frac{p_{n}}{P_{n}}\right)^{k} |t_{n}|^{k}$$

$$= O(1) \sum_{n=1}^{m} |\lambda_{n}| \theta_{n}^{k-1} \left(\frac{1}{X_{v}}\right)^{k-1} \left(\frac{p_{n}}{P_{n}}\right)^{k} |t_{n}|^{k}$$

$$= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_{n}| \sum_{v=1}^{n} \theta_{v}^{k-1} \left(\frac{p_{v}}{P_{v}}\right)^{k} \frac{|t_{v}|^{k}}{X_{v}^{k-1}} + O(1) |\lambda_{m}| \sum_{n=1}^{m} \theta_{n}^{k-1} \left(\frac{p_{n}}{P_{n}}\right)^{k} \frac{|t_{n}|^{k}}{X_{n}^{k-1}}$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_{n}| X_{n} + O(1) |\lambda_{m}| X_{m}$$

$$= O(1) \text{ as } m \longrightarrow \infty, \tag{4.4}$$

by virtue of the hypotheses of the theorem and lemma. Now, when k > 1 applying Hölder's inequality with indices k and k', where (1/k) + (1/k') = 1, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v|^k |t_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_v |t_v|^k |\lambda_v| \left(\frac{1}{X_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
= O(1) \sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k |\lambda_v| \left(\frac{1}{X_v}\right)^{k-1} |t_v|^k \\
= O(1) \sum_{v=1}^{m} |\lambda_v| \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k \frac{|t_v|^k}{X_v^{k-1}} = O(1) \quad \text{as } m \longrightarrow \infty.$$

Again we have that

$$\sum_{n=2}^{m+1} \theta_{n}^{k-1} |T_{n,3}|^{k} = O(1) \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left(\frac{p_{n}}{P_{n}}\right)^{k} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} \frac{P_{v}}{v} |\Delta \lambda_{v}|^{k} v^{k} |t_{v}|^{k} \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v}}{v} \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{P_{v}}{v} |t_{v}|^{k} v^{k} |\Delta \lambda_{v}|^{k-1} |\Delta \lambda_{v}| \sum_{n=v+1}^{m+1} \left(\frac{\theta_{n} p_{n}}{P_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\theta_{v} p_{v}}{P_{v}}\right)^{k-1} v^{k-1} \left(\frac{1}{v X_{v}}\right)^{k-1} |\Delta \lambda_{v}| |t_{v}|^{k}$$

$$= O(1) \left(\frac{\theta_{1} p_{1}}{P_{1}}\right)^{k-1} \sum_{v=1}^{m} v |\Delta \lambda_{v}| \frac{|t_{v}|^{k}}{v X_{v}^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v |\Delta \lambda_{v}|) \sum_{i=1}^{v} \frac{|t_{i}|^{k}}{i X_{i}^{k-1}} + O(1) m |\Delta \lambda_{m}| \sum_{v=1}^{m} \frac{|t_{v}|^{k}}{v X_{v}^{k-1}}$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta(v |\Delta \lambda_{v}|) |X_{v} + O(1) m |\Delta \lambda_{m}| X_{m}$$

$$= O(1) \sum_{v=1}^{m-1} |\langle v + 1 \rangle |\Delta^{2} \lambda_{v}| - |\Delta \lambda_{v}| |X_{v} + O(1) m |\Delta \lambda_{m}| X_{m}$$

$$= O(1) \sum_{v=1}^{m-1} v |\Delta^{2} \lambda_{v}| X_{v} + O(1) \sum_{v=1}^{m-1} |\Delta \lambda_{v}| X_{v}$$

$$+ O(1) m |\Delta \lambda_{m}| X_{m} = O(1) \quad \text{as } m \longrightarrow \infty,$$

by virtue of the hypotheses of the theorem and lemma. Finally, we have that

$$\sum_{n=2}^{m+1} \theta_{n}^{k-1} |T_{n,4}|^{k} \leq \sum_{n=2}^{m+1} \theta_{n}^{k-1} \left(\frac{p_{n}}{P_{n}}\right)^{k} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_{v} |\lambda_{v+1}|^{k} |t_{v}|^{k} \frac{1}{v} \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v}}{v}\right\}^{k-1} \\
= O(1) \sum_{v=1}^{m} P_{v} |\lambda_{v+1}|^{k-1} |\lambda_{v+1}| |t_{v}|^{k} \frac{1}{v} \sum_{n=v+1}^{m+1} \left(\frac{\theta_{n} p_{n}}{P_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}} \\
= O(1) \sum_{v=1}^{m} P_{v} \left(\frac{1}{X_{v}}\right)^{k-1} |\lambda_{v+1}| |t_{v}|^{k} \frac{1}{v} \sum_{n=v+1}^{m+1} \left(\frac{\theta_{n} p_{n}}{P_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}} \\
= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \left(\frac{\theta_{v} p_{v}}{P_{v}}\right)^{k-1} \frac{|t_{v}|^{k}}{v X_{v}^{k-1}} \\
= O(1) \left(\frac{\theta_{1} p_{1}}{P_{1}}\right)^{k-1} \sum_{v=1}^{m} |\lambda_{v+1}| \frac{|t_{v}|^{k}}{v X_{v}^{k-1}} \\
= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_{v+1}| \sum_{v=1}^{v} \frac{|t_{r}|^{k}}{r X_{r}^{k-1}} + O(1) |\lambda_{m+1}| \sum_{v=1}^{m} \frac{|t_{v}|^{k}}{v X_{v}^{k-1}} \\
= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_{v+1}| X_{v+1} + O(1) |\lambda_{m+1}| X_{m+1} \\
= O(1) \text{ as } m \longrightarrow \infty,$$

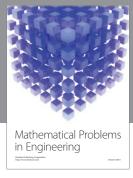
by virtue of the hypotheses of the theorem and lemma. This completes the proof of the theorem. If we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a result dealing with $|C,1|_k$ summability factors. Also, if we take $p_n = 1$ for all values of n, then we have a new result for $|C,1,\theta_n|_k$ summability. Finally, if we take $\theta_n = n$, then we have another new result for $|R,p_n|_k$ summability factors.

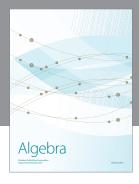
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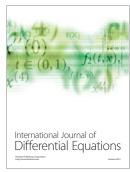


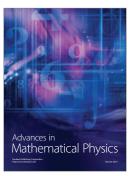


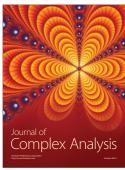




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