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Research Article

On the Self-Intersection Local Time of Subfractional Brownian Motion

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We study the problem of self-intersection local time of d-dimensional subfractional Brownian motion based on the property of chaotic representation and the white noise analysis.

1. Introduction

As an extension of Brownian motion, Bojdecki et al. [1] introduced and studied a rather special class of self-similar Gaussian process. This process arises from occupation time fluctuations of branching particles with Poisson initial condition. It is called the *subfractional Brownian motion*. The so-called subfractional Brownian motion with index $H \in (0,1)$ is a mean zero Gaussian process $S_0^H = \{S_0^H(t), t \geq 0\}$ with the covariance function

$$R_H(t,s) = \mathbb{E}\left[S_0^H(t)S_0^H(s)\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right], \tag{1.1}$$

for all $s,t \geq 0$. For $H = 1/2$, S_0^H coincides with the standard Brownian motion. S_0^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from classical stochastic analysis are not available when dealing with S_0^H. The subfractional Brownian motion has properties analogous to those of fractional Brownian motion, such as self-similarity, Hölder continuous paths, and so forth. But its increments are not stationary, because, for $s \le t$, we have the following estimates:

$$\left[\left(2-2^{2H-1}\right)\wedge 1\right](t-s)^{2H} \le \mathbb{E}\left[\left(S_0^H(t)-S_0^H(s)\right)^2\right] \le \left[\left(2-2^{2H-1}\right)\vee 1\right](t-s)^{2H}. \qquad (1.2)$$

Let $\mathbf{S}^H = \{\mathbf{S}^H(t), t \in \mathbb{N}_+\}$ be a d-dimensional subfractional Brownian motion with multiparameters $H = (H_1, H_2, \dots, H_d)$. Suppose that $d \ge 2$, we are interested in, when it exists, the self-intersection local time of subfractional Brownian motion $\mathbf{S}^H$ which is formally defined as

$$\ell_T^H = \int_0^T \int_0^t \delta_0\left(\mathbf{S}^H(t)-\mathbf{S}^H(s)\right) ds dt, \qquad (1.3)$$

where δ_0 is the Dirac delta function. It measures the amount of time that the processes spend intersecting itself on the time interval $[0,T]$ and has been an important topic of the theory of stochastic process.

More precisely, we study the existence of the limit when ε tends to zero, of the following sequence of processes

$$\ell_{T,\varepsilon}^H = \int_0^T \int_0^t p_\varepsilon\left(\mathbf{S}^H(t)-\mathbf{S}^H(s)\right) ds dt, \qquad (1.4)$$

where

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-d/2}\exp\left(-\frac{|x|^2}{2\varepsilon}\right) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\xi\cdot x-(\varepsilon|\xi|^2/2)} d\xi, \quad x \in \mathbb{R}^d. \qquad (1.5)$$

For $H = 1/2$, the process S_0^H is a classical Brownian motion. The self-intersection local time of the Brownian motion has been studied by many authors such as Albeverio et al. [2], Calais and Yor [3], He et al. [4], Hu [5], Varadhan [6], and so forth. In the case of planar Brownian motion, Varadhan [6] has proved that $\ell_{T,\varepsilon}^{1/2}$ does not converge in L^2 but it can be renormalized so that $\ell_{T,\varepsilon}^{1/2} - (T/2\pi)\log(1/\varepsilon)$ converges in L^2 as ε tends to zero. The limit is called the renormalized self-intersection local time of the planar Brownian motion. This result has been extended by Rosen [7] to the (planar) fractional Brownian motion, where it is proved that for $1/2 < H < 3/4$, $\ell_{T,\varepsilon}^{H,fBm} - C_H T\varepsilon^{-1+(1/2H)}$ converges in L^2 as ε tends to zero, where C_H is a constant depending only on H. Hu [8] showed that, under the condition $H < \min(3/(2d),\ 2/(d+2))$, the (renormalized) self-intersection local time of fractional Brownian motion is in the Meyer-Watanabe test functional space, that is, the L^2 space of "differentiable" functionals. In 2005, Hu and Nualart [9] proved that the renormalized self-intersection local time of d-dimensional fractional Brownian motion exists in L^2 if and only if $H < 3/2d$, which generalizes the Varadhan renormalization theorem to any dimension and with any Hurst parameter. They also showed that in the case $3/4 > H \ge 3/2d$, $r(\varepsilon)\ell_{T,\varepsilon}^{H,fBm}$ converges in distribution to a normal law $N(0, T\sigma^2)$, as ε tends to zero, and $r(\varepsilon) = |\log\varepsilon|^{-1}$ if $H = 3/(2d)$, and $r(\varepsilon) = \varepsilon^{d-3/(2H)}$ if $3/(2d) < H$. Wu and Xiao [10] proved the existence of the intersection local times of (N_i, d), $i = 1,2$-fractional Brownian motions, and had a continuous version. They also established Hölder conditions for the intersection local times and determined the Hausdorff and packing dimensions of the sets of intersection times and intersection points. They extended the results of Nualart and Ortiz-Latorre [11], where the existence of the intersection local times of two independent $(1, d)$-fractional Brownian motions with the same Hurst index was studied by using a different method. Moreover, Wu and Xiao [10] also showed that anisotropy brings subtle differences into the analytic properties of the intersection local times as well as rich geometric structures into the sets of intersection times and intersection points. Oliveira et al. [12] presented expansions of intersection local times of fractional Brownian motions in $\mathbb{R}^d$, for any dimension $d \geq 1$, with arbitrary Hurst coefficients in $(0,1)^d$. The expansions are in terms of Wick powers of white noises (corresponding to multiple Wiener integrals), being well-defined in the sense of generalized white noise functionals. As an application of their approach, a sufficient condition on d for the existence of intersection local times in L^2 was also derived. For the case of subfractional Brownian motion, Yan and Shen [13] studied the so-called collision local time $\ell_T = \int_0^T \delta(S_0^{H_1}(t) - S_0^{H_2}(t))dt$ of two independent subfractional Brownian motion with respective indices $H_i \in (0,1)$, $i = 1,2$. By an elementary method, they showed that ℓ_T is smooth in the sense of Meyer-Watanabe if and only if $\min(H_1, H_2) < 1/3$.

Motivated by all these results, we will study the self-intersection local time of the so-called subfractional Brownian motion (see below for a precise definition), which has been proposed by Bojdecki et al. [1]. Recently, the long-range dependence property has become an important aspect of stochastic models in various scientific area including hydrology, telecommunication, turbulence, image processing, and finance. It is well known that fractional Brownian motion (fBm in short) is one of the best known and most widely used processes that exhibits the long-range dependence property, self-similarity, and stationary increments. It is a suitable generalization of classical Brownian motion. On the other hand, many authors have proposed to use more general self-similar Gaussian process and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which does not have stationary increments. The subfractional Brownian motion has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths, the variation, and the renormalized variation). However, in comparison with fractional Brownian motion, the subfractional Brownian motion has nonstationary increments and the increments over nonoverlapping intervals are more weakly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki et al. [1] is called subfractional Brownian motion). The above mentioned properties make subfractional Brownian motion a possible candidate for models which involve long-range dependence, self-similarity, and nonstationary. Therefore, it seems interesting to study the self-intersection local time of subfractional Brownian motion. And we need more precise estimates to prove our results because of the nonstationary increments. We will view the self-intersection local time of subfractional Brownian motion as the generalized white noise functionals. Furthermore, we discuss the existence and expansions of the self-intersection local times in L^2. We have organized our paper as follows: Section 2 contains the notations, definitions, and results for Gaussian white noise analysis. In Section 3, we present the main results and their demonstrations.

Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C, which may not be the same in each occurrence. Sometimes we will emphasize the dependence of these constants upon parameters.

2. Gaussian White Noise Analysis

In this section, we briefly recall the concepts and results of white noise analysis used through out this work, and for details, see Kuo [15], Obata [16], and so forth.

2.1. Subfractional Brownian Motion

The starting point of white noise analysis for the construction of d-dimensional, $d \geq 1$, subfractional Brownian motion is the real Gélfand triple

$$\mathcal{S}_d(\mathbb{R}) \subset L^2\left(\mathbb{R}, \mathbb{R}^d\right) \subset \mathcal{S}'_d(\mathbb{R}), \tag{2.1}$$

where $L^2(\mathbb{R}, \mathbb{R}^d)$ is the real Hilbert space of all vector-valued square integrable functions with respect to Lebesgue measure on $\mathbb{R}$ and $\mathcal{S}_d(\mathbb{R}), \mathcal{S}'_d(\mathbb{R})$ are the Schwartz spaces of the vectors valued test functions and tempered distributions, respectively. Denote the norm in $L^2(\mathbb{R}, \mathbb{R}^d)$ by $|\cdot|_d$ or if there is no risk of confusion simply by $|\cdot|$ and the dual pairing between $\mathcal{S}'_d$ and $\mathcal{S}_d(\mathbb{R})$ by $\langle \cdot, \cdot \rangle$, which is defined as the bilinear extension of the inner product on $L^2(\mathbb{R}, \mathbb{R}^d)$, that is

$$\langle \mathbf{g}, \mathbf{f} \rangle = \sum_{i=1}^{d} \int_{\mathbb{R}} g_i(x) f_i(x) dx, \tag{2.2}$$

for all $\mathbf{g} = (g_1, g_2, \ldots g_d) \in L^2(\mathbb{R}, \mathbb{R}^d)$ and all $\mathbf{f} = (f_1, f_2, \ldots, f_d) \in \mathcal{S}_d(\mathbb{R})$. By the Minlos theorem, there is a unique probability measure μ on the σ-algebra $\mathcal{B}$ generated by the cylinder sets on $\mathcal{S}'_d(\mathbb{R})$ with characteristic function given by

$$C(\mathbf{f}) := \int_{\mathcal{S}'_d(\mathbb{R})} e^{i\langle \vec{\omega}, \mathbf{f} \rangle} d\mu(\vec{\omega}) = e^{-(1/2)|\mathbf{f}|^2}, \quad \mathbf{f} \in \mathcal{S}_d(\mathbb{R}). \tag{2.3}$$

In this way, we have defined the white noise measure space $(\mathcal{S}'_d(\mathbb{R}), \mathcal{B}, \mu)$. Then a realization of vector of independent subfractional Brownian motion S^{H_j}, $j = 1, 2, \ldots, d$, is given by

$$S^{H_j}(t) = \langle \omega_j, K_{H_j} \rangle = \int_0^t K_{H_j}(t, s) \omega_j(s) ds, \quad \omega_i \in \mathcal{S}'_d(\mathbb{R}). \tag{2.4}$$

We recall the explicit formula for the kernel $K_H(t,s)$ of a one-dimensional subfractional Brownian motion with parameter $H \in (0,1)$

$$\begin{aligned} K_H(t,s) &= -\frac{\sqrt{\pi}s^{1/2-H}}{2^H\Gamma(H+1/2)}\frac{d}{ds}\left(\int_s^t \left(u^2-s^2\right)^{H-1/2}du\right)1_{[0,t]}(s) \\ &= \frac{\sqrt{\pi}s^{3/2-H}}{2^H\Gamma(H+1/2)}\left(\frac{\left(t^2-s^2\right)^{H-1/2}}{t}+\int_s^t\frac{\left(u^2-s^2\right)^{H-1/2}}{u^2}du\right)1_{[0,t]}(s). \end{aligned} \tag{2.5}$$

Especially, for $H > 1/2$, we have

$$K_H(t,s) = \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-1/2)}s^{3/2-H}\left(\int_s^t\left(u^2-s^2\right)^{H-3/2}du\right)1_{[0,t]}(s). \tag{2.6}$$

We refer to Bojdecki et al. [1, 17–19], Dzhaparidze and van Zanten [20], Liu and Yan [21], Liu et al. [22], Shen and Yan [23], Tudor [24–27], Yan and Shen [13, 14], and the references therein for a complete description of subfractional Brownian motion.

2.2. Hida Distributions and Characterization Results

Let us now consider the complex Hilbert space $(L^2) := L^2(\mathcal{S}'_d(\mathbb{R}),\mathcal{B},\mu)$. This space is canonically isomorphic to the symmetric Fock space of symmetric square integrable functions

$$L^2(\mathcal{S}'_d(\mathbb{R}),\mathcal{B},\mu) \simeq \left(\bigoplus_{k=0}^{\infty}\operatorname{Sym}L^2(\mathbb{R}^k,k!d^kx)\right)^{\otimes 2d} = \mathcal{G} \tag{2.7}$$

leading to the chaos expansion of the elements in $L^2(\mathcal{S}'_d(\mathbb{R}),\mathcal{B},\mu)$

$$F(\omega_1,\omega_2,\ldots,\omega_d) = \sum_{\mathbf{n}=(n_1,n_2,\ldots,n_d)\in\mathbb{N}^d}\left\langle :\omega_1^{\otimes n_1}:\otimes\cdots\otimes:\omega_d^{\otimes n_d}:,\mathbf{f_n}\right\rangle, \tag{2.8}$$

with kernel functions $\mathbf{f_n}$ in the Fock space, that is, square integrable functions of the m arguments and symmetric in each n_i-tuple.

For simplicity, in the sequel, we will use the notations

$$\mathbf{n} = (n_1,\ldots,n_d)\in\mathbb{N}^d, \qquad n = \sum_{i=1}^d n_i, \qquad \mathbf{n}! = \prod_{i=1}^d n_i!, \tag{2.9}$$

which reduces expansion (2.8) to

$$F(\omega) = \sum_{\mathbf{n}\in\mathbb{N}^d}\left\langle :\omega^{\otimes\mathbf{n}}:,\mathbf{f_n}\right\rangle, \quad \omega\in\mathcal{S}'_d(\mathbb{R}). \tag{2.10}$$

The norm of F is given by

$$\|F\|^2_{(L^2)} = \sum_{\mathbf{n}\in\mathbb{N}^d} \mathbf{n}!|\mathbf{f_n}|^2_{2,n}, \tag{2.11}$$

where $|\cdot|_{2,n}$ is the norm in $L^2(\mathbb{R}^n, dt)$.

To proceed further, we have to consider a Gelfand triple around the space (L^2). We will use the space $(S)^*$ of Hida distributions (or generalized Brownian functionals) and the corresponding Gelfand triple $(S) \subset (L^2) \subset (S)^*$. Here (S) is the space of white noise test functions such that its dual space (with respect to (L^2)) is the space $(S)^*$. Instead of reproducing the explicit construction of $(S)^*$ in Theorem 2.2 below we characterize this space through its S-transform. We recall that given a $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$, let us consider the Wick exponential

$$\begin{aligned} :\exp\langle\omega, \mathbf{f}\rangle :&= \exp\left(\langle\omega, \mathbf{f}\rangle - \frac{1}{2}(\mathbf{f}, \mathbf{f})\right) \\ &= \sum_{\mathbf{n}\in\mathbb{N}^d} \frac{1}{\mathbf{n}!}\left\langle :\omega^{\otimes\mathbf{n}}:, \mathbf{f}^{\otimes\mathbf{n}}\right\rangle = C(\mathbf{f})e^{\langle\omega,\mathbf{f}\rangle}, \quad \omega \in \mathcal{S}'_d(\mathbb{R}). \end{aligned} \tag{2.12}$$

We define the S-transform of a $\Phi \in (S)^*$ by

$$S\Phi(\mathbf{f}) := \left\langle\left\langle \Phi, :\exp(\langle\cdot, \mathbf{f}\rangle):\right\rangle\right\rangle, \quad \forall \mathbf{f} \in \mathcal{S}_d(\mathbb{R}). \tag{2.13}$$

Here $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the dual pairing between $(S)^*$ and (S) which is defined as the bilinear extension of the sesquilinear inner product on (L^2). We observe that the multilinear expansion of (2.13)

$$S\Phi(\mathbf{f}) := \sum_{\mathbf{n}} \left\langle F_{\mathbf{n}}, \mathbf{f_n}^{\otimes\mathbf{n}}\right\rangle, \tag{2.14}$$

extends the chaos expansion to $\Phi \in (S)^*$ with distribution valued kernels $F_{\mathbf{n}}$ such that

$$\left\langle\left\langle \Phi, \varphi\right\rangle\right\rangle = \sum_{\mathbf{n}} \mathbf{n}!\left\langle F_{\mathbf{n}}, \varphi_{\mathbf{n}}\right\rangle, \tag{2.15}$$

for every generalized test function $\varphi \in (S)$ with kernel function $\varphi_{\mathbf{n}}$.

In order to characterize the space $(S)^*$ through its S-transform, we need the following definition:

Definition 2.1. A function $F : \mathcal{S}_d(\mathbb{R}) \to \mathbb{C}$ is called a U-functional whenever

(1) for every $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{S}_d(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the mapping $\lambda \to F(\lambda\mathbf{f}_1 + \mathbf{f}_2)$ has an entire extension to $\lambda \in \mathbb{C}$,

(2) there are constants $K_1, K_2 > 0$ such that

$$|F(z\mathbf{f})| \le K_1 \exp\left(K_2 |z|^2 \|\mathbf{f}\|^2\right), \quad \forall z \in \mathbb{C}, \mathbf{f} \in \mathcal{S}_d(\mathbb{R}), \tag{2.16}$$

for some continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$.

We are now ready to state the aforementioned characterization result.

Theorem 2.2. *The S-transform defines a bijection between the space $(S)^*$ and the space of U-functionals.*

As a consequence of Theorem 2.2, one may derive the next two statements. The first one concerns the convergence of sequences of Hida distributions and the second one the Bochner integration of families of distributions of the same type.

Corollary 2.3. *Let $(\Phi_n, n \in \mathbb{N})$ be a sequence in $(S)^*$ such that*

(1) *for all $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$, $((S\Phi_n)(\mathbf{f}))_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{C}$;*

(2) *there are $K_1, K_2 > 0$ such that for some continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ one has*

$$|(S\Phi_n)(z\mathbf{f})| \le K_1 e^{K_2|z|^2\|\mathbf{f}\|^2}, \quad z \in \mathbb{C}, \ \mathbf{f} \in \mathcal{S}_d(\mathbb{R}), \ n \in \mathbb{N}, \tag{2.17}$$

then $(\Phi_n, n \in \mathbb{N})$ converges strongly in $(S)^*$ to a unique Hida distribution.

Corollary 2.4. *Let $(\Omega, \mathcal{B}, m)$ be a measure space and $\lambda \to \Phi_\lambda$ be a mapping from Ω to $(S)^*$. We assume that the S-transform of Φ_λ fulfills the following two properties:*

(1) *the mapping $\lambda \to (S\phi_\lambda)(\mathbf{f})$ is measurable for every $\mathbf{f} \in \mathcal{S}_d(\mathbb{R})$,*

(2) *the $(S\Phi_\lambda)(\mathbf{f})$ obeys a U-estimate*

$$|(S\Phi_\lambda)(z\mathbf{f})| \le C_1(\lambda) e^{C_2(\lambda)|z|^2\|\mathbf{f}\|^2}, \quad z \in \mathbb{C}, \ \mathbf{f} \in \mathcal{S}_d(\mathbb{R}), \tag{2.18}$$

for some continuous $\|\cdot\|$ on $\mathcal{S}_{2d}(\mathbb{R})$ and for $C_1 \in L^1(\Omega, m)$, $C_2 \in L^\infty(\Omega, m)$. Then

$$\begin{gathered} \int_\Omega dm(\lambda)\Phi_\lambda \in (S)', \\ S\left(\int_\Omega dm(\lambda)\Phi_\lambda\right)(\mathbf{f}) = \int_\Omega dm(\lambda) S\Phi_\lambda(\mathbf{f}). \end{gathered} \tag{2.19}$$

3. Self-Intersection Local Time

Let us now consider the d-dimensional subfractional Brownian motion $\mathbf{S}^H(t)$ with parameter $H = (H_1, H_2, \ldots, H_d)$. In view of (2.4), for $j = 1, \ldots, d$, $S^{H_j}(t) - S^{H_j}(s) = \langle \Delta K_j, \omega_j \rangle$ with $\Delta K_j = K_{H_j}(t, u) - K_{H_j}(s, u)$.

Proposition 3.1. *For each t and s strictly positive real numbers, the Bochner integral*

$$\delta\Big(\mathbf{S}^H(t)-\mathbf{S}^H(s)\Big) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} d\lambda e^{i\lambda(\mathbf{S}^H(t)-\mathbf{S}^H(s))} \tag{3.1}$$

is a Hida distribution with S-transform given by

$$S\delta\Big(\mathbf{S}^H(t)-\mathbf{S}^H(s)\Big)(\mathbf{f}) = (2\pi)^{-(d/2)}\prod_{j=1}^{d}\frac{1}{\left(\delta^{H_j}(t,s)\right)^{1/2}}\exp\left\{-\frac{\left\langle f_j,\Delta K_j\right\rangle^2}{2\delta^{H_j}(t,s)}\right\} \tag{3.2}$$

for all $\mathbf{f}=(f_1,\ldots,f_d)\in\mathcal{S}_d(\mathbb{R})$ and $\delta^{H_j}(t,s)=|\Delta K_j|_2^2$.

Proof. The proof of this result follows from an applications of Corollary 2.4 to the S-transform of the integrand function

$$\Phi(\vec{\omega}) := e^{i\lambda(\mathbf{S}^H(t)-\mathbf{S}^H(s))},\quad \vec{\omega}=(\omega_1,\ldots,\omega_d). \tag{3.3}$$

With respect to the Lebesgue measure on $\mathbb{R}^d$. By symmetry, we assume that $s\le t$. For this purpose, we begin by observing that, for $\mathbf{f}=(f_1,\ldots,f_d)\in\mathcal{S}_d(\mathbb{R})$, and all $\lambda\in\mathbb{C}$, one has

$$\begin{aligned} S\Phi(\mathbf{f}) &= \prod_{j=1}^{d} S\Big(e^{i\lambda_j\langle\Delta K_j,\omega_j\rangle}\Big)(f_j),\\ S\Big(e^{i\lambda_j\langle\Delta K_j,\omega_j\rangle}\Big)(f_j) &= e^{-(|f_j|_2^2/2)}\int_{S'(\mathbb{R})} d\mu_j(\omega_j)e^{i\lambda_j\langle\Delta K_j,\omega_j\rangle+\langle f_j,\omega_j\rangle}\\ &= e^{-(|f_j|_2^2/2)}\int_{S'(\mathbb{R})} d\mu_j(\omega_j)e^{i\langle\lambda_j\Delta K_j-if_j,\omega_j\rangle} = e^{-(\lambda_j^2|\Delta K_j|_2^2/2)}e^{i\lambda_j\langle\Delta K_j,f_j\rangle}. \end{aligned} \tag{3.4}$$

The last equality was obtained by the definition of μ. Then we obtain

$$S\Phi(\mathbf{f}) = \prod_{j=1}^{d} e^{-(\lambda_j^2/2)|\Delta K_j|_2^2}e^{i\lambda_j\langle\Delta K_j,f_j\rangle}, \tag{3.5}$$

which clearly fulfills the measurability condition. Moreover, for all $z\in\mathbb{C}$, we find

$$\begin{aligned} |S\Phi(z\mathbf{f})| &= \prod_{j=1}^{d} e^{-(\lambda_j^2|\Delta K_j|_2^2/4)}\prod_{j=1}^{d}\left|e^{-(\lambda_j^2|\Delta K_j|_2^2/4)+iz\lambda_j\langle\Delta K_j,f_j\rangle}\right|\\ &\le \prod_{j=1}^{d} e^{-(\lambda_j^2|\Delta K_j|_2^2/4)}\prod_{j=1}^{d} e^{-(\lambda_j^2|\Delta K_j|_2^2/4)+|z||\lambda_j||\langle\Delta K_j,f_j\rangle|}, \end{aligned} \tag{3.6}$$

where, for each $j = 1, \ldots, d$, the corresponding term in the second product is bounded by

$$\exp\left(\frac{|z|^2}{|\Delta K_j|_2^2} \cdot |\langle \Delta K_j, f_j \rangle|^2\right) \tag{3.7}$$

because

$$\begin{aligned}
&-\frac{\lambda_j^2}{4}|\Delta K_j|_2^2 + |z||\lambda_j||\langle \Delta K_j, f_j\rangle| \\
&\quad = -\left(\frac{|\lambda_j|}{2}|\Delta K_j|_2 - \frac{|z|}{|\Delta K_j|_2}|\langle \Delta K_j, f_j\rangle|\right)^2 + \frac{|z|^2}{|\Delta K_j|_2^2}|\langle \Delta K_j, f_j\rangle|^2.
\end{aligned} \tag{3.8}$$

As a result

$$|S\Phi(z\mathbf{f})| \le e^{-(1/4)\sum_{j=1}^d \lambda_j^2 |\Delta K_j|_2^2} e^{|z|^2 \sum_{j=1}^d (1/|\Delta K_j|_2^2)|\langle \Delta K_j, f_j\rangle|^2}, \tag{3.9}$$

where, as a function of λ, the first exponential is integrable on $\mathbb{R}^d$ and the second exponential is constant.

An application of the result mentioned above completes the proof. In particular, it yields (3.2) by integrating (3.5) over λ. □

For the sequence of processes $\ell_{T,\varepsilon}^H$ given by (1.4) and (3.5), combining

$$\begin{aligned}
&(2\pi)^{-d}\int_{\mathbb{R}^d} d\lambda e^{-\varepsilon(|\lambda|^2/2)} S\left(e^{i\lambda(\mathbf{S}^H(t)-\mathbf{S}^H(s))}\right)(\mathbf{f}) \\
&\quad = (2\pi)^{-d}\int_{\mathbb{R}^d} d\lambda \prod_{j=1}^d e^{-(\lambda_j^2/2)(|\Delta K_j|_2^2+\varepsilon)} e^{i\lambda_j\langle \Delta K_j, f_j\rangle}
\end{aligned} \tag{3.10}$$

and the elementary equality

$$\int_{\mathbb{R}} e^{-\alpha(x^2/2)+i\beta x} dx = \sqrt{\frac{2\pi}{\alpha}} e^{-(\beta^2/2\alpha)}, \tag{3.11}$$

then we have the following.

Corollary 3.2. *For each t and s strictly positive real numbers,*

$$\ell_{T,\varepsilon}^H = \int_0^T \int_0^t ds dt \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\lambda e^{i\lambda\cdot(\mathbf{S}^H(t)-\mathbf{S}^H(s))-(\varepsilon\lambda^2/2)}, \tag{3.12}$$

is a Hida distribution with S-transform given by

$$S\ell_{T,\varepsilon}^{H}(\mathbf{f}) = (2\pi)^{-(d/2)} \int_0^T \int_0^t dsdt \prod_{j=1}^{d} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{1/2}} \exp\left\{ -\frac{\langle f_j, \Delta K_j\rangle^2}{2\left(\delta^{H_j}(t,s)+\varepsilon\right)} \right\} \tag{3.13}$$

for all $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{S}_d(\mathbb{R})$ *and* $\delta^{H_j}(t,s) = |\Delta K_j|_2^2$.

Theorem 3.3. *For* $H \in (0,1)$, *every positive integer* $d \geq 1$, *and* $\varepsilon > 0$, *the self-intersection local time* $\ell_{T,\varepsilon}^{H}$ *has the following chaos expansion:*

$$\ell_{T,\varepsilon}^{H} = \frac{1}{(2\pi)^{d/2}} \sum_{m=0}^{\infty} \frac{1}{\mathbf{m}!} \left(-\frac{1}{2}\right)^{\mathbf{m}} \left\langle :\omega^{\otimes 2\mathbf{m}}:, \mathbf{G}_{\mathbf{m},\varepsilon} \right\rangle, \tag{3.14}$$

where

$$\mathbf{G}_{\mathbf{m},\varepsilon} = \int_0^T \int_0^t dsdt \prod_{j=1}^{d} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{m_j+(1/2)}} \Delta K^{\otimes 2\mathbf{m}}. \tag{3.15}$$

Proof. For every $\mathbf{f} \in \mathcal{S}(\mathbb{R})$, we calculate the S-transform of $\ell_{T,\varepsilon}$ as follows:

$$\begin{aligned}
S\ell_{T,\varepsilon}(\mathbf{f}) &= (2\pi)^{-(d/2)} \int_0^T \int_0^t dsdt \prod_{j=1}^{d} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{1/2}} \exp\left\{ -\frac{\langle f_j, \Delta K_j\rangle^2}{2\left(\delta^{H_j}(t,s)+\varepsilon\right)} \right\} \\
&= (2\pi)^{-(d/2)} \int_0^T \int_0^t dsdt \prod_{j=1}^{d} \sum_{m_j=0}^{\infty} \frac{1}{m_j!} \left(-\frac{1}{2}\right)^{m_j} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{m_j+(1/2)}} \langle f_j, \Delta K_j\rangle^{2m_j} \\
&= (2\pi)^{-(d/2)} \int_0^T \int_0^t dsdt \sum_{\mathbf{m}=0}^{\infty} \frac{1}{\mathbf{m}!} \left(-\frac{1}{2}\right)^{\mathbf{m}} \prod_{j=1}^{d} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{m_j+(1/2)}} \left\langle f_j^{\otimes 2\mathbf{m}}, \Delta K_j^{\otimes 2\mathbf{m}} \right\rangle.
\end{aligned} \tag{3.16}$$

Comparing with the general form of the chaos expansion, we find that the kernel functions are equal to

$$\mathbf{G}_{\mathbf{m},\varepsilon} = \int_0^T \int_0^t dsdt \prod_{j=1}^{d} \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{m_j+(1/2)}} \Delta K^{\otimes 2\mathbf{m}}. \tag{3.17}$$

□

For simplicity, we assume that the notation $F \asymp G$ means that there are positive constants C_1 and C_2 such that

$$C_1 G(x) \leq F(x) \leq C_2 G(x), \tag{3.18}$$

in the common domain of definition for F and G.

Let us now compute the expectation of the self-intersection local time of the subfractional Brownian motion, $\mathbb{E}(\ell^H_{T,\varepsilon})$, it is just the first chaos. So in view of Theorem 3.3

$$\mathbb{E}\left(\ell^H_{T,\varepsilon}\right) = (2\pi)^{-(d/2)} \int_0^T \int_0^t ds dt \prod_{j=1}^d \frac{1}{\left(\delta^{H_j}(t,s)+\varepsilon\right)^{1/2}}. \tag{3.19}$$

Moreover, for all $s \le t$, the second moment of increments $\delta^{H_j}(t,s) = \mathbb{E}(S^{H_j}(t) - S^{H_j}(s))^2$ satisfying the following estimate:

$$\left[\beta_{H_j} \wedge 1\right](t-s)^{2H_j} \le \delta^{H_j}(t,s) \le \left[\beta_{H_j} \vee 1\right](t-s)^{2H_j}, \tag{3.20}$$

with $\beta_{H_j} = 2 - 2^{2H_j - 1}$. As the integrand in $\mathbb{E}(\ell^H_{T,\varepsilon})$ is always positive, we have

$$\begin{aligned}\mathbb{E}\left(\ell^H_{T,\varepsilon}\right) &\le (2\pi)^{-(d/2)} \int_0^T \int_0^t ds dt \prod_{j=1}^d \frac{1}{\left(\left[\beta_{H_j} \wedge 1\right](t-s)^{2H_j}+\varepsilon\right)^{1/2}} \\ &= (2\pi)^{-(d/2)} \int_0^T ds \frac{T-s}{\prod_{j=1}^d \left(\left[\beta_{H_j} \wedge 1\right] s^{2H_j}+\varepsilon\right)^{1/2}}.\end{aligned} \tag{3.21}$$

We use the change of variables $s = \varepsilon^{d/2H^*} z := \alpha(\varepsilon) z$ with $H^* = \sum_{j=1}^d H_j$,

$$\begin{aligned}\mathbb{E}\left(\ell^H_{T,\varepsilon}\right) &\le \frac{\alpha(\varepsilon)}{(2\pi)^{d/2}} \int_0^{T/\alpha(\varepsilon)} dz \frac{T - \alpha(\varepsilon) z}{\prod_{j=1}^d \left(\left[\beta_{H_j} \wedge 1\right] \varepsilon^{dH_j/H^*} z^{2H_j}+\varepsilon\right)^{1/2}} \\ &= \frac{\alpha(\varepsilon)}{(2\pi\varepsilon)^{d/2}} \int_0^{T/\alpha(\varepsilon)} dz \frac{T - \alpha(\varepsilon) z}{\prod_{j=1}^d \left(\left[\beta_{H_j} \wedge 1\right] \varepsilon^{(dH_j/H^*)-1} z^{2H_j}+1\right)^{1/2}}.\end{aligned} \tag{3.22}$$

We divide the integral in two parts

$$\begin{aligned}\mathbb{E}\left(\ell^H_{T,\varepsilon}\right) \le \frac{\alpha(\varepsilon)}{(2\pi\varepsilon)^{d/2}} \Bigg\{ &\int_0^1 dz \frac{T - \alpha(\varepsilon) z}{\prod_{j=1}^d \left(\left[\beta_{H_j} \wedge 1\right] \varepsilon^{(dH_j/H^*)-1} z^{2H_j}+1\right)^{1/2}} \\ &+ \int_1^{T/\alpha(\varepsilon)} dz \frac{T - \alpha(\varepsilon) z}{\prod_{j=1}^d \left(\left[\beta_{H_j} \wedge 1\right] \varepsilon^{(dH_j/H^*)-1} z^{2H_j}+1\right)^{1/2}} \Bigg\}.\end{aligned} \tag{3.23}$$

The first integral in braces is bounded. Set I the second integral in braces, so

$$I \leq \frac{T}{\prod_{j=1}^{d}\left[\beta_{H_j} \wedge 1\right]^{1/2}} \int_{1}^{T/\alpha(\varepsilon)} dz \frac{1}{z^{H^*}} = \begin{cases} \dfrac{T^{2-H^*}\varepsilon^{(d/2)-(d/2H^*)} - 1}{(1-H^*)\prod_{j=1}^{d}\left[\beta_{H_j} \wedge 1\right]^{1/2}}, & H^* \neq 1 \\ \dfrac{T\left[\log T - (d/2H^*)\log \varepsilon\right]}{\prod_{j=1}^{d}\left[\beta_{H_j} \wedge 1\right]^{1/2}}, & H^* = 1. \end{cases} \tag{3.24}$$

Proposition 3.4. *Denote by $H^* = \sum_{j=1}^{d} H_j$. Let $T > 0$ and $d \geq 1$, then if $H^* \geq 1$,*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\ell_{T,\varepsilon}^{H}\right) = \infty. \tag{3.25}$$

Moreover if $H^* = 1$,

$$\mathbb{E}\left(\ell_{T,\varepsilon}^{H}\right) \leq C_{T,H_j,d}\left|\log \varepsilon\right|. \tag{3.26}$$

If $H^* > 1$

$$\mathbb{E}\left(\ell_{T,\varepsilon}^{H}\right) \leq C_{T,H_j,d}\,\varepsilon^{(d/2)-(d/2H^*)}. \tag{3.27}$$

If $H^* < 1$, there is no blow up, that is,

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\ell_{T,\varepsilon}^{H}\right) \in \left[\frac{1}{(2\pi)^{d/2}\prod_{j=1}^{d}\left(\left[\beta_{H_j} \vee 1\right]\right)^{1/2}} \int_{0}^{T} ds \frac{T-s}{s^{H^*}}, \frac{1}{(2\pi)^{d/2}\prod_{j=1}^{d}\left(\left[\beta_{H_j} \wedge 1\right]\right)^{1/2}} \int_{0}^{T} ds \frac{T-s}{s^{H^*}}\right]. \tag{3.28}$$

In particular, we have

Proposition 3.5. *Suppose that all $H_j = H$, let $T > 0$ and $d \geq 1$.*

(1) *If $H = 1/d$, then $\mathbb{E}\ell_{T,\varepsilon}^{H} = C\ln(1/\varepsilon) + o(\varepsilon)$.*

(2) *If $1/d < H < 3/(2d)$, then $\mathbb{E}\ell_{T,\varepsilon}^{H} = C\varepsilon^{(1/2H)-(d/2)} + o(\varepsilon)$.*

From the above results, if $H^* < 1$, the self-intersection local time ℓ_T^H is well defined in $(S)^*$. This is same as the case of fractional Brownian motion mainly because the covariance structure and the property (1.2) of the increments of the subfractional Brownian motion. Suppose now that $H^* \geq 1$. The idea is that if we subtract some of the first term in the expansion of the exponential function in the expression of the S-transform of $\delta(\mathbf{S}^H(t)-\mathbf{S}^H(s))$, we could obtain an integrable function in factor of the remaining part, then the second condition of Corollary 2.4 will be satisfied. And so we could define a renormalization of the self-intersection local time in $(S)^*$.

Let us denote the truncated exponential series by

$$\exp_N(x) = \sum_{n=N}^{\infty} \frac{x^n}{n!}. \tag{3.29}$$

It follows from (3.2) that the S-transform of $\delta^{(N)}$ is given by

$$S\delta^{(N)}\Big(\mathbf{S}^H(t) - \mathbf{S}^H(s)\Big)(\mathbf{f}) = (2\pi)^{-(d/2)} \sum_{\mathbf{n}, n\ge N} \left(-\frac{1}{2}\right)^{\mathbf{n}} \prod_{j=1}^{d} \frac{\langle f_j, \Delta K_j\rangle^{2n_j}}{\delta^{H_j}(t,s)^{n_j+(1/2)}} \cdot \frac{1}{(n_j)!}, \tag{3.30}$$

so

$$\begin{aligned} &\left|S\delta^{(N)}\Big(\mathbf{S}^H(t) - \mathbf{S}^H(s)\Big)(\mathbf{f})\right| \\ &\quad \le (2\pi)^{-(d/2)} \sum_{\mathbf{n}, n\ge N} \left(\frac{1}{2}\right)^{\mathbf{n}} \prod_{j=1}^{d} \frac{|f_j|_\infty^{2n_j} \cdot |\Delta K_j|_1^{2n_j}}{\left(\beta_{H_j} \wedge 1\right)^{n_j+(1/2)} |t-s|^{(2n_j+1)H_j}} \cdot \frac{1}{(n_j)!}. \end{aligned} \tag{3.31}$$

We need to estimate the L^1-norm of ΔK_j, $|\Delta K_j|_1$ for fixed j.

We will treat the case when all $H_j > 1/2$. For the case all $H_j < 1/2$, we do not have a good estimation of $|K_j|_1$ and so we do not have a result. Let $H_j > 1/2$, in view of (2.8)

$$\begin{aligned} \Delta K_j &= K_{H_j}(t,u) - K_{H_j}(s,u) \\ &= C_{H_j} u^{3/2-H_j} \left\{ \int_u^t \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[0,t]}(u) - \int_u^s \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[0,s]}(u) \right\} \\ &= C_{H_j} u^{3/2-H_j} \left\{ \int_u^t \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[s,t]}(u) \right. \\ &\qquad \left. + \int_u^t \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[0,s]}(u) - \int_u^s \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[0,s]}(u) \right\} \\ &= C_{H_j} u^{3/2-H_j} \left\{ \int_u^t \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[s,t]}(u) + \int_s^t \left(x^2-u^2\right)^{H_j-3/2} dx 1_{[0,s]}(u) \right\}. \end{aligned} \tag{3.32}$$

We obtain

$$\begin{aligned} \left|\Delta K_j\right|_1 &= \int_0^T \Delta K_j(u)du \\ &= C_{h_j}\left\{\int_0^T u^{3/2-H_j}du\int_s^t \left(x^2-u^2\right)^{H_j-3/2}dx1_{[0,s]}(u)\right. \\ &\qquad \left. + \int_0^T u^{3/2-H_j}du\int_u^t \left(x^2-u^2\right)^{H_j-3/2}dx1_{[s,t]}(u)\right\} \\ :&= C_{H_j}(I_1+I_2). \end{aligned} \tag{3.33}$$

For I_1, we obtain

$$\begin{aligned} I_1 &= \int_0^T u^{3/2-H_j}du\int_s^t \left(x^2-u^2\right)^{H_j-3/2}dx1_{[0,s]}(u) \\ &\le \int_0^T u^{3/2-H_j}du\int_s^t (x-u)^{H_j-3/2}x^{H_j-3/2}dx1_{[0,s]}(u) \\ &\le \int_s^t x^{H_j-3/2}dx\int_0^x (x-u)^{H_j-3/2}u^{3/2-H_j}1_{[0,s]}(u)du \\ &\le \frac{B\left(5/2-H_j, H_j-1/2\right)}{H_j+1/2}|t-s|^{H_j+1/2}. \end{aligned} \tag{3.34}$$

For I_2, we obtain

$$\begin{aligned} I_2 &= \int_0^T u^{3/2-H_j}du\int_u^t \left(x^2-u^2\right)^{H_j-3/2}dx1_{[s,t]}(u) \\ &\le \int_0^T u^{3/2-H_j}du\int_u^t (x-u)^{H_j-3/2}x^{H_j-3/2}dx1_{[s,t]}(u) \\ &\le \int_s^t x^{H_j-3/2}dx\int_0^x (x-u)^{H_j-3/2}u^{3/2-H_j}du \\ &\le \frac{B\left(5/2-H_j, H_j-1/2\right)}{H_j+1/2}|t-s|^{H_j+1/2}. \end{aligned} \tag{3.35}$$

So

$$\left|\Delta K_j\right|_1 \le C_{H_j}|t-s|^{H_j+1/2}. \tag{3.36}$$

Suppose $|t-s|$ is small enough, we get

$$\begin{aligned}
&\left|S\delta^{(N)}\left(\mathbf{S}^H(t)-\mathbf{S}^H(s)\right)(\mathbf{f})\right| \\
&\quad\le (2\pi)^{-(d/2)}\sum_{\mathbf{n},n\ge N}\left(\frac{1}{2}\right)^n\prod_{j=1}^{d}C_{H_j}\frac{\left|f_j\right|_\infty^{2n_j}}{\left(\beta_{H_j}\wedge 1\right)^{n_j+(1/2)}|t-s|^{n_j(H_j-1/2)+H_j}}\cdot\frac{1}{(n_j)!} \\
&\quad\le C_H\sum_{\mathbf{n},n\ge N}\prod_{j=1}^{d}\frac{1}{|t-s|^{n_j(H_j-1/2)+H_j}}\frac{1}{2^n}\prod_{j=1}^{d}\left|f_j\right|_\infty^{2n_j}\cdot\frac{1}{(n_j)!} \\
&\quad\le C_H\frac{1}{|t-s|^{N(H_{\max}-1/2)+H^*}}\exp\left\{\frac{1}{2}\sum_{j=1}^{d}\left|f_j\right|_\infty^2\right\},
\end{aligned}\tag{3.37}$$

with $H^*=\sum_{j=1}^{d}H_j$. Then we have.

Theorem 3.6. *Let $T>0$ and $n\in\mathbb{N}$, suppose that*

$$H^*+N\left(H_{\max}-\frac{1}{2}\right)<1,\tag{3.38}$$

then

$$\ell_T^{H,(N)}=\int_0^T\int_0^t\delta^{(N)}\left(\mathbf{S}^H(t)-\mathbf{S}^H(s)\right)ds\,dt,\tag{3.39}$$

is well defined as an element of $(S)^*$ and

$$\lim_{\varepsilon\to 0}\ell_{T,\varepsilon}^{H,(N)}=\ell_T^{H,(N)},\quad in\ (S)^*.\tag{3.40}$$

Theorem 3.7. *For $H\in(0,1)$, every positive integer $d\ge 1$, and $\varepsilon>0$, suppose that (3.38) holds, then the truncated self-intersection local time $\ell_T^{H,(N)}$ has the following chaos expansion:*

$$\ell_T^{H,(N)}=\frac{1}{(2\pi)^{d/2}}\sum_{m=0}^{\infty}\frac{1}{\mathbf{m}!}\left(-\frac{1}{2}\right)^{\mathbf{m}}\left\langle :\omega^{\otimes 2\mathbf{m}}:,\mathbf{G}_{\mathbf{m}}\right\rangle,\tag{3.41}$$

where

$$\mathbf{G}_{\mathbf{m}}=\int_0^T\int_0^t ds\,dt\prod_{j=1}^{d}\frac{1}{\left(\delta^{H_j}(t,s)\right)^{m_j+(1/2)}}\Delta K^{\otimes 2\mathbf{m}},\tag{3.42}$$

for each $\mathbf{m}\in\mathbb{N}^d$ and $m\ge N$. All other kernel functions $\mathbf{G}_{\mathbf{m}}$ are identically equal to zero.

Proof. By Corollary 2.4, the S-transform of the truncated self-intersection local time is given as an integral over (3.30). Then given $\mathbf{f} = (f_1, \dots, f_d) \in \mathcal{S}_d(\mathbb{R})$, we have

$$\left(S\ell_T^{H,(N)}\right)(\mathbf{f}) = (2\pi)^{-(d/2)} \int_0^T \int_0^t ds dt \sum_{\mathbf{n}, n \geq N} \left(-\frac{1}{2}\right)^{\mathbf{n}} \prod_{j=1}^{d} \frac{\langle f_j, \Delta K_j \rangle^{2n_j}}{\delta^{H_j}(t,s)^{n_j+(1/2)}} \cdot \frac{1}{(n_j)!}. \tag{3.43}$$

Comparing with the general form of chaos expansion, the result is proved. □

Next we will estimate the L^2-norm of the chaos of the self-intersection local time of subfractional Brownian motion. Now we state the result.

Theorem 3.8. *Suppose all $H_j = H$, let $T > 0$, $\mathbf{n} \neq 0$ and $d \geq 2$.*

(1) *If $Hd = 1$, then*

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{|\log \varepsilon|}} |G_{2\mathbf{n},\varepsilon}|_{2,2n} \tag{3.44}$$

exists.

(2) *If $1 < dH < 3/2$, then*

$$\lim_{\varepsilon \to 0} |G_{2\mathbf{n},\varepsilon}|_{2,2n} \tag{3.45}$$

exists.

(3) *If $H > 3/2$, then $\lim_{\varepsilon \to 0} \varepsilon^{d/2-(3/4H)} |G_{2\mathbf{n},\varepsilon}|_{2,2n}$ exists.*

Proof. For $\mathbf{n} \in \mathbb{N}^d$, the $2\mathbf{n}$th chaos is given by

$$\ell_{T,2\mathbf{n},\varepsilon} = (2\pi)^{-d/2} \frac{1}{\mathbf{n}!} \frac{1}{2^n} \left\langle : \omega^{\otimes 2\mathbf{n}} :, G_{2\mathbf{n},\varepsilon} \right\rangle, \tag{3.46}$$

where

$$G_{2\mathbf{n},\varepsilon} = \int_0^T dt \int_0^t ds \prod_{j=1}^{d} \frac{\Delta K_j^{\otimes 2\mathbf{n}}}{\left(\delta^{H_j}(t,s) + \varepsilon\right)^{n_j+1/2}}. \tag{3.47}$$

So

$$\mathbb{E}(\ell_{T,2\mathbf{n},\varepsilon})^2 = \frac{(2\mathbf{n})!}{(2\pi)^d (\mathbf{n}!)^2 2^{2n}} |G_{2\mathbf{n},\varepsilon}|^2_{L^2(\mathbb{R}^n)}. \tag{3.48}$$

In view of (3.48), we need to estimate $|G_{2\mathbf{n},\varepsilon}|^2_{L^2(\mathbb{R}^n)} = |G_{2\mathbf{n},\varepsilon}|^2_{2,2n}$, where

$$|G_{2\mathbf{n},\varepsilon}|^2_{2,2n} = \int_{\mathbb{R}^{2n}} d^{2n}u \int_0^T \int_0^t \int_0^T \int_0^{t'} dt ds dt' ds' \prod_{j=1}^{d} \frac{\Delta K_j(t,s)^{\otimes 2n_j} \Delta K_j(t',s')^{\otimes 2n_j}}{\left(\left(\delta^{H_j}(t,s)+\varepsilon\right)\left(\delta^{H_j}(t',s')+\varepsilon\right)\right)^{n_j+1/2}}. \tag{3.49}$$

By using Fubini theorem, we first get

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} d^{2n}u \prod_{j=1}^{d} \Delta K_j(t,s)^{\otimes 2n_j} \Delta K_j(t',s')^{\otimes 2n_j} \\
&\quad = \prod_{j=1}^{d}\prod_{i=1}^{2n_j} \int_{\mathbb{R}} du_i^j \Delta K_j(t,s)(t,s) \Delta K_j(t,s)(t',s') \\
&\quad = \prod_{j=1}^{d}\prod_{i=1}^{2n_j} \mathbb{E}\left(S^{H_j}(t) - S^{H_j}(s)\right)\left(S^{H_j}(t') - S^{H_j}(s')\right) = \prod_{j=1}^{d}\left(C^{H_j}_{t,s,t's'}\right)^{2n_j},
\end{aligned} \tag{3.50}$$

with $C^{H_j}_{t,s,t's'} = \mathbb{E}(S^{H_j}(t) - S^{H_j}(s))(S^{H_j}(t') - S^{H_j}(s'))$. Moreover denote by $R^{H_j}_{t,s,t's'} = \mathbb{E}(B^{H_j}(t) - B^{H_j}(s))(B^{H_j}(t') - B^{H_j}(s'))$ with B^H a fractional Brownian motion with Hurst index $H \in (0,1)$. Then, from Bojdecki et al. [1], we know that

$$\begin{aligned}
&0 < C^{H_j}_{t,s,t's'} < R^{H_j}_{t,s,t's'}, \quad H > \frac{1}{2}, \\
&R^{H_j}_{t,s,t's'} < C^{H_j}_{t,s,t's'} < 0, \quad H < \frac{1}{2}.
\end{aligned} \tag{3.51}$$

So

$$|G_{2\mathbf{n},\varepsilon}|^2_{2,2n} \le \int_0^T \int_0^t \int_0^T \int_0^{t'} dt ds dt' ds' \prod_{j=1}^{d} \frac{\left(R^{H_j}_{t,s,t's'}\right)^{2n_j}}{\left(\left(\delta^{H_j}(t,s)+\varepsilon\right)\left(\delta^{H_j}(t',s')+\varepsilon\right)\right)^{n_j+1/2}}. \tag{3.52}$$

In view of the symmetry of the domain and integrand function, it suffices to integrate only on

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3, \tag{3.53}$$

where $\mathcal{T}_1 = \{0 < s' < t' < s < t\}$, $\mathcal{T}_2 = \{0 < s' < s < t' < t\}$, $\mathcal{T}_3 = \{0 < s < s' < t' < t\}$. Let us first integrate over $\mathcal{T}_1$. We make the following change of variables $x = t - s, t = t' - s', z = s - t'$ and $x + y + z = t - s' < t$, where t is considered as a parameter. Set $|G^{(i)}_{2\mathbf{n},\varepsilon}|^2_{2,2n}$ the integral over $\mathcal{T}_i$, $i = 1,2,3$.

Step 1. For $|G^{(1)}_{2\mathbf{n},\varepsilon}|^2_{2,2n}$, we obtain

$$\left|G^{(1)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} \leq C_H \int_0^T dt \int_{0\leq x+y+z\leq t} dxdydz \cdot \prod_{j=1}^{d} \frac{\left[(x+z)^{2H_j} + (y+z)^{2H_j} - (x+y+z)^{2H_j} - z^{2H_j}\right]^{2n_j}}{\left[\left(x^{2H_j}+\varepsilon\right)\left(y^{2H_j}+\varepsilon\right)\right]^{n_j+(1/2)}}. \tag{3.54}$$

It is almost possible to compute the integral when all H_j are different, so let us suppose that all H_j are equal to some H.

Denote by $\theta_t(\varepsilon) = \varepsilon^{-(1/2H)}t$ and make the following change of variables $(x, y, z) = \varepsilon^{-(1/2H)}(x', y', z')$, we get

$$\begin{aligned}
&\left|G^{(1)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} \\
&\quad\leq C_H \varepsilon^{(3/2H)-d} \int_0^T dt \int_{0\leq x+y+z\leq\theta_t(\varepsilon)} dxdydz \\
&\qquad \cdot \frac{\left[(x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} \\
&\quad= 2C_H \varepsilon^{(3/2H)-d} \int_0^T dt \int_{0\leq x+y+z\leq\theta_t(\varepsilon), 0\leq x\leq y\leq\theta_t(\varepsilon)} dxdydz \\
&\qquad \cdot \frac{\left[(x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}.
\end{aligned} \tag{3.55}$$

Denote by

$$f_H(x, y, z) = (x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}, \tag{3.56}$$

and by

$$f(t, H, d, n, \varepsilon) = \int_{0\leq x+y+z\leq\theta_t(\varepsilon), 0\leq x\leq y\leq\theta_t(\varepsilon)} dxdydz \frac{\left[f_H(x, y, z)\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}. \tag{3.57}$$

First note that $|f_H(x, y, z)| \leq x^{2H}$ and for $dH > 1$,

$$\int_{0\leq x\leq y} dxdy \frac{x^{4Hn}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} < \infty. \tag{3.58}$$

This implies that for every $x \in \mathbb{R}_+$,

$$\int_{0\leq x\leq y} dxdy \frac{f_H(x,y,z)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}, \tag{3.59}$$

is finite. On the other hand, for $\alpha \in (1, 2-2H)$, $z^{\alpha} f_H(x,y,z)^{2n}$ decreases to zero when z tends to zero, then

$$f(H,d,n) = \int_{\mathbb{R}_+} dz \int_{0\leq x\leq y} dxdy \frac{f_H(x,y,z)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} < \infty, \tag{3.60}$$

and so that

$$\lim_{\varepsilon\to 0} f(t,H,d,n,\varepsilon) = f(H,d,n). \tag{3.61}$$

Therefore, we obtain, if $Hd \in (1, 3/2)$,

$$\lim_{\varepsilon\to 0}\left|G^{(1)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} = 0, \tag{3.62}$$

and if $Hd \geq 3/2$,

$$\lim_{\varepsilon\to 0}\varepsilon^{d-(3/2H)}\left|G^{(1)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} = 2Tf(d,H,n). \tag{3.63}$$

Suppose now $Hd = 1$,

$$\left|G^{(1)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} = 2\varepsilon^{1/2H}\int_0^T f(t,H,d,n,\varepsilon)dt, \tag{3.64}$$

with

$$f(t,H,d,n,\varepsilon) = \theta_t(\varepsilon)\int_{0\leq x\leq y\leq\theta_t(\varepsilon)} dxdy \frac{x^{4Hn}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}, \tag{3.65}$$

and then

$$\frac{f(t,H,d,n,\varepsilon)}{|\log\varepsilon|} \leq \theta_t(\varepsilon)\left\{\frac{a(H,d,n)}{|\log\varepsilon|} + b(H,d,n)\right\}, \tag{3.66}$$

where

$$a(H,d,n) = \int_{0\le x\le y, x\le 1} dxdy \frac{x^{4Hn}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}},$$
$$b(H,d,n) = \sup_{0\le t\le T}\sup_{\varepsilon>0} \frac{1}{|\log\varepsilon|}\int_{1\le x\le y\le\theta_t(\varepsilon)} dxdy \frac{x^{4Hn}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}. \tag{3.67}$$

So we obtain

$$\frac{1}{|\log\varepsilon|}\left|G^{(1)}_{T,2\mathbf{n},\varepsilon}\right|^2_{2,2n} \le T^2\left\{\frac{a(H,d,n)}{|\log\varepsilon|} + b(H,d,n)\right\}. \tag{3.68}$$

Step 2. Let us now treat $|G^{(2)}_{2\mathbf{n},\varepsilon}|^2_{2,2n}$.

$$\left|G^{(2)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} = 2\varepsilon^{(3/2H)-d}\int_0^T dt g(t,H,d,n,\varepsilon), \tag{3.69}$$

where

$$g(t,H,d,n,\varepsilon) \le C_H \int_{0\le x+y-z\le\theta_t(\varepsilon), 0\le z\le x\le y\le\theta_t(\varepsilon)} dxdydz \frac{\left[g_H(x,y,z)\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}, \tag{3.70}$$

where

$$g_H(x,y,z) = (x-z)^{2H} + (y+z)^{2H} - (x+y-z)^{2H} - z^{2H}. \tag{3.71}$$

We have that $|g_H(x,y,z)| \le 2x^{2H}$ and so

$$z \longrightarrow \int_{0\le z\le x\le y} dxdy \frac{\left[g_H(x,y,z)\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}$$
$$= z^{4Hn+2}\int_{1\le x\le y} dxdy \frac{\left[g_H(x,y,z)\right]^{2n}}{\left[(z^{2H}x^{2H}+1)(z^{2H}y^{2H}+1)\right]^{n+(d/2)}}, \tag{3.72}$$

is well defined on $\mathbb{R}_+$ and for $Hd \ge 3/2$, one can choose $\alpha \in (1, 2-2H)$ such that when $z \longrightarrow +\infty$

$$z^{\alpha}\int_{0\le z\le x\le y} dxdy \frac{\left[g_H(x,y,z)\right]^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} \longrightarrow 0. \tag{3.73}$$

And then

$$\int_{\mathbb{R}_+}\int_{0\le z\le x\le y} dxdy\frac{[g_H(x,y,z)]^{2n}}{[(x^{2H}+1)(y^{2H}+1)]^{n+(d/2)}} = g(H,d,n) < \infty. \tag{3.74}$$

So that

$$\lim_{\varepsilon\to 0} g(t,H,d,n,\varepsilon) = g(H,d,n). \tag{3.75}$$

Suppose $Hd = 1$, the same computations as in the case of $|G^{(1)}_{2\mathbf{n},\varepsilon}|^2_{2,2n}$ leads to

$$\frac{1}{|\log\varepsilon|}\left|G^{(2)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} \le 2^{2n}T^2\left\{\frac{a(H,d,n)}{|\log\varepsilon|} + b(H,d,n)\right\}. \tag{3.76}$$

Suppose now $Hd \in (1,3/2)$, we have

$$g(t,H,d,n,\varepsilon) \le 2^{2n}\int_0^{\theta_t(\varepsilon)} dz z^{2-2Hd}\int_{1\le x\le y} dxdy\frac{x^{4Hn}}{(xy)^{H(2n+d)}} := 2^{2n}C(H,d,n). \tag{3.77}$$

Then

$$\left|G^{(2)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} \le C_H\frac{2^{2nC(H,d,n)}}{(3-2Hd)(2-Hd)}T^{4-2Hd}. \tag{3.78}$$

We know that $\lim_{\varepsilon\to 0}|G^{(1)}_{2\mathbf{n},\varepsilon}|^2_{2,2n} = 0$, then $\lim_{\varepsilon\to 0}|G^{(2)}_{2\mathbf{n},\varepsilon}|^2_{2,2n}$ exists. Let us suppose that $Hd = 3/2$,

$$\left|G^{(2)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} = 2\int_0^T dt h(t,H,d,n,\varepsilon) \tag{3.79}$$

with

$$\begin{aligned} h(t,H,d,n,\varepsilon) \le &\int_0^1 dz\int_{z\le x\le y\le\theta_t(\varepsilon)} dxdy\frac{[g_H(x,y,z)]^{2n}}{[(x^{2H}+1)(y^{2H}+1)]^{n+(d/2)}} \\ &+\int_1^{\theta_t(\varepsilon)} dz\int_{z\le x\le y\le\theta_t(\varepsilon)} dxdy\frac{[g_H(x,y,z)]^{2n}}{[(x^{2H}+1)(y^{2H}+1)]^{n+(d/2)}}. \end{aligned} \tag{3.80}$$

The first term is bounded by

$$2^{2n}\int_{0\le x\le y} dxdy\frac{x^{4Hn}}{[(x^{2H}+1)(y^{2H}+1)]^{n+(d/2)}} = 2^{2n}e(H,d,n). \tag{3.81}$$

The second term is bounded by

$$\int_1^{\theta_t(\varepsilon)} dz \frac{1}{z} 2^{2n} \int_{1\le x\le y} dxdy \frac{x^{4Hn}}{(xy)^{H(2n+(d/2))}} = 2^{2n} C(H,d,n) \log|\theta_t(\varepsilon)|. \tag{3.82}$$

So

$$\begin{aligned} \frac{1}{|\log\varepsilon|} \left| G^{(2)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} \le{}& \frac{2^{2n+1}}{|\log\varepsilon|} e(H,d,n)T \\ &+ \frac{2^{2n}}{H} C(H,d,n)T + \frac{2^{2n+1}}{|\log\varepsilon|} \int_0^T \log t dt. \end{aligned} \tag{3.83}$$

Therefore

$$\frac{1}{|\log\varepsilon|} \left| G^{(2)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} \tag{3.84}$$

has a finite nontrivial limit and

$$\frac{1}{|\log\varepsilon|} \left| G^{(2)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} \le \frac{2^{2n}}{H} C(H,d,n)T. \tag{3.85}$$

Step 3. Finally, let us treat $|G^{(3)}_{\mathbf{2n},\varepsilon}|^2_{2,2n}$. Let $x = t - t'$, $y = t' - s'$, $z = s' - s$. We know

$$\left| G^{(3)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} \le 2\varepsilon^{(3/2H)-d} \int_0^T dt j(t,H,d,n,\varepsilon), \tag{3.86}$$

where

$$j(t,H,d,n,\varepsilon) = \int_{0\le x+y+z\le\theta_t(\varepsilon)} dxdydz \frac{k_H(x,y,z)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}, \tag{3.87}$$

with

$$k_H(x,y,z) = (x+y)^{2H} - x^{2H} + (y+z)^{2H} - z^{2H}. \tag{3.88}$$

It is obvious that $|k_H(x,y,z)| \le 2y^{2H}$. So

$$j(t,H,d,n,\varepsilon) \le 2\int_{0\le x+y+z\le\theta_t(\varepsilon), 0\le x\le y\le\theta_t(\varepsilon)} dxdydz \frac{\left(2y^{2H}\right)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}. \tag{3.89}$$

For $Hd > 1$,

$$\int_{0\le x\le y} \frac{\left(2y^{2H}\right)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} dxdy < \infty, \tag{3.90}$$

this implies that for every $z \in \mathbb{R}_+$,

$$\int_{0\le x\le y} \frac{k_H(x,y,z)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} dxdy < \infty. \tag{3.91}$$

On the other hand, for $\alpha \in (1, 2-2H)$, $z^\alpha k_H(x,y,z)^{2n}$ decreases to zero when z tends to infinity, then

$$g(H,d,n) = \int_{\mathbb{R}_+} dz \int_{0\le x\le y} dxdy \frac{k_H(x,y,z)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}} < \infty, \tag{3.92}$$

and so that

$$\lim_{\varepsilon\to 0} g(t,H,d,n,\varepsilon) = g(H,d,n). \tag{3.93}$$

Therefore, if $Hd \in (1.3/2)$, we obtain

$$\lim_{\varepsilon\to 0} \left| G^{(3)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} = 0. \tag{3.94}$$

If $Hd \ge 3/2$,

$$\lim_{\varepsilon\to 0} \varepsilon^{d-(3/2H)} \left| G^{(3)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} = 2Tg(H,d,n). \tag{3.95}$$

Suppose now $Hd = 1$,

$$\left| G^{(3)}_{\mathbf{2n},\varepsilon} \right|^2_{2,2n} = 2\varepsilon^{1/2H} \int_0^T dt g(t,H,d,n,\varepsilon), \tag{3.96}$$

where

$$g(t,H,d,n,\varepsilon) \le \theta_t(\varepsilon) \int_{0\le x\le y\le \theta_t(\varepsilon)} dxdy \frac{\left(2y^{2H}\right)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}, \tag{3.97}$$

and then

$$\frac{g(t,H,d,n,\varepsilon)}{|\log\varepsilon|} \le \theta_t(\varepsilon) \left\{ \frac{a(H,d,n)}{|\log\varepsilon|} + b(H,d,n) \right\}, \tag{3.98}$$

where

$$a(H,d,n) = \int_{0\le x\le y, x\le 1} dxdy \frac{\left(2y^{2H}\right)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}},$$
$$b(H,d,n) = \sup_{0\le t\le T}\sup_{\varepsilon>0} \frac{1}{|\log\varepsilon|} \int_{1\le x\le y\le \theta_t(\varepsilon)} dxdy \frac{\left(2y^{2H}\right)^{2n}}{\left[\left(x^{2H}+1\right)\left(y^{2H}+1\right)\right]^{n+(d/2)}}. \tag{3.99}$$

So we obtain

$$\frac{1}{|\log\varepsilon|}\left|G^{(3)}_{2\mathbf{n},\varepsilon}\right|^2_{2,2n} \le T^2\left\{\frac{a(H,d,n)}{|\log\varepsilon|} + b(H,d,n)\right\}. \tag{3.100}$$

□

Theorem 3.9. *Assume that $H_i = H, i = 1,\ldots,d$, then for $H\in(0,1)$ and $d\ge 1$ satisfying $Hd<1$,*

$$\ell^H_{T,\varepsilon} \longrightarrow \ell^H_T, \quad \textit{in } \left(L^2\right), \tag{3.101}$$

as ε tends to zero.

Proof. By Theorems 3.3 and 3.7, we only need to consider chaos expansion of $\ell^H_{T,\varepsilon}$ and ℓ^H_T. Similar techniques in Oliveira et al. [12] allow us to write

$$I = \mathbb{E}\left(\ell^H_{T,2\mathbf{n}}\right)^2 = \frac{(2\mathbf{n})!}{(2\pi)^d(\mathbf{n}!)^2 2^{2n}}|G_{2\mathbf{n},\varepsilon}|^2_{L^2(\mathbb{R}^n)}. \tag{3.102}$$

In view of (3.102), we need to estimate $|G_{2\mathbf{n}}|^2_{L^2(\mathbb{R}^n)} = |G_{2\mathbf{n}}|^2_{2,2n}$, where

$$|G_{2\mathbf{n}}|^2_{2,2n} = \sum_{\mathbf{m}}\mathbf{m}!\left(\frac{1}{2\pi}\right)^d\left(\frac{1}{2}\right)^{2\mathbf{m}}\frac{1}{(\mathbf{m}!)^2}$$
$$\cdot\int_{\mathbb{R}^{2n}} d^{2n}u\int_{\mathcal{T}} dtdsdt'ds'\prod_{j=1}^{d}\frac{\Delta K_j(t,s)^{\otimes 2n_j}\Delta K_j(t',s')^{\otimes 2n_j}}{\left(\left(\delta^{H_j}(t,s)\right)\left(\delta^{H_j}(t',s')\right)\right)^{n_j+1/2}} \tag{3.103}$$
$$= \left(\frac{1}{2\pi}\right)^d\int_{\mathcal{T}}\left[\lambda_{t,s}\lambda_{t',s'}-\mu^2\right]^{-(d/2)}dtdsdt'ds',$$

where

$$\begin{aligned}
\lambda_{t,s} &= |t+s|^{2H} + |t-s|^{2H} - 2^{2H-1}\left(t^{2H} + s^{2H}\right), \\
\lambda_{t',s'} &= \left|t'+s'\right|^{2H} + \left|t'-s'\right|^{2H} - 2^{2H-1}\left(\left(t'\right)^{2H} + \left(s'\right)^{2H}\right), \\
\mu &= \frac{1}{2}\left(\left|s-t'\right|^{2H} + \left|s'-t\right|^{2H} - \left|t-t'\right|^{2H} - \left|s-s'\right|^{2H}\right. \\
&\qquad \left. + \left|s-t'\right|^{2H} + \left|s'+t\right|^{2H} - \left|t+t'\right|^{2H} - \left|s+s'\right|^{2H}\right).
\end{aligned} \tag{3.104}$$

In order to show $|I| < \infty$, we need some preliminaries. Recall

$$\mathbb{T} = \left\{\left(s,t,s',t'\right) : 0 < s < t < T, 0 < s' < t' < T\right\}. \tag{3.105}$$

Without loss of generality, one can assume $t < t'$. For any $(s,t,s',t') \in \mathbb{T}$, we denote $\mathbb{T}_1 = \{0 \leq s < s' < t < t' \leq T\}$, $\mathbb{T}_2 = \{0 \leq s' < s < t < t' \leq T\}$, $\mathbb{T}_3 = \{0 \leq s < t < s' < t' \leq T\}$. From Lemma 2.1 in Yan and Shen [13], we know that there exists a constant $\kappa > 0$ such that the following three statements hold:

(1) for $(s,t,s',t') \in \mathbb{T}_1$,

$$\lambda_{t,s}\lambda_{t',s'} - \mu^2 \geq \kappa\left[(t-s)^{2H}\left(t'-t\right)^{2H} + \left(t'-s'\right)^{2H}\left(s'-s\right)^{2H}\right]; \tag{3.106}$$

(2) for $(s,t,s',t') \in \mathbb{T}_2$,

$$\lambda_{t,s}\lambda_{t',s'} - \mu^2 \geq \kappa\left[(t-s)^{2H}\left(t'-s'\right)^{2H}\right]; \tag{3.107}$$

(3) for $(s,t,s',t') \in \mathbb{T}_3$,

$$\lambda_{t,s}\lambda_{t',s'} - \mu^2 \geq \kappa\left[(t-s)^{2H}\left(t'-s'\right)^{2H}\right]. \tag{3.108}$$

Then we can easily check that, if $Hd < 1$, $|I| < \infty$. Moreover, $\ell_{T,\varepsilon}$ converges to ℓ_T^H in (L^2) as ε tends to zero. □

Remark 3.10. In this work, we only study the existence of self-intersection of subfractional Brownian motion in L^2 under mild conditions. The asymptotic behavior and the central limit theorem of the difference $\ell_{T,\varepsilon}^H - \mathbb{E}\ell_{T,\varepsilon}^H$ in L^2 as ε tends to zero will be discussed in future works.

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