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Research Article

A Two-Grid Method for Finite Element Solutions of Nonlinear Parabolic Equations

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A two-grid method is presented and discussed for a finite element approximation to a nonlinear parabolic equation in two space dimensions. Piecewise linear trial functions are used. In this two-grid scheme, the full nonlinear problem is solved only on a coarse grid with grid size H. The nonlinearities are expanded about the coarse grid solution on a fine gird of size h, and the resulting linear system is solved on the fine grid. A priori error estimates are derived with the H^1-norm $O(h+H^2)$ which shows that the two-grid method achieves asymptotically optimal approximation as long as the mesh sizes satisfy $h = O(H^2)$. An example is also given to illustrate the theoretical results.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary Γ and consider the initial-boundary value problem for the following nonlinear parabolic equations:

$$\begin{gathered} u_t - \nabla \cdot (A(u)\nabla u) = f(x), \quad x \in \Omega,\ 0 < t \leq T, \\ u(x,t) = 0, \quad x \in \Gamma,\ 0 < t \leq T, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where u_t denotes $\partial u/\partial t$. $x = (x_1, x_2)$, $f(x)$ is a given real-valued function on Ω. We assume that $A(u)$ is a symmetric, uniformly positive definite second-order diagonal tensor. $A(u)$ and $A'(u)$ satisfy the Lipschitz continuous condition with respect to u.

$$|A(u_1) - A(u_2)| \leq L|u_1 - u_2|, \tag{1.2}$$

$$\left|A'(u_1) - A'(u_2)\right| \leq L|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R}, \tag{1.3}$$

where L is a positive constant. We also assume that each element of A is twice continuously differentiable in space and

$$|A(u)| + |A'(u)| \leq M, \quad \forall u \in \mathbb{R}, \tag{1.4}$$

where M is a positive constant.

It is assumed that the functions f, u_0 have enough regularity, and they satisfy appropriate compatibility conditions so that the initial-boundary value problem (1.1) has a unique solution satisfying the regularity results as demanded by our subsequent analysis.

Two-grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems and then use it as the initial guess for one Newton-like iteration on the fine grid. This method involves a nonlinear solve on the coarse grid with grid size H and a linear solve on the fine grid with grid size $h \ll H$. Two-grid method was first introduced by Xu [1, 2] for linear (nonsymmetric or indefinite) and especially nonlinear elliptic partial differential equations. Later on, two-grid method was further investigated by many authors. Dawson and Wheeler [3] Dawson et al. [4] have applied this method combined with mixed finite element method and finite difference method to nonlinear parabolic equations. Wu and Allen [5] have applied two-grid method combined with mixed finite element method to reaction-diffusion equations. Chen et al. [6–9] have constructed two-grid methods for expanded mixed finite element solution of semilinear and nonlinear reaction-diffusion equations and nonlinear parabolic equations. Bi and Ginting [10] have studied two-grid finite volume element method for linear and nonlinear elliptic problems. Chen et al. [11] and Chen and Liu [12, 13] have studied two-grid method for semilinear parabolic and second-order hyperbolic equations using finite volume element.

In this paper, based on two conforming piecewise linear finite element spaces S_H and S_h on one coarse grid with grid size H and one fine grid with grid size h, respectively, we consider the two-grid finite element discretization techniques for the nonlinear parabolic problems. With the proposed techniques, solving the nonlinear problems on the fine space is reduced to solving a linear problems on the fine space and a nonlinear problems on a much smaller space. This means that solving a nonlinear problem is not much more difficult than solving one linear problem, since $\dim S_H \ll \dim S_h$ and the work for solving the nonlinear problem is relatively negligible. A remarkable fact about this simple approach is, as shown in [1], that the coarse mesh can be quite coarse and still maintain a good accuracy approximation.

The rest of this paper is organized as follows. In Section 2, we describe the finite element scheme for the nonlinear parabolic problem (1.1). Section 3 contains the error estimates for the finite element method. Section 4 is devoted to the two-grid finite element and its error analysis. A numerical example is presented to confirm the theoretical results in the last section.

Throughout this paper, the letter C or with its subscript denotes a generic positive constant which does not depend on the mesh parameters and may be different at its different occurrences.

2. Finite Element Method

We adopt the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1 \leq p \leq \infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_{\Omega} \sum_{|\alpha|\leq s} |D^{\alpha} u|^{p} dx\right)^{1/p}, \tag{2.1}$$

with the standard modification for $p = \infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible, that is, $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$. We denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ of functions vanishing on the boundary Γ.

The corresponding variational form of (1.1) is to find $u(\cdot, t) \in H_0^1(\Omega), 0 < t \leq T$ such that

$$\begin{gathered}(u_t, v) + a(u; u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \\ u(x, 0) = u_0(x), \quad x \in \Omega,\end{gathered} \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot; \cdot, \cdot)$ is defined by

$$a(w; u, v) = \int_{\Omega} A(w) \nabla u \cdot \nabla v dx. \tag{2.3}$$

Henceforth, it will be assumed that the problem (2.2) has a unique solution u, and in the appropriate places to follow, additional conditions on the regularity of u which guarantee the convergence results will be imposed.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \mathcal{T}_h$. Denote the continuous piecewise linear finite element space associated with the triangulation $\mathcal{T}_h$ by

$$S_h = \left\{v \in C\left(\overline{\Omega}\right) : v \text{ linear in } K \text{ for each } K \in \mathcal{T}_h, \ v = 0 \text{ on } \Gamma\right\}. \tag{2.4}$$

With the above assumptions on $\mathcal{T}_h$, it is easy to see that S_h is a finite-dimensional subspace of the Hilbert space $H_0^1(\Omega)$ [14].

Thus, the continuous-time finite element approximation is defined as to find a solution $u_h(t) \in S_h$, $0 < t \leq T$, such that

$$\begin{gathered}(u_{h,t}, v_h) + a(u_h; u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h, \\ u_h(0) = u_0,\end{gathered} \tag{2.5}$$

with $u_{h,t} = \partial u_h / \partial t$. Since we have discretized only in the space variables, this is referred to as a spatially semidiscrete problem. By means of Brouwer fixed-point iteration, Li [15] has proved the existence and uniqueness of the solution u_h of this problem.

3. Error Analysis for the Finite Element Method

To describe the error estimates for the finite element scheme (2.5), we will give some useful lemmas. In [16], it was shown that the bilinear form $a(\cdot;\cdot,\cdot)$ is symmetric and positive definite and the following lemma was proved, which indicates that the bilinear form $a(\cdot;\cdot,\cdot)$ is continuous and coercive on S_h.

Lemma 3.1. *For sufficiently small h, there exist two positive constants $C_1, C_2 > 0$ such that, for all $u_h, v_h, w_h \in S_h$, the coercive property*

$$a(w_h; u_h, u_h) \geq C_1 \|u_h\|_1^2 \tag{3.1}$$

and the boundedness property

$$|a(w_h; u_h, v_h)| \leq C_2 \|u_h\|_1 \|v_h\|_1 \tag{3.2}$$

hold true.

Lemma 3.2. *Let $\widetilde{u} \in S_h$ be the standard Ritz projection such that*

$$a(u(x,t); (\widetilde{u} - u)(x,t), v_h) = 0, \quad \forall v_h \in S_h. \tag{3.3}$$

Thus $\widetilde{u}$ is the finite element approximation of the solution of the corresponding elliptic problem whose exact solution is u. From [16–18], we have

$$\|u - \widetilde{u}\| + h\|u - \widetilde{u}\|_1 \leq Ch^2 \|u\|_2, \tag{3.4}$$

$$\|(u - \widetilde{u})_t\| + h\|(u - \widetilde{u})_t\|_1 \leq Ch^2 \|u_t\|_2, \tag{3.5}$$

for some positive constant C independent of h and u. In addition, Yuan and Wang [16] have proved that $\|\nabla \widetilde{u}\|_\infty$ and $\|\nabla \widetilde{u}_t\|_\infty$ are bounded by a positive constant.

Now we turn to describe the estimates for the finite element method. We give the error estimates in the L^2-norm between the exact solution and the semidiscrete finite element solution.

Theorem 3.3. *Let u and u_h be the solutions of problem* (1.1) *and the semidiscrete finite element scheme* (2.5)*, respectively. Under the assumptions given in Section 1, if $u_h(0) = \widetilde{u}_0$, for $0 < t \leq T$, we have*

$$\|u(t) - u_h(t)\| \leq Ch^2, \tag{3.6}$$

where $C = C(\|u\|_{L^2(H^2)}, \|u_t\|_{L^2(H^2)})$ independent of h.

Proof. For convenience, let $u - u_h = (u - \widetilde{u}) + (\widetilde{u} - u_h) =: \eta + \xi$. Then from (1.1), (2.5), and (3.3), we get the following error equation:

$$(\xi_t, v_h) + a(u_h; \xi, v_h) = -(\eta_t, v_h) + a(u_h; \widetilde{u}, v_h) - a(u; \widetilde{u}, v_h), \quad \forall v_h \in S_h. \tag{3.7}$$

Choosing $v_h = \xi$ in (3.7) to get

$$(\xi_t, \xi) + a(u_h; \xi, \xi) = -(\eta_t, \xi) + a(u_h; \widetilde{u}, \xi) - a(u; \widetilde{u}, \xi). \tag{3.8}$$

For the first term of (3.8), we have

$$(\xi_t, \xi) = \frac{1}{2}\frac{d}{dt}(\xi, \xi) = \frac{1}{2}\frac{d}{dt}\|\xi\|^2. \tag{3.9}$$

Integrating (3.8) from 0 to t, by (3.9) and noting that $\xi(0) = 0$, we have

$$\begin{aligned} \frac{1}{2}\|\xi\|^2 + \int_0^t a(u_h; \xi, \xi)dt &= -\int_0^t (\eta_t, \xi)dt + \int_0^t (a(u_h; \widetilde{u}, \xi) - a(u; \widetilde{u}, \xi))dt \\ &\equiv Q_1 + Q_2. \end{aligned} \tag{3.10}$$

Now let's estimate the right-hand terms of (3.10), for Q_1, there is

$$|Q_1| \le \int_0^t \|\eta_t\| \|\xi\| dt \le C \int_0^t \left(\|\eta_t\|^2 + \|\xi\|^2\right) dt. \tag{3.11}$$

For Q_2, by (1.2), we obtain

$$\begin{aligned} |Q_2| &\le \int_0^t \int_\Omega |(A(u_h) - A(u))||\nabla\widetilde{u} \cdot \nabla\xi| dx dt \\ &\le L \int_0^t \|\nabla\widetilde{u}\|_\infty \|u - u_h\| \|\nabla\xi\| dt \\ &\le C \int_0^t (\|\eta\| + \|\xi\|)\|\xi\|_1 dt \\ &\le C(\epsilon) \int_0^t \left(\|\eta\|^2 + \|\xi\|^2\right) dt + \epsilon \int_0^t \|\xi\|_1^2 dt, \end{aligned} \tag{3.12}$$

with ϵ a small positive constant. By Lemma 3.1, from (3.10)–(3.12), we get

$$\begin{aligned} &\|\xi\|^2 + C_1 \int_0^t \|\xi\|_1^2 dt \\ &\quad \le C_2 \int_0^t \left(\|\eta_t\|^2 + \|\xi\|^2\right) dt + C_3(\epsilon) \int_0^t \left(\|\eta\|^2 + \|\xi\|^2\right) dt + \epsilon \int_0^t \|\xi\|_1^2 dt. \end{aligned} \tag{3.13}$$

Choosing proper ϵ and kicking the last term into the left side of (3.13), and applying Gronwall lemma, for $t \leq T$, we have

$$\|\xi\|^2 + C_1 \int_0^T \|\xi\|_1^2 dt \leq C_2 \int_0^T \left(\|\eta\|^2 + \|\eta_t\|^2 \right) dt. \tag{3.14}$$

By (3.4) and (3.5), we obtain

$$\|\xi\|^2 \leq Ch^4 \int_0^T \left(\|u\|_2^2 + \|u_t\|_2^2 \right) dt, \tag{3.15}$$

which yields (3.6). □

4. Two-Grid Finite Element Method

In this section, we will present two-grid finite element algorithm for problem (1.1) based on two different finite element spaces. The idea of the two-grid method is to reduce the nonlinear problem on a fine grid into a linear system on a fine grid by solving a nonlinear problem on a coarse grid. The basic mechanisms are two quasi-uniform triangulations of Ω, $\mathcal{T}_H$, and $\mathcal{T}_h$, with two different mesh sizes H and h $(H > h)$, and the corresponding piecewise linear finite element spaces S_H and S_h which satisfies $S_H \subset S_h$ and will be called the coarse-grid space and the fine-grid space, respectively.

To solve problem (1.1), we introduce a two-grid algorithm into finite element method. This method involves a nonlinear solve on the coarse-grid space and a linear solve on the fine-grid space. We present the two-grid finite element method as two steps.

Algorithm 4.1. Step 1. On the coarse grid $\mathcal{T}_H$, find $u_H \in S_H$, such that

$$\begin{gathered} (u_{H,t}, v_H) + a(u_H; u_H, v_H) = (f, v_H), \quad \forall v_H \in S_H, \\ u_H(0) = \tilde{u}_0, \end{gathered} \tag{4.1}$$

where $\tilde{u}_0$ is defined by (3.3).
Step 2. On the fine grid $\mathcal{T}_h$, find $u_h \in S_h$, such that

$$\begin{gathered} (u_{h,t}, v_h) + a(u_H; u_h, v_h) = (f, v_h), \quad \forall v_h \in S_h, \\ u_h(0) = \tilde{u}_0. \end{gathered} \tag{4.2}$$

The main feature of this method is that it replaces the resolution of a nonlinear problem on the fine grid with the resolution of a nonlinear problem on the coarse grid coupled with the resolution of a linear system on the fine grid. Now we consider the error estimates in H^1-norm for the two-grid finite element method Algorithm 4.1.

Theorem 4.2. *Let u and u_h be the solutions of problem* (1.1) *and the two-grid scheme Algorithm 4.1, respectively. Under the assumptions given in Section 1, if $u_h(0) = \tilde{u}_0$, for $0 < t \le T$, one has*

$$\|u(t) - u_h(t)\|_1 \le C\left(h + H^2\right), \tag{4.3}$$

where $C = C(\|u\|_{L^2(H^2)}, \|u_t\|_{L^2(H^2)})$ is independent of h.

Proof. Once again, we set $u - u_h = (u - \tilde{u}) + (\tilde{u} - u_h) =: \eta + \xi$ and choose $v_h = \xi_t$. Then for Algorithm 4.1, we get the error equation

$$(\xi_t, \xi_t) + a(u_H; \xi, \xi_t) = -(\eta_t, \xi_t) + a(u_H; \tilde{u}, \xi_t) - a(u; \tilde{u}, \xi_t). \tag{4.4}$$

For the terms of (4.4), we have

$$\begin{aligned} a(u_H; \xi, \xi_t) &= \int_\Omega A(u_H)\nabla\xi \cdot \nabla\xi_t dx \\ &= \frac{1}{2}\frac{d}{dt} a(u_H; \xi, \xi) - \frac{1}{2}\int_\Omega A'(u_H)u_{H,t}\nabla\xi \cdot \nabla\xi dx. \end{aligned} \tag{4.5}$$

$$\begin{aligned} a(u_H; \tilde{u}, \xi_t) - a(u; \tilde{u}, \xi_t) &= \int_\Omega [A(u_H) - A(u)]\nabla\tilde{u} \cdot \nabla\xi_t dx \\ &= \frac{d}{dt}\left(\int_\Omega [A(u_H) - A(u)]\nabla\tilde{u} \cdot \nabla\xi dx\right) - \int_\Omega [A(u_H) - A(u)]\nabla\tilde{u}_t \cdot \nabla\xi dx \\ &\quad - \int_\Omega \left(A'(u_H)u_{H,t} - A'(u)u_t\right)\nabla\tilde{u} \cdot \nabla\xi dx. \end{aligned} \tag{4.6}$$

Integrating (4.4) from 0 to t, combining with (4.4)–(4.6) and noting that $\xi(0) = 0$ and $\xi_t(0) = 0$, we have

$$\begin{aligned} &\int_0^t \|\xi_t\|^2 dt + \frac{1}{2} a(u_H; \xi, \xi) \\ &\quad = -\int_0^t (\eta_t, \xi_t) dt + \frac{1}{2}\int_0^t \left(\int_\Omega A'(u_H)u_{H,t}\nabla\xi \cdot \nabla\xi dx\right) dt \\ &\qquad + \int_\Omega [A(u_H) - A(u)]\nabla\tilde{u} \cdot \nabla\xi dx - \int_0^t \left(\int_\Omega [A(u_H) - A(u)]\nabla\tilde{u}_t \cdot \nabla\xi dx\right) dt \\ &\qquad - \int_0^t \left(\int_\Omega \left(A'(u_H)u_{H,t} - A'(u)u_t\right)\nabla\tilde{u} \cdot \nabla\xi dx\right) dt \equiv \sum_{i=1}^5 T_i. \end{aligned} \tag{4.7}$$

Now let's estimate the right-hand terms of (4.7), for T_1, there is

$$|T_1| \le C\int_0^t \|\eta_t\|\|\xi_t\|dt \le C(\epsilon_1)\int_0^t \|\eta_t\|^2 dt + \epsilon_1\int_0^t \|\xi_t\|^2 dt, \tag{4.8}$$

with ϵ_1 a small positive constant. For T_2, by (1.4), we obtain

$$\begin{aligned} |T_2| &\le C\int_0^t \left|A'(u_H)u_{H,t}\right|_\infty \|\nabla\xi\|^2 dt \\ &\le CM\int_0^t |u_{H,t}|_\infty \|\nabla\xi\|^2 dt \le C\int_0^t \|\xi\|_1^2 dt, \end{aligned} \tag{4.9}$$

where we used the fact $|u_{H,t}|_\infty$ is bounded by a positive constant [15].

For T_3–T_5, by (1.2) and Theorem 3.3, we get

$$\begin{aligned} |T_3| &\le C\|\nabla\tilde{u}\|_\infty \|A(u_H) - A(u)\|\|\nabla\xi\| \\ &\le C\|\nabla\tilde{u}\|_\infty L\|u_H - u\|\|\xi\|_1 \\ &\le C(\epsilon_2)H^4 + \epsilon_2\|\xi\|_1^2, \end{aligned} \tag{4.10}$$

with ϵ_2 a small positive constant.

$$\begin{aligned} |T_4| &\le C\int_0^t \|\nabla\tilde{u}_t\|_\infty \|A(u_H) - A(u)\|\|\nabla\xi\| dt \\ &\le C\int_0^t \|\nabla\tilde{u}_t\|_\infty L\|u_H - u\|\|\xi\|_1 dt \\ &\le C_1H^4 + C_2\int_0^t \|\xi\|_1^2 dt. \end{aligned} \tag{4.11}$$

$$\begin{aligned} |T_5| &\le C\int_0^t \|\nabla\tilde{u}\|_\infty \|u_{H,t} - u_t\|\|\nabla\xi\| dt \\ &\le C_1H^4 + C_2\int_0^t \|\xi\|_1^2 dt. \end{aligned} \tag{4.12}$$

By Lemma 3.1, from (4.7)–(4.12), we get

$$\begin{aligned} \int_0^t \|\xi_t\|^2 dt + C_0\|\xi\|_1^2 \le{}& C_1(\epsilon_1)\int_0^t \|\eta_t\|^2 dt + C_2\int_0^t \|\xi\|_1^2 dt + C_3(\epsilon_2)H^4 \\ &+ C_4H^4 + \epsilon_1\int_0^t \|\xi_t\|^2 dt + \epsilon_2\|\xi\|_1^2. \end{aligned} \tag{4.13}$$

Choosing proper ϵ_1 and ϵ_2 and kicking the last term into the left side of (4.13), by (3.5) and the Gronwall lemma, for $t \leq T$, we have

$$\int_0^T \|\xi_t\|^2 dt + C_0\|\xi\|_1^2 \leq C_1 \int_0^T \|\eta_t\|^2 dt + C_2 H^4 \leq C\left(h^4 + H^4\right). \tag{4.14}$$

Together with (3.4), this yields (4.3). □

Remark 4.3. We consider the spatial discretization to focus on the two-grid method. Algorithm 4.1 is only a semidiscrete two-grid finite element method. In practical computations, the method should be combined with a time-stepping algorithm. We consider a time step Δt and approximate the solutions at $t^n = n\Delta t$, $\Delta t = T/N$, $n = 0, 1, \ldots, N$. Denote $u_h^n = u(\cdot, t^n)$, $u_{h,t}^n = (u_h^n - u_h^{n-1})/\Delta t$, we can get an implicit backward Euler two-grid finite element algorithm as follows.

(1) On the coarse grid $\mathcal{T}_H$, find $u_H^n \in S_H$ $(n = 1, 2, \ldots)$, such that

$$\begin{gathered}\left(u_{H,t}^n, v_H\right) + a\left(u_H^n; u_H^n, v_H\right) = \left(f^n, v_H\right), \quad \forall v_H \in S_H, \\ u_H^0 = \tilde{u}_0,\end{gathered} \tag{4.15}$$

where $\tilde{u}_0$ is defined by (3.3).

(2) On the fine grid $\mathcal{T}_h$, find $u_h^n \in S_h$ $(n = 1, 2, \ldots)$, such that

$$\begin{gathered}\left(u_{h,t}^n, v_h\right) + a\left(u_H^n; u_h^n, v_h\right) = \left(f^n, v_h\right), \quad \forall v_h \in S_h, \\ u_h^0 = \tilde{u}_0.\end{gathered} \tag{4.16}$$

Higher order temporal discretization methods such as Crank-Nicolson method or Runge-Kutta method can also be used. For the space discretization, from a practical point of view, we just need to choose space grid $h < H$ to obtain a considerable error reduction in spite of the demanding requirement $h = O(H^2)$.

5. Numerical Example

In this section, we consider the following nonlinear parabolic problem:

$$\begin{gathered}\frac{\partial u}{\partial t} - \nabla \cdot (A(u)\nabla u) = f(x,t), \quad x \in \Omega = [0,1]^2, \ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = x_1 x_2 (1 - x_1)(1 - x_2),\end{gathered} \tag{5.1}$$

where $x = (x_1, x_2)$, $A = u$ and $f(x,t)$ is decided by the exact solution of (5.1).

Table 1: H^1 error and CPU time of the finite element method.

h	H_1 error	CPU time (s)
1/9	4.746×10^{-3}	301.68
1/16	3.222×10^{-4}	1142.05

Table 2: H^1 error and CPU time of the two-grid finite element method.

h	H	H_1 error	CPU time (s)
1/9	1/3	4.989×10^{-3}	5.7660
1/16	1/4	3.264×10^{-4}	16.1812
1/36	1/6	1.286×10^{-4}	414.875
1/64	1/8	8.548×10^{-5}	2831.70

Let the exact solution of (5.1) be

$$u(x,t) = e^{-t} x_1 x_2 (1 - x_1)(1 - x_2). \tag{5.2}$$

Our main interest is to verify the performances of the two-grid finite element method Algorithm 4.1. Choose the space step H and obtain the coarse grids. Let $h = H^2$ and then obtain the fine grids. We further discretize time t of Algorithm 4.1 and consider a time step Δt and approximate the solutions at $t^n = n\Delta t$, $\Delta t = T/N$, $n = 0, 1, \ldots, N$. The mesh consists of triangular elements and the backward Euler scheme is used for the time discretization. We use Newton iteration method for the solutions on the coarse grid. In order to prove the efficiency of the two-grid finite element method, we compare this method with the standard finite element method. Computational results are shown in Tables 1 and 2.

From Tables 1 and 2, we can see that the numerical results coincide with the theoretical analysis, and the two-grid finite element method spends less time than the standard finite element method, that is to say, the two-grid algorithm is effective for saving a large amount of computational time and still keeping good accuracy.

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