Research Article

Fixed Point Theorems and Uniqueness of the Periodic Solution for the Hematopoiesis Models

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We present some results to the existence and uniqueness of the periodic solutions for the hematopoiesis models which are described by the functional differential equations with multiple delays. Our methods are based on the equivalent norm techniques and a new fixed point theorem in the continuous function space.

1. Introduction

In this paper, we aim to establish the existence and uniqueness result for the periodic solutions to the following functional differential equations with multiple delays:

$$x'(t) = -a(t)x(t) + f(t, x(t - \tau_1(t)), \dots, x(t - \tau_m(t))), \qquad (1.1)$$

where $a, \tau_i \in C(R, R^+)$, $f \in C(R^{m+1}, R)$ are T-periodic functions on variable t for $T > 0$ and m is a positive integer.

Recently, many authors investigate the dynamics for the various hematopoiesis models, which includ the attractivity and uniqueness of the periodic solutions. For examples, Mackey and Glass in [1] have built the following delay differential equation:

$$x'(t) = -ax(t) + \frac{\beta \theta^n}{\theta^n + x^n(t - \tau)}, \qquad (1.2)$$

where $a, n, \beta, \theta, \tau$ are positive constants, $x(t)$ denotes the density of mature cells in blood circulation, and τ is the time between the production of immature cells in the bone marrow

and their maturation for release in the circulating bloodstream; Liu et al. [2],Yang [3], Saker [4], Zaghrout et al. [5], and references therein, also investigate the attractivity and uniqueness of the periodic solutions for some hematopoiesis models.

This paper is organized as follows. In Section 2, we present two new fixed point theorems in continuous function spaces and establish the existence and uniqueness results for the periodic solutions of (1.1). An illustrative example to the hematopoiesis models is exhibited in the Section 3.

2. Fixed Point Theorems and Existence Results

2.1. Fixed Point Theorems

In this subsection, we will present two new fixed point theorems in continuous function spaces. More details about the fixed point theorems in continuous function spaces can be found in the literature [6–8] and references therein.

Let E be a Banach space equipped with the norm $\| \cdot \|_E$. $BC(R, E)$ which denotes the Banach space consisting of all bounded continuous mappings from R into E with norm $\|u\|_C = \max\{\|u(t)\|_E : t \in R\}$ for $u \in BC(R, E)$.

Theorem 2.1. *Let F be a nonempty closed subset of $BC(R, E)$ and $A : F \to F$ an operator. Suppose the following:*

(H_1) there exist $\beta \in [0, 1)$ and $G : R \times R \to R$ such that for any $u, v \in F$,

$$\|Au(t) - Av(t)\|_E \leq \beta\|u(t) - v(t)\|_E + \int_{t-T}^{t} G(t, s)\|u(s) - v(s)\|_E ds \quad \text{for } t \in R, \tag{2.1}$$

(H_2) there exist an $\alpha \in [0, 1 - \beta)$ and a positive bounded function $y \in C(R, R)$ such that

$$\int_{t-T}^{t} G(t, s)y(s)ds \leq \alpha y(t) \quad \forall t \in R. \tag{2.2}$$

Then A has a unique fixed point in F.

Proof. For any given $x_0 \in F$, let $x_n = Ax_{n-1}$, $(n = 1, 2, \ldots)$. By (H_1), we have

$$\|Ax_{n+1}(t) - Ax_n(t)\|_E \leq \beta\|x_{n+1}(t) - x_n(t)\|_E + \int_{t-T}^{t} G(t, s)\|x_{n+1}(s) - x_n(s)\|_E ds. \tag{2.3}$$

Set $a_n(t) = \|x_{n+1}(t) - x_n(t)\|_E$, then we get

$$a_{n+1}(t) \leq \beta a_n(t) + \int_{t-T}^{t} G(t, s)a_n(s)ds. \tag{2.4}$$

In order to prove that the sequence $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$, we introduce an equivalent norm and show that $\{x_n\}$ is a Cauchy sequence with respect to the new one. Basing on the condition (H_2), we see that there are two positive constants M and m such that $m \leq y(t) \leq M$ for all $t \in R$. Define the new norm $\|\cdot\|_1$ by

$$\|u\|_1 = \sup\left\{\frac{1}{y(t)}\|u(t)\|_E : t \in R\right\}, \quad u \in BC(R,E). \tag{2.5}$$

Then,

$$\frac{1}{M}\|u\|_C \leq \|u\|_1 \leq \frac{1}{m}\|u\|_C. \tag{2.6}$$

Thus, the two norms $\|\cdot\|_1$ and $\|\cdot\|_C$ are equivalent.

Set $a_n = \|x_{n+1} - x_n\|_1$, then we have $a_n(t) \leq y(t)a_n$ for $t \in R$. By (2.13), we have

$$\begin{aligned}
\frac{1}{y(t)}a_{n+1}(t) &\leq \beta a_n + \frac{1}{y(t)}\int_{t-T}^{t} G(t,s)a_n(s)ds \\
&\leq \beta a_n + \frac{a_n}{y(t)}\int_{t-T}^{t} G(t,s)y(s)ds \leq (\beta + \alpha)a_n.
\end{aligned} \tag{2.7}$$

Thus,

$$a_{n+1} \leq (\beta + \alpha)a_n \leq (\beta + \alpha)^2 a_{n-1} \leq \cdots \leq (\beta + \alpha)^{n+1}a_0. \tag{2.8}$$

This means $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_1$. Therefore, also, $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$. Thus, we see that $\{x_n\}$ has a limit point in F, say u. It is known that u is the fixed point of A in F.

Suppose both u and v $(u \neq v)$ are the fixed points of A, then $Au = u$, $Av = v$. Following the similar arguments, we prove that

$$\|u - v\|_1 = \|Au - Av\|_1 \leq (\beta + \alpha)\|u - v\|_1. \tag{2.9}$$

It is impossible. Thus the fixed point of A is unique. This completes the proof of Theorem 2.1. $\square$

Let $PC(R,E)$ be a Banach space consisting of all T-periodic functions in $BC(R,E)$ with the norm $\|u\|_P = \max\{\|u(t)\|_E : t \in [0,T]\}$ for $u \in PC(R,E)$. Then, following the similar arguments in Theorem 2.1, we deduce Theorem 2.2 which is a useful result for achieving the existence of periodic solutions of functional differential equations.

Theorem 2.2. *Let* $A : PC(R, E) \rightarrow PC(R, E)$ *be an operator. Suppose the following:*

$(\widetilde{H_1})$ *there exist* $\beta \in [0, 1)$ *and* $G : R \times R \rightarrow R$ *such that for any* $u, v \in PC(R, E)$,

$$\|Au(t) - Av(t)\|_E \leq \beta \|u(t) - v(t)\|_E + \int_{t-T}^{t} G(t, s) \sum_{i=1}^{n} \|u(\eta_i(s)) - v(\eta_i(s))\|_E ds, \qquad (2.10)$$

where $\eta_i \in C([0, T], R^+)$ *with* $\eta_i(s) \leq s$ *and* n *is a positive integer;*

$(\widetilde{H_2})$ *there exist two constants* α, K *and a positive function* $y \in C(R, R)$ *such that* $nK\alpha \in [0, 1 - \beta)$, $y(\eta_i(s)) \leq Ky(s)$, *and*

$$\int_{t-T}^{t} G(t, s) y(s) ds \leq \alpha y(t) \quad \forall t \in [0, T]. \qquad (2.11)$$

Then A *has a unique fixed point in* $PC(R, E)$.

Proof. For any given $x_0 \in F$, let $x_k = Ax_{k-1}$, $(k = 1, 2, \ldots)$. By $(\widetilde{H_1})$, we have

$$\|Ax_{k+1}(t) - Ax_k(t)\|_E \leq \beta \|x_{k+1}(t) - x_k(t)\|_E + \int_{t-T}^{t} G(t, s) \sum_{i=1}^{n} \|x_{k+1}(\eta_i(s)) - x_k(\eta_i(s))\|_E ds. \qquad (2.12)$$

Set $a_k(t) = \|x_{k+1}(t) - x_k(t)\|_E$, then we get

$$a_{k+1}(t) \leq \beta a_k(t) + \int_{t-T}^{t} G(t, s) \sum_{i=1}^{n} a_k(\eta_i(s)) ds. \qquad (2.13)$$

Basing on the condition $(\widetilde{H_2})$, we see that there are two positive constants M and m such that $m \leq y(t) \leq M$ for all $t \in [0, T]$. Define the new norm $\| \cdot \|_2$ by

$$\|u\|_2 = \sup \left\{ \frac{1}{y(t)} \|u(t)\|_E : t \in [0, T] \right\}, \quad u \in PC(R, E). \qquad (2.14)$$

Then,

$$\frac{1}{M} \|u\|_p \leq \|u\|_2 \leq \frac{1}{m} \|u\|_p. \qquad (2.15)$$

Thus, the two norms $\| \cdot \|_2$ and $\| \cdot \|_p$ are equivalent.

Set $a_k = \|x_{k+1} - x_k\|_2$, then we have $a_k(t) \leq y(t)a_k$ for $t \in [0, T]$. By (2.13), we have

$$
\frac{1}{y(t)} a_{k+1}(t) \leq \beta a_k + \frac{1}{y(t)} \int_{t-T}^t G(t, s) \sum_{i=1}^n a_k(\eta_i(s)) ds
$$

$$
\leq \beta a_k + \frac{a_k}{y(t)} \int_{t-T}^t G(t, s) \sum_{i=1}^n y(\eta_i(s)) ds \leq (\beta + nK\alpha) a_k.
$$

(2.16)

Thus,

$$
a_{k+1} \leq (\beta + nK\alpha) a_k \leq (\beta + nK\alpha)^2 a_{k-1} \leq \cdots \leq (\beta + nK\alpha)^{k+1} a_0.
$$

(2.17)

This means $\{x_k\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_2$. Therefore, $\{x_k\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_p$. Therefore, we see that $\{x_k\}$ has a limit point in $PC(R, E)$, say u. It is easy to prove that u is the fixed point of A in $PC(R, E)$. The uniqueness of the fixed point is obvious. This completes the proof of Theorem 2.2. □

2.2. Existence and Uniqueness of the Periodic Solution

In order to show the existence of periodic solutions of (1.1), we assume that the function f is fulfilling the following conditions:

(H_f) there exist $L_i > 0$ $(i = 1, 2, \ldots, m)$ such that for any $x_i, y_i \in R$,

$$
|f(t, x_1, \ldots, x_m) - f(t, y_1, \ldots, y_m)| \leq \sum_{i=1}^m L_i |x_i - y_i|,
$$

(2.18)

$(H_{f\tau})$ for all $t \in [0, T]$, $t \geq \tau_i(t) \geq 0$ $(i = 1, 2, \ldots, m)$.

Theorem 2.3. *Suppose (H_f) and $(H_{f\tau})$ hold. Then the equation (1.1) has a unique T-periodic solution in $C[0, T]$.*

Proof. By direction computations, we see that $\varphi(t)$ is the T-periodic solution if and only if $\varphi(t)$ is solution of the following integral equation:

$$
x(t) = \frac{e^{\lambda T}}{e^{\lambda T} - 1} \int_{t-T}^t e^{-\lambda(t-s)} [(\lambda - a(s))x(s) + f(s, x(s - \tau_1(s)), \ldots, x(s - \tau_m(s)))] ds, \quad (2.19)
$$

where $\lambda = \max\{|a(t)| : t \in [0, T]\}$.

Thus, we would transform the existence of periodic solution of (1.1) into a fixed point problem. Considering the map $A : PC(R, R) \to PC(R, R)$ defined by, for $t \in [0, T]$,

$$
(Ax)(t) = \frac{e^{\lambda T}}{e^{\lambda T} - 1} \int_{t-T}^t e^{-\lambda(t-s)} [(\lambda - a(s))x(s) + f(s, x(s - \tau_1(s)), \ldots, x(s - \tau_m(s)))] ds.
$$

(2.20)

Then, u is a T-periodic solution of (1.1) if and only if u is a fixed point of the operator A in $PC(R, R)$.

At this stage, we should check that A fulfill all conditions of **Theorem 2.2**. In fact, for $x, y \in PC(R, R)$, by assumption (H_f), we have

$$
\begin{aligned}
&\left|(Ax)(t) - (Ay)(t)\right| \\
&\quad \leq \frac{e^{\lambda T}}{e^{\lambda T} - 1} \int_{t-T}^{t} e^{-\lambda(t-s)} \left[2\lambda |x(s) - y(s)| + \sum_{i=1}^{m} L_i |x(s - \tau_i(s)) - y(s - \tau_i(s))| \right] ds \\
&\quad \leq \frac{L e^{\lambda T}}{e^{\lambda T} - 1} \int_{t-T}^{t} e^{-\lambda(t-s)} \left[\sum_{i=1}^{m+1} |x(\eta_i(s)) - y(\eta_i(s))| \right] ds,
\end{aligned}
\tag{2.21}
$$

where $\eta_i(s) = s - \tau_i(s)$ $(i = 1, 2, \ldots, m)$, $\eta_{m+1}(s) = s$ and $L = \max\{2\lambda, L_1, \ldots, L_m\}$.

Thus, the condition $(\widetilde{H}_1)$ in **Theorem 2.2** holds for $\beta = 0, n = m + 1$, and $G(t, s) = (L e^{\lambda T} / (e^{\lambda T} - 1)) e^{-\lambda(t-s)}$.

On the other hand, we choose a constant $c > 0$ such that $0 < (m + 1)(L e^{\lambda T} / (e^{\lambda T} - 1))(1/(c + \lambda)) < 1$. Take $\alpha = (L e^{\lambda T} / (e^{\lambda T} - 1))(1/(c + \lambda))$ and $y(t) = e^{ct}$ for $t \in [0, T]$, then $y(\eta_i(t)) \leq y(t)$, and we have

$$
\int_{t-T}^{t} G(t, s) y(s) ds = \int_{t-T}^{t} \frac{L e^{\lambda T}}{e^{\lambda T} - 1} e^{-\lambda(t-s)} e^{cs} ds \leq \alpha y(t).
\tag{2.22}
$$

This implies the condition $(\widetilde{H}_2)$ in **Theorem 2.2** holds for $K = 1$.

Following **Theorem 2.2**, we conclude that the operator A has a unique fixed point, say φ, in $PC(R, R)$. Thus, (1.1) has a unique T-periodic solution in $PC(R, R)$. This completes the proof of **Theorem 2.3**. $\qquad\square$

3. Application to the Hematopoiesis Model

In this section, we consider the periodic solution of following hematopoiesis model with delays:

$$
x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))},
\tag{3.1}
$$

where $a, b_i, \tau_i \in C(R, R^+)$ are T-periodic functions, τ_i satisfies conditions $(H_{f\tau})$, and $n \geq 1$ is a real number $(i = 1, \ldots, m)$.

Theorem 3.1. *The delayed hematopoiesis model* (3.1) *has a unique positive T-periodic solution.*

Proof. Let $C_T^+ = \{y : y \in C_T \text{ and } y(t) \geq 0 \text{ for } t \geq 0\}$, define the operator $F : C_T^+ \to C_T^+$ by

$$
(Ax)(t) = \frac{e^{\lambda T}}{e^{\lambda T} - 1} \int_{t-T}^{t} e^{-\lambda(t-s)} \left[(\lambda - a(s))x(s) + \sum_{i=1}^{m} \frac{b_i(s)}{1 + x^n(s - \tau_i(s))} \right] ds.
\tag{3.2}
$$

It is easy to show that A is welldefined. Furthermore, since the function

$$g(t, x_1, \ldots, x_m) = \sum_{i=1}^{m} \frac{b_i(t)}{1 + x_i^n} \quad \text{for } x_i \in R^+ \tag{3.3}$$

with the bounded partial derivative

$$\frac{\partial g(t, x_1, \ldots, x_m)}{\partial x_i} \leq \max\left\{ \frac{b_i(t) n x_i^{n-1}}{1 + x_i^n} : t \in [0, T], 1 \leq i \leq m \right\}, \tag{3.4}$$

then it is easy to prove that the condition (H_f) holds. Following the similar arguments of Theorem 2.3, we claim that the operator A has a unique fixed point in C_T^+, which is the unique positive T-periodic solution for equation (3.1). This completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 exhibits that the periodic coefficients hematopoiesis model admits a unique positive periodic solution without additional restriction. Also, **Theorem 3.1** improves Theorem 2.1 in [2] and Corollary 1 in [3].

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