

## Research Article

# Bounded Approximate Identities in Ternary Banach Algebras

Madjid Eshaghi Gordji,<sup>1,2</sup> Ali Jabbari,<sup>3</sup> and Gwang Hui Kim<sup>4</sup>

<sup>1</sup> Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran

<sup>2</sup> Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, P.O. Box 35195-363, Semnan, Iran

<sup>3</sup> Young Researchers Club, Ardabil Branch, Islamic Azad University, P.O. Box 15655-461, Ardabil, Iran

<sup>4</sup> Department of Mathematics, Kangnam University, Youngin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to Gwang Hui Kim, ghkim@kangnam.ac.kr

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Let  $A$  be a ternary Banach algebra. We prove that if  $A$  has a left-bounded approximating set, then  $A$  has a left-bounded approximate identity. Moreover, we show that if  $A$  has bounded left and right approximate identities, then  $A$  has a bounded approximate identity. Hence, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras.

## 1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] who introduced the notion of cubic matrix which in turn was generalized by Kapranov et al. in [2]. The comments on physical applications of ternary structures can be found in [3–7].

A nonempty set  $G$  with a ternary operation  $[\cdot, \cdot, \cdot] : G \times G \times G \rightarrow G$  is called a ternary groupoid and denoted by  $(G, [\cdot, \cdot, \cdot])$ . The ternary groupoid  $(G, [\cdot, \cdot, \cdot])$  is called a ternary semigroup if the operation  $[\cdot, \cdot, \cdot]$  is associative, that is, if

$$[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]] \quad (1.1)$$

holds for all  $x, y, z, u, v \in G$ . A ternary semigroup  $(G, [\cdot, \cdot, \cdot])$  is a ternary group if for all  $a, b, c \in G$ , there are  $x, y, z \in G$  such that

$$[x, a, b] = [a, y, b] = [a, b, z] = c, \quad (1.2)$$

where the elements  $x, y, z$  are uniquely determined (see [8]).

A ternary Banach algebra is a complex Banach space  $A$ , equipped with a ternary product  $(x, y, z) \mapsto [x, y, z]$  of  $A^3$  into  $A$ , which is associative in the sense that  $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [y, z, u], v]$  and satisfies  $\|[x, y, z]\| \leq \|x\|\|y\|\|z\|$ . An element  $e \in A$  is an identity of  $A$  if  $x = [x, e, e] = [e, e, x]$  for all  $x \in A$ .

Assume that  $A$  is a ternary Banach algebra a bounded net  $(e_\alpha, f_\alpha)$  is a left-bounded approximate identity for  $A$  if  $\lim_\alpha [e_\alpha, f_\alpha, a] = a$  for all  $a \in A$ . Similarly, a bounded net  $(e_\alpha, f_\alpha)$  is a right-bounded approximate identity for  $A$  if  $\lim_\alpha [a, e_\alpha, f_\alpha] = a$  for all  $a \in A$ . Also,  $(e_\alpha, f_\alpha)$  is a middle-bounded approximate identity for  $A$  if  $\lim_\alpha [e_\alpha, a, f_\alpha] = a$  for all  $a \in A$ . A net  $(e_\alpha, f_\alpha)$  is a bounded approximate identity for  $A$  if  $(e_\alpha, f_\alpha)$  is a left-, right-, and middle-bounded approximate identity for  $A$ .

For ternary Banach algebra  $A$ , a set  $U \times V$  is said to be an approximating set for  $A$  ( $U$  and  $V$  are bounded subsets of  $A$ ) if for every  $\epsilon > 0$ , and every  $a \in A$ , there exist  $u \in U, v \in V$  such that  $\|[u, v, a] - a\| < \epsilon, \|[u, a, v] - a\| < \epsilon, \|[a, u, v] - a\| < \epsilon$ .

Existence of bounded approximating set for binary Banach algebras guarantees existing of bounded approximate identity (Altman's Theorem [9, Proposition 2, page 58] or [10]) and also this notion generalized for commutative Fréchet algebras [11]. For normed algebra  $A$  with left-bounded approximate identity and right-bounded approximate identity, Dixon [12] proved that  $A$  has a bounded approximate identity [13, Proposition 2.9.3].

In this paper, we prove Altman's Theorem and Dixon's Theorem for ternary Banach algebras. By " $\circ$ ", we mean the quasiproduct between elements  $x, y$  of binary algebra  $A$  which are defined by  $x \circ y = x + y - xy$ .

## 2. Main Results

We start our work with the following theorem which can be regarded as a version of Altman's Theorem for ternary Banach algebras.

**Theorem 2.1.** *Let  $A$  be a ternary Banach algebra and  $U, V$  be bounded subsets of  $A$  such that for given  $a \in A$  and  $\epsilon > 0$  there are  $u \in U$  and  $v \in V, \|[u, v, a] - a\| < \epsilon$ . Then  $A$  possess a left-bounded approximate identity.*

*Proof.* Let  $\epsilon > 0$ , and set

$$W = UV \circ UV = \{(u_1 v_1) \circ (u_2 v_2) : u_1, u_2 \in U, v_1 v_2 \in V\}. \quad (2.1)$$

For proof of theorem, it is sufficient to show that for every finite subset  $F \subset A$ , there exists  $w = uv \circ st \in W$  such that  $\|[uv \circ st, a] - a\| < \epsilon$  for every  $a \in F$ .

*Step 1.* Let  $F = \{a\}$  be singleton. Then, there are  $u \in U$  and  $v \in V$  such that  $\|uv\| < M$ , and

$$\|[u, v, a] - a\| < \frac{\epsilon}{(M+1)}. \quad (2.2)$$

Letting  $w = uv \circ uv$ , then

$$\|[uv \circ uv, a] - a\| = \|[u, v, [u, v, a] - a] - ([u, v, a] - a)\| < \epsilon. \quad (2.3)$$

*Step 2.* Let  $F = \{a_1, a_2\}$ . There is a  $(u_1, v_1) \in U \times V$  such that  $\|[u_1, v_1, a_1] - a_1\| < \epsilon/(1+M)$ , and for  $[u_1, v_1, a_2] - a_2 \in A$  there is a  $(u_2, v_2) \in U \times V$  such that

$$\|[u_2, v_2, [u_1, v_1, a_2] - a_2] - ([u_1, v_1, a_2] - a_2)\| < \epsilon. \quad (2.4)$$

Put  $w_1 = u_1v_1$  and  $w_2 = u_2v_2$ . Then

$$\begin{aligned} \|[w_2 \circ w_1, a_i] - a_i\| &= \|[u_2, v_2, a_i] + [u_1, v_1, a_i] - [u_2, v_2, [u_1, v_1, a_i]] - a_i\| \\ &= \|[u_2, v_2, a_i - [u_1, v_1, a_i]] - (a_i - [u_1, v_1, a_i])\| < \epsilon, \end{aligned} \quad (2.5)$$

for  $i = 1, 2$ .

*Step 3.* Now, suppose that obtained results in Steps 1 and 2 are true for  $i = 1, 2, \dots, n$ . Let  $F = \{a_1, a_2, \dots, a_{n+1}\}$ , and let  $K = \max\{\|a_i\| : i = 1, \dots, n\}$ . There exist  $w_1 \circ w_2 \in W$  such that  $\|a_i - [w_2 \circ w_1, a_i]\| < \epsilon/3(M+1)^2$ , for  $i = 1, 2, \dots, n$ , where  $w_1$  and  $w_2$  are defined as in Step 2. Also, we can choose  $\alpha_1 = \theta_1\eta_1$  and  $\alpha_2 = \theta_2\eta_2$  such that  $\alpha_1 \circ \alpha_2 \in W$ ,

$$\begin{aligned} &\|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \\ &= \|[\theta_2, \eta_2, w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1]] - (w_2 \circ w_1 - [\theta_1, \eta_1, w_2 \circ w_1])\| \\ &< \frac{\epsilon}{3K}, \end{aligned} \quad (2.6)$$

and  $\|[\alpha_2 \circ \alpha_1, a_{n+1}] - a_{n+1}\| < \epsilon$ . Then for every  $j = 1, 2, \dots, n$  we have

$$\begin{aligned} \|\alpha_2 \circ \alpha_1, a_j\| - a_j\| &\leq \|a_j - [w_2 \circ w_1, a_j]\| + \|[\alpha_2 \circ \alpha_1, a_j] - [\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]]\| \\ &\quad + \|[\alpha_2 \circ \alpha_1, [w_2 \circ w_1, a_j]] - [w_2 \circ w_1, a_j]\| \\ &\leq \|a_j - [w_2 \circ w_1, a_j]\| + \|\alpha_2 \circ \alpha_1\| \|a_j - [w_2 \circ w_1, a_j]\| \\ &\quad + \|[\alpha_2 \circ \alpha_1, w_2 \circ w_1] - w_2 \circ w_1\| \|a_j\| \\ &< \epsilon. \end{aligned} \quad (2.7)$$

Let  $F(A)$  be the collection of all finite subsets of  $A$  and  $\Lambda = \mathbb{N} \times \mathbb{N} \times F(A)$ . Then  $\Lambda$  is a direct set with the following partial order:

$$(n_1, m_1, F_1) \leq (n_2, m_2, F_2) \quad \text{iff } F_1 \subseteq F_2, n_1 \leq n_2, m_1 \leq m_2. \quad (2.8)$$

Now, we can choose a bounded approximate identity  $(e_\lambda, f_\lambda)_{\lambda \in \Lambda}$  for  $A$ . □

Now, we prove Dixon's Theorem for ternary Banach algebras. Hence, we prove that if a ternary Banach algebra has both left- and right-bounded approximate identities, then it has a bounded approximate identity.

**Theorem 2.2.** *Let  $A$  be a ternary Banach algebra and  $(e_\alpha, f_\alpha)$  and  $(e_\beta, f_\beta)$  be bounded left and right approximate identities of  $A$ , respectively. Then  $A$  has a bounded approximate identity.*

*Proof.* Consider  $(c_{\alpha,\beta}, d_{\alpha,\beta}) = (e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha)$ . We claim that  $(c_{\alpha,\beta}, d_{\alpha,\beta})$  is a bounded approximate identity for  $A$ . Boundedness of mentioned net is clear. Therefore, we have to show that  $(c_{\alpha,\beta}, d_{\alpha,\beta})$  is a right, left, and middle approximate identity for  $A$ .

*Step 1.*  $(c_{\alpha,\beta}, d_{\alpha,\beta})$  is a left approximate identity. Because

$$\begin{aligned} & \| [e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \| \\ &= \| [e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha, a] + [e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha, a] - a \| \\ &\leq \| [e_\alpha f_\alpha, e_\alpha f_\alpha, a] - a \| + \| [e_\alpha f_\alpha, e_\beta f_\beta, a] - [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\ &\quad + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha, a] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \| + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta, a] - [e_\beta f_\beta, e_\beta f_\beta, a] \| \\ &\quad + \| [e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\ &\leq \| [e_\alpha, f_\alpha, [e_\alpha, f_\alpha, a]] - [e_\alpha, f_\alpha, a] \| + \| [e_\alpha, f_\alpha, a] - a \| + \| e_\alpha f_\alpha \| \| e_\beta f_\beta \| \| a - [e_\alpha, f_\alpha, a] \| \\ &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\alpha f_\alpha, a]] - [e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\ &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta, a]] - [e_\beta f_\beta, e_\beta f_\beta, a] \| \\ &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha, a]] - [e_\beta f_\beta e_\beta f_\beta, e_\alpha f_\alpha, a] \| \\ &\leq \frac{5\epsilon}{MN+1} + MN \frac{\epsilon}{MN+1} < \epsilon, \end{aligned} \quad (2.9)$$

where  $\|e_\alpha f_\alpha\| \leq \|e_\alpha\| \|f_\alpha\| \leq M$ , and  $\|e_\beta f_\beta\| \leq \|e_\beta\| \|f_\beta\| \leq N$ .

Step 2.  $(c_{\alpha,\beta}, d_{\alpha,\beta})$  is a right approximate identity because

$$\begin{aligned}
 & \| [a, e_\alpha f_\alpha \circ e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \| \\
 &= \| [a, e_\alpha f_\alpha, e_\beta f_\beta \circ e_\alpha f_\alpha] + [a, e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \| \\
 &\leq \| [a, e_\beta f_\beta, e_\beta f_\beta] - a \| + \| [a, e_\alpha f_\alpha, e_\beta f_\beta] - [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta] \| \\
 &\quad + \| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\alpha f_\alpha] \| + \| [a, e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\beta f_\beta, e_\alpha f_\alpha] \| \\
 &\quad + \| [a, e_\alpha f_\alpha e_\beta f_\beta, e_\beta f_\beta e_\alpha f_\alpha] - [a, e_\alpha f_\alpha, e_\beta f_\beta e_\alpha f_\alpha] \| \\
 &\leq \| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \| + \| [a, e_\beta, f_\beta] - a \| \\
 &\quad + \| [a, e_\alpha, f_\alpha] - [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] \| \| e_\beta f_\beta \| + \| [[a, e_\alpha, f_\alpha], e_\beta, f_\beta] - [a, e_\alpha, f_\alpha] \| \| e_\alpha f_\alpha \| \\
 &\quad + \| [[a, e_\beta, f_\beta], e_\beta, f_\beta] - [a, e_\beta, f_\beta] \| \| e_\alpha f_\alpha \| \\
 &\quad + \| [[a, e_\alpha f_\alpha, e_\beta f_\beta], e_\beta f_\beta] - [a, e_\alpha f_\alpha, e_\beta f_\beta] \| \| e_\alpha f_\alpha \| \\
 &\leq \frac{2\epsilon}{MN+1} + \frac{3M\epsilon}{MN+1} + \frac{N\epsilon}{MN+1} < \epsilon.
 \end{aligned} \tag{2.10}$$

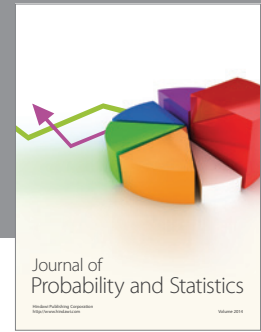
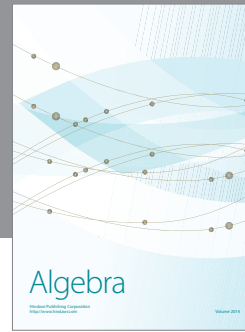
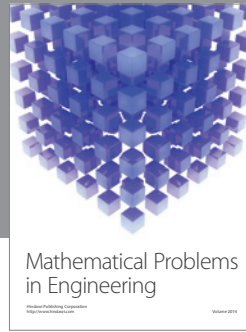
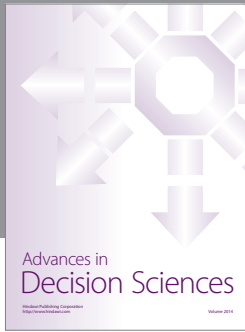
Step 3. By the similar method, we show that the net  $(c_{\alpha,\beta}, d_{\alpha,\beta})$  is a middle approximate identity:

$$\begin{aligned}
 & \| [e_\alpha f_\alpha \circ e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \| \\
 &= \| [e_\alpha f_\alpha, a, e_\beta f_\beta \circ e_\alpha f_\alpha] + [e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta \circ e_\alpha f_\alpha] - a \| \\
 &\leq \| [e_\alpha f_\alpha, a, e_\beta f_\beta] - a \| + \| [e_\alpha f_\alpha, a, e_\alpha f_\alpha] - [e_\alpha f_\alpha, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
 &\quad + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta] - [e_\beta f_\beta, a, e_\beta f_\beta] \| + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \| \\
 &\quad + \| [e_\alpha f_\alpha e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
 &\leq \| [e_\alpha, f_\alpha, [a, e_\beta, f_\beta]] - [a, e_\beta, f_\beta] \| + \| [a, e_\beta, f_\beta] - a \| \\
 &\quad + \| [e_\alpha, f_\alpha, a] - [[e_\alpha, f_\alpha, a], e_\beta, f_\beta] \| \| e_\alpha f_\alpha \| \\
 &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta]] - [e_\beta f_\beta, a, e_\beta f_\beta] \| \\
 &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\alpha f_\alpha] \| \\
 &\quad + \| [e_\alpha, f_\alpha, [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha]] - [e_\beta f_\beta, a, e_\beta f_\beta e_\alpha f_\alpha] \| \\
 &\leq \frac{5\epsilon}{MN+1} + \frac{M\epsilon}{MN+1} < \epsilon.
 \end{aligned} \tag{2.11}$$

This completes the proof of theorem. □

## References

- [1] A. Cayley, "On the 34 concomitants of the ternary cubic," *American Journal of Mathematics*, vol. 4, no. 1–4, pp. 1–15, 1881.
- [2] M. M. Kapranov, I. M. Gelfand, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhauser, Berlin, Germany, 1994.
- [3] V. Abramov, R. Kerner, O. Liivapuu, and S. Shitov, "Algebras with ternary law of composition and their realization by cubic matrices," *Journal of Generalized Lie Theory and Applications*, vol. 3, no. 2, pp. 77–94, 2009.
- [4] R. Kerner, "Ternary and non-associative structures," *International Journal of Geometric Methods in Modern Physics*, vol. 5, no. 8, pp. 1265–1294, 2008.
- [5] R. Kerner, "The cubic chessboard," *Geometry and Physics. Classical Quantum Gravity*, vol. 14, no. 1A, pp. A203–A225, 1997.
- [6] G. L. Sewell, *Quantum Mechanics and Its Emergent Macrophysics*, Princeton University Press, Princeton, NJ, USA, 2002.
- [7] H. Zettl, "A characterization of ternary rings of operators," *Advances in Mathematics*, vol. 48, no. 2, pp. 117–143, 1983.
- [8] E. L. Post, "Polyadic groups," *Transactions of the American Mathematical Society*, vol. 48, pp. 208–350, 1940.
- [9] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, New York, NY, USA, 1973.
- [10] M. Altman, "Contractors, approximate identities and factorization in Banach algebras," *Pacific Journal of Mathematics*, vol. 48, pp. 323–334, 1973.
- [11] E. Ansari-piri and A. Zohri, "Approximating sets in commutative Fréchet algebras," *Journal of Analysis and Applications*, vol. 7, no. 2, pp. 85–90, 2009.
- [12] P. G. Dixon, "Approximate identities in normed algebras," *Proceedings of the London Mathematical Society*, vol. 26, pp. 485–496, 1973.
- [13] H. G. Dales, *Banach Algebras and Automatic Continuity*, vol. 24 of *London Mathematical Society Monographs*, The Clarendon Press, Oxford, UK, 2000.



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