Research Article

Ulam Stability of a Quartic Functional Equation

Abasalt Bodaghi,[1,2] Idham Arif Alias,[2] and Mohammad Hosein Ghahramani[1]

[1] *Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[2] *Institute for Mathematical Research, University Putra Malaysia, 43400 UPM, Serdang, Selangor Darul Ehsan, Malaysia*

Correspondence should be addressed to Abasalt Bodaghi, abasalt.bodaghi@gmail.com

The oldest quartic functional equation was introduced by J. M. Rassias in (1999), and then was employed by other authors. The functional equation $f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$ is called a *quartic functional equation,* all of its solution is said to be a *quartic function.* In the current paper, the Hyers-Ulam stability and the superstability for quartic functional equations are established by using the fixed-point alternative theorem.

1. Introduction

We say a functional equation $\mathcal{F}$ is *stable* if any function f satisfying the equation $\mathcal{F}$ approximately is near to true solution of $\mathcal{F}$. Moreover, a functional equation $\mathcal{F}$ is *superstable* if any function f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$ (see [1] for another notion of the superstability which may be called *superstability modulo the bounded functions*).

The stability problem for functional equations originated from a question by Ulam [2] in 1940, concerning the stability of group homomorphisms: let $(G_1, \cdot)$ be a group, and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that, if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(s \cdot t), h(s) * h(t)) < \delta$ for all $s, t \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(s), H(s)) < \epsilon$ for all $s \in G_1$? In other words, under what condition a functional equation is stable? In the following year, Hyers [3] gave a partial affirmative answer to the question of Ulam for Banach spaces. In 1978, the generalized Hyers' theorem was independently rediscovered by Th. M. Rassias [4] by obtaining a unique linear mapping under certain continuity assumption.

The functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$
$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.1}$$

are called *quadratic* and *cubic* functional equations, respectively. During the last decades, several stability problems for functional equations especially the quadratic and cubic and their generalized have been extensively investigated by many mathematicians (for instances, [5–9]).

In [10], Lee et al. considered the following quartic functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to check that for every $a \in \mathbb{R}$, the function $f(x) = ax^4$ is a solution of the above functional equation. They solved (1.2) and in fact showed that a function $f : \mathcal{X} \rightarrow \mathcal{Y}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are real vector spaces is quadratic if and only if there exists a symmetric biquadratic function $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ such that $f(x) = F(x,x)$ for all $x \in \mathcal{X}$. They also proved the stability of (1.2). Zhou Xu et al. in [11] used the fixed-point alternative (Theorem 2.1 of the current paper) to establish Hyers-Ulam-Rassias stability of the general mixed additive-cubic functional equation, where functions map a linear space into a complete quasifuzzy p-normed space. The generalized Hyers-Ulam stability of a general mixed AQCQ-functional in multi-Banach spaces is also proved by using the mentioned theorem in [12].

Recently, Bodaghi et al. in [13, 14] investigated the stability and the superstability of quadratic and cubic functional equations by a fixed-point method and applied this method to prove the stability of (quadratic, cubic) multipliers on Banach algebras.

In this paper we prove the generalized Hyers-Ulam stability and the superstability for quartic functional equation (1.2) by using the alternative fixed point (Theorem 2.1) under certain conditions.

2. Main Results

Throughout this paper, assume that $\mathcal{X}$ is a normed vector space and $\mathcal{Y}$ is a Banach space. For a given mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$, we consider

$$Df(x,y) := f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y), \tag{2.1}$$

for all $x, y \in \mathcal{X}$.

To achieve our aim, we need the following known fixed-point theorem which has been proved in [15].

Theorem 2.1. *Suppose that (Δ, d) is a complete generalized metric space, and let $\mathcal{J} : \Delta \rightarrow \Delta$ be a strictly contractive mapping with Lipschitz constant $L < 1$, Then for each element $g \in \Delta$, either $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

(i) *$d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) < \infty$, for all $n \geq n_0$,*

(ii) *the sequence $\{\mathcal{J}^n g\}$ is convergent to a fixed-point g^* of $\mathcal{J}$,*

(iii) g^* is the unique fixed point of $\mathcal{Q}$ in the set

$$\Omega = \{g \in \Delta : d(\mathcal{Q}^{n_0}g, g) < \infty\}; \tag{2.2}$$

(iv) $d(g, g^*) \le (1/(1-L))d(g, \mathcal{Q}g)$, for all $g \in \Omega$.

Theorem 2.2. *Assume that $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function satisfying*

$$\|Df(x, y)\| \le \phi(x, y), \tag{2.3}$$

for all $x, y \in \mathcal{X}$. Let a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfy $f(0) = 0$. If there exists $K \in (0, 1)$ such that

$$\phi(x, y) \le 2^4 K \phi\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.4}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{32(1-K)}\phi(x, 0), \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. By recurrence method, we can conclude from (2.4) that $\phi(2^n x, 2^n y)/2^{4n} \le K^n \phi(x, y)$ for all $x, y \in \mathcal{X}$. Passing to the limit, we get

$$\lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.6}$$

for all $x, y \in \mathcal{X}$. Here, we intend to build the conditions of **Theorem 2.1** and so consider the set $\Delta := \{h : \mathcal{X} \to \mathcal{Y} \mid h(0) = 0\}$ and the mapping d defined on $\Delta \times \Delta$ by

$$d(g, h) := \inf\{C \in (0, \infty) : \|g(x) - h(x)\| \le C\phi(x, 0) \ \forall x \in \mathcal{X}\} \tag{2.7}$$

if there exists such constant C, and $d(g, h) = \infty$ otherwise. It is easy to see that $d(h, h) = 0$ and $d(g, h) = d(h, g)$, for all $g, h \in \Delta$. For each $g, h, p \in \Delta$, we have

$$\begin{aligned}
\inf\{C \in (0, \infty) &: \|g(x) - h(x)\| \le C\phi(x, 0) \ \forall x \in \mathcal{X}\} \\
&\le \inf\{C \in (0, \infty) : \|g(x) - p(x)\| \le C\phi(x, 0) \ \forall x \in \mathcal{X}\} \\
&\quad + \inf\{C \in (0, \infty) : \|p(x) - h(x)\| \le C\phi(x, 0) \ \forall x \in \mathcal{X}\}.
\end{aligned} \tag{2.8}$$

Hence, $d(g, h) \le d(g, p) + d(p, h)$. Now if $d(g, h) = 0$, then for every fixed $x_0 \in \mathcal{X}$, we have $\|g(x_0) - h(x_0)\| \le C\phi(x_0, 0)$, for all $C > 0$. This implies $g = h$. Let $\{h_n\}$ be a d-Cauchy sequence in Δ, then $d(h_m, h_n) \to 0$, and thus $\|h_m(x) - h_n(x)\| \to 0$, for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is

complete, then there exists $h \in \Delta$ such that $h_n \xrightarrow{d} h$ in Δ. Therefore, d is a generalized metric on Δ, and the metric space (Δ, d) is complete. Now, we define the mapping $\mathcal{Q} : \Delta \to \Delta$ by

$$\mathcal{Q}g(x) = \frac{1}{2^4}g(2x), \quad (x \in \mathcal{X}). \tag{2.9}$$

Fix a $C \in (0, \infty)$ and take $g, h \in \Delta$ such that $d(g, h) < C$. The definitions of d and $\mathcal{Q}$ show that

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \leq \frac{1}{2^4}C\phi(2x, 0), \tag{2.10}$$

for all $x \in \mathcal{X}$. By using (2.4), we have

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \leq CK\phi(x, 0), \tag{2.11}$$

for all $x \in \mathcal{X}$. It follows from the above inequality that $d(\mathcal{Q}g, \mathcal{Q}h) \leq Kd(g, h)$, for all $g, h \in \Delta$. Hence, $\mathcal{Q}$ is a strictly contractive mapping on Δ with a Lipschitz constant K. Putting $y = 0$ in (2.3) and dividing both sides of the resulting inequality by 32, we have

$$\left\| f(x) - \frac{1}{16}f(2x) \right\| \leq \frac{1}{32}\phi(x, 0), \tag{2.12}$$

for all $x \in X$. Thus, $d(f, \mathcal{Q}f) \leq 1/32 < \infty$. Note that by Theorem 2.1, $d(\mathcal{Q}^n g, \mathcal{Q}^{n+1} g) < \infty$, for all $n \geq 0$. Thus, we get $n_0 = 0$ in this theorem, so (iii) and (iv) of Theorem 2.1 are true on the whole Δ. However, the sequence $\{\mathcal{Q}^n f\}$ converges to a unique fixed-point $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\{g \in \Delta; d(f, g) < \infty\}$, that is,

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^{4n}}, \tag{2.13}$$

for all $x \in X$. By the part (iv) of Theorem 2.1, we have

$$d(f, Q) \leq \frac{d(f, \mathcal{Q}f)}{1 - K} \leq \frac{1}{32(1 - K)}. \tag{2.14}$$

From (2.14), we observe that the inequality (2.5) holds for all $x \in \mathcal{X}$. Substituting x, y by $2^n x, 2^n y$ in (2.3), respectively, and applying (2.6) and (2.13), we have

$$\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{2^{4n}} \|Df(2^n x, 2^n y)\| \leq \lim_{n \to \infty} \frac{1}{2^{4n}} \phi(2^n x, 2^n y) = 0, \tag{2.15}$$

for all $x \in \mathcal{X}$. Therefore, Q is a quartic mapping which is unique by part (iii) of Theorem 2.1.

$\square$

Corollary 2.3. *Let p, θ be nonnegative real numbers such that $p < 4$, and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a mapping (with $f(0) = 0$ when $p = 0$) satisfying*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{2.16}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{32 - 2^{p+1}} \|x\|^p, \tag{2.17}$$

for all $x \in \mathcal{X}$.

Proof. The result follows from Theorem 2.2 by using $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$.   □

Now, we establish the superstability of quartic mapping on Banach spaces under some conditions.

Corollary 2.4. *Let p, q, θ be nonnegative real numbers such that $p + q \in (0, 4)$. Suppose that a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies*

$$\|Df(x, y)\| \leq \theta \|x\|^p \|y\|^q, \tag{2.18}$$

for all $x, y \in \mathcal{X}$, then f is a quartic mapping on $\mathcal{X}$.

Proof. Letting $\phi(x, y) = \theta \|x\|^p \|y\|^q$ in Theorem 2.2, we have

$$\lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.19}$$

which shows (2.6) holds for ϕ. Putting $x = y = 0$ in (2.18), we get $f(0) = 0$. Furthermore, if we put $y = 0$ in (2.18), then we have $f(2x) = 2^4 f(x)$, for all $x \in \mathcal{X}$. It is easy to see that by induction, we have $f(2^n x) = 2^{4n} f(x)$, and so $f(x) = f(2^n x)/2^{4n}$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 2.2 that f is a quartic mapping.   □

Let θ and p be positive real numbers. Suppose that a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies

$$\|Df(x, y)\| \leq \theta \|y\|^p, \tag{2.20}$$

for all $x, y \in \mathcal{X}$, then by considering $\phi(x, y) = \theta \|y\|^p$ in Theorem 2.2, the mapping f is again a quartic mapping on $\mathcal{X}$.

The following result is proved in [16, Theorem 1].

Theorem 2.5. *Let $\mathcal{X}$ be a linear space, and let $\mathcal{Y}$ be a Banach space. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-4k} \varphi\left(2^k x, 2^k y\right) < \infty, \tag{2.21}$$

$$\|Df(x, y)\| \leq \delta + \varphi(x, y)$$

for all $x, y \in \mathcal{X}$, where $\delta \geq 0$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - Q(x) + \frac{1}{5} f(0) \right\| \leq \frac{1}{30} \delta + \frac{1}{32} \widetilde{\varphi}(x, 0) \tag{2.22}$$

for all $x \in \mathcal{X}$.

One should note that in the above theorem, $f(0)$ is not necessarily zero, but in the following result, we assume that $f(0) = 0$ and also consider the case $\delta = 0$. By these hypotheses and by applying Theorem 2.1, we obtain the specific result which is a way to prove the superstability of a quartic functional equation.

Theorem 2.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$, and let $\psi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying*

$$\lim_{n \to \infty} 2^{4n} \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{2.23}$$

$$\|Df(x, y)\| \leq \psi(x, y), \tag{2.24}$$

for all $x, y \in \mathcal{X}$. If there exists $L \in (0, 1)$ such that

$$\psi(x, 0) \leq 2^{-4} L \psi(2x, 0), \tag{2.25}$$

for all $x \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{32(1 - L)} \psi(x, 0), \tag{2.26}$$

for all $x \in \mathcal{X}$.

Proof. We take the set $\Omega := \{g : \mathcal{X} \to \mathcal{Y} \mid g(0) = 0\}$ and consider the generalized metric on Ω,

$$d(g_1, g_2) := \inf\{C \in (0, \infty) : \|g_1(x) - g_2(x)\| \leq C\psi(x, 0) \ \forall x \in \mathcal{X}\}, \tag{2.27}$$

if there exists such a constant C, and $d(g_1, g_2) = \infty$ otherwise. It follows from the proof of Theorem 2.2 that the metric space (Ω, d) is complete (see the proof of Theorem 2.2).

We will show that the mapping $\mathcal{J} : \Omega \to \Omega$ defined by $\mathcal{J}g(x) = 2^4 g(x/2) (x \in \mathcal{X})$ is strictly contractive. Fix a $C \in (0, \infty)$ and take $g_1, g_2 \in \Omega$ such that $d(g_1, g_2) < C$, then we have

$$\left\| 2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right) \right\| \le 2^4 C \psi\left(\frac{x}{2}, 0\right), \tag{2.28}$$

for all $x \in \mathcal{X}$. By using (2.25), we obtain

$$\left\| 2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right) \right\| \le CL\psi(x, 0), \tag{2.29}$$

for all $x \in \mathcal{X}$. It follows from the last inequality that $d(\mathcal{J}g_1, \mathcal{J}g_2) \le Ld(g_1, g_2)$, for all $g_1, g_2 \in \Omega$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Ω with a Lipschitz constant L. By putting $y = 0$, replacing x by $x/2$ in (2.24) and using (2.25), and then dividing both sides of the resulting inequality by 2, we have

$$\left\| 2^4 f\left(\frac{x}{2}\right) - f(x) \right\| \le \frac{1}{2}\psi\left(\frac{x}{2}, 0\right) \le 2^{-5}L\psi(x, 0), \tag{2.30}$$

for all $x \in \mathcal{X}$. Hence, $d(f, \mathcal{J}f) \le 2^{-5}L < \infty$. By applying the fixed-point alternative Theorem 2.1, there exists a unique mapping $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\Omega_1 = \{g \in \Omega; d(f, g) < \infty\}$ such that

$$Q(x) = \lim_{n \to \infty} 2^{4n} f\left(\frac{x}{2^n}\right), \tag{2.31}$$

for all $x \in \mathcal{X}$. Again Theorem 2.1 shows that

$$d(f, Q) \le \frac{d(f, \mathcal{J}f)}{1 - L} \le \frac{2^{-5}L}{1 - L}. \tag{2.32}$$

Hence, inequality (2.32) implies (2.26). Replacing x, y by $2^n x, 2^n y$ in (2.24), respectively, and using (2.23) and (2.31), we conclude that

$$\begin{aligned} \|DQ(x, y)\| &= \lim_{n \to \infty} 2^{4n} \left\| Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\le \lim_{n \to \infty} 2^{4n} \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{X}$. Therefore, Q is a quartic mapping. $\square$

Corollary 2.7. *Let p and λ be nonnegative real numbers such that $p > 4$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\|Df(x, y)\| \le \lambda(\|x\|^p + \|y\|^p), \tag{2.34}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\lambda}{2(2^p - 2^4)} \|x\|^p \qquad (2.35)$$

for all $x \in \mathcal{X}$.

Proof. It is enough to let $\psi(x, y) = \lambda(\|x\|^p + \|y\|^p)$ in Theorem 2.6. ☐

Corollary 2.8. *Let p, q, λ be nonnegative real numbers such that $p + q \in (4, \infty)$. Suppose that a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies*

$$\|Df(x, y)\| \leq \lambda \|x\|^p \|y\|^q \qquad (2.36)$$

for all $x, y \in \mathcal{X}$. Then f is a quartic mapping on $\mathcal{X}$.

Proof. Putting $\psi(x, y) = \theta \|x\|^p \|y\|^q$ in Theorem 2.6, we have

$$\lim_{n \to \infty} \frac{\psi(2^n x, 2^n y)}{2^{4n}} = 0, \qquad (2.37)$$

and thus, (2.6) holds. If we put $x = y = 0$ in (2.36), then we get $f(0) = 0$. Again, letting $y = 0$ in (2.36), we conclude that $f(x) = 2^4 f(x/2)$, and thus, $f(x) = 2^{4n} f(x/2^n)$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, we can obtain the desired result by Theorem 2.6.

From Corollaries 2.4 and 2.8 we deduce the following result. ☐

Corollary 2.9. *Let $p, q,$ and λ be nonnegative real numbers such that $p + q > 0$ and $p + q \neq 4$. Suppose that a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies (2.36), for all $x, y \in \mathcal{X}$ then f is a quartic mapping on $\mathcal{X}$.*

Acknowledgments

References

[1] J. Baker, "The stability of the cosine equation," *Proceedings of the American Mathematical Society*, vol. 80, no. 3, pp. 411–416, 1980.

[2] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1940.

[3] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[4] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] I. S. Chang, K. W. Jun, and Y. S. Jung, "The modified Hyers-Ulam-Rassias stability of a cubic type functional equation," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 675–683, 2005.

[6] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[7] M. Eshaghi Gordji and A. Bodaghi, "On the Hyers-Ulam-Rassias stability problem for quadratic functional equations," *East Journal on Approximations*, vol. 16, no. 2, pp. 123–130, 2010.

[8] K. W. Jun and H. M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.

[9] J. Lee, J. An, and C. Park, "On the stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2008, Article ID 628178, 8 pages, 2008.

[10] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.

[11] T. Zhou Xu, J. M. Rassias, and W. Xin Xu, "A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces," *International Journal of the Physical SciencesInt*, vol. 6, no. 2, pp. 313–324, 2011.

[12] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Generalized Ulam-Hyers stability of a general mixed AQCQ-functional equation in multi-Banach spaces: a fixed point approach," *European Journal of Pure and Applied Mathematics*, vol. 3, no. 6, pp. 1032–1047, 2010.

[13] A. Bodaghi, I. A. Alias, and M. Eshaghi Gordji, "On the stability of quadratic double centralizers and quadratic multipliers: a fixed point approach," *Journal of Inequalities and Applications*, vol. 2011, Article ID 957541, 9 pages, 2011.

[14] A. Bodaghi, I. A. Alias, and M. H. Ghahramani, "Approximately cubic functional equations and cubic multipliers," *Journal of Inequalities and Applications*, vol. 2011, 53 pages, 2011.

[15] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

[16] A. Najati, "On the stability of a quartic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 569–574, 2008.