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Research Article

The Tikhonov Regularization Method for Set-Valued Variational Inequalities

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This paper aims to establish the Tikhonov regularization theory for set-valued variational inequalities. For this purpose, we firstly prove a very general existence result for set-valued variational inequalities, provided that the mapping involved has the so-called variational inequality property and satisfies a rather weak coercivity condition. The result on the Tikhonov regularization improves some known results proved for single-valued mapping.

1. Introduction

This paper discusses the generalized variational inequality problem (in short, GVIP(F, K)) which is to find $x \in K$ and $x^* \in F(x)$ such that

$$\langle x^*, y - x \rangle \geq 0, \quad \forall y \in K, \tag{1.1}$$

where K is a nonempty closed convex set in $\mathbb{R}^n$ and $F : K \to 2^{\mathbb{R}^n}$ is a set-valued mapping with nonempty values. We use GVIP(F, K) and SOL(F, K) to denote the problem (1.1) and its solution set, respectively.

Generalized variational inequality has been extensively studied in the literature; see [1–7] and the references therein.

The Tikhonov regularization method is an important method for the ill-posed variational inequalities; see Pages 307 and 1224 in [8]. To our best knowledge, the Tikhonov regularization method has been discussed only for the case where the mapping F is single valued. This paper develops the Tikhonov regularization method for set-valued variational inequality GVIP(F, K).

As a preparation, we firstly give an existence result for $\mathrm{GVIP}(F,K)$. It is well known if F is upper semicontinuous with nonempty compact convex values and the set K is compact and convex, then $\mathrm{GVIP}(F,K)$ has a solution. If K is noncompact, one usually requires some kind of coerciveness conditions for the existence of solutions to $\mathrm{GVIP}(F,K)$. Thus many researchers have attempted to search coerciveness condition as weak as possible; see [2, 3, 8, 9] and the references therein. In particular, [9] proved the following result.

Theorem 1.1. *If F is single valued and continuous, and if the following condition is satisfied:*

(H) *There exists $n \in \mathbb{N}$ such that for every $x \in K \setminus K_n$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying $\langle F(x), x-y\rangle \geq 0$,*

then the variational inequality has a solution.

Example 3.1 in [9] shows that the condition (H) is strictly weaker than many known coerciveness conditions. So far, it is not known whether Theorem 1.1 could be extended to the situation where F is a set-valued mapping. An affirmative answer is given in Corollary 3.9 of this paper, which says that if F is upper semicontinuous with nonempty compact convex values, then the coerciveness condition (A), which reduces to the condition (H) when F is single-valued, implies that $\mathrm{GVIP}(F,K)$ has a solution. Actually, a more general existence result is verified in Theorem 3.8 which does not require that F have any kind of continuity. Theorem 3.8 shows that if F has the so-called variational inequality property, then the condition (A) implies that $\mathrm{GVIP}(F,K)$ has a solution. If the mapping F is either upper semicontinuous (with nonempty compact convex values), or quasimonotone and upper hemicontinuous (with nonempty compact convex values), then it has the variational inequality property. Thus Theorem 3.8 unifies many known existence results for $\mathrm{GVIP}(F,K)$.

The Tikhonov regularization method has been much discussed in the literature. In particular, assuming that K is a box and F is a single-valued mapping, [10] proved that $\mathrm{SOL}(F+\varepsilon I, K)$ is nonempty for any $\varepsilon > 0$ (here I stands for the identity mapping), provided that F is a continuous P_0-function and $\mathrm{SOL}(F,K)$ is nonempty and bounded. This result was extended by [11] to the situation where K is a closed convex set, the mapping F is single valued, and a coercivity condition is used to replace the assumption of $\mathrm{SOL}(F,K)$ being bounded. It should be noted that the coercivity condition assumed in [11] does not necessarily imply that $\mathrm{SOL}(F,K)$ is bounded. [12] further improves the result of [11] by assuming a weaker coercivity condition. All the above results on the Tikhonov regularization assume that the mapping F is single valued. The last part of this paper aims to establish the Tikhonov regularization theory for set-valued variational inequality (1.1). Theorem 4.1 improves the main result of [11, 12] by assuming a weaker coercivity condition and by allowing F to be a set-valued mapping (without monotonicity).

General variational inequalities have been extensively discussed; see [13–16]. It should be interesting to discuss the Tikhonov regularization method for general variational inequality in a similar way.

2. Preliminaries

Unless stated otherwise, we assume that $K \subset \mathbb{R}^n$ is a nonempty closed convex set and $F : K \to 2^{\mathbb{R}^n}$ is a set-valued mapping with nonempty values. For $r > 0$, $K_r := \{x \in K : \|x\| \leq r\}$.

Definition 2.1. Let $F : K \to 2^{\mathbb{R}^n}$ be a set-valued mapping. F is said to be

(i) monotone on K if for each pair of points $x, y \in K$ and for all $x^* \in F(x)$ and $y^* \in F(y)$, $\langle y^* - x^*, y - x\rangle \geq 0$,

(ii) maximal monotone on K if, for any $u \in K$, $\langle \xi - x^*, u - x\rangle \geq 0$ for all $x \in K$ and all $x^* \in F(x)$ implies $\xi \in F(u)$,

(iii) quasimonotone on K if for each pair of points $x, y \in K$ and for all $x^* \in F(x)$ and $y^* \in F(y)$, $\langle x^*, y - x\rangle > 0$ implies that $\langle y^*, y - x\rangle \geq 0$,

(iv) F is said to be upper semicontinuous at $x \in K$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for all $y \in K \cap U$; if F is upper semicontinuous at every $x \in K$, we say F is upper semicontinuous on K,

(v) upper hemicontinuous on K if the restriction of F to every line segment of K is upper semicontinuous.

The following result is celebrated; see [17].

Lemma 2.2. *Let K be a nonempty convex subset of a Hausdorff topological vector space E, and let $G : K \to 2^E$ be a set-valued mapping from K into E satisfying the following properties:*

(i) *G is a KKM mapping: for every finite subset A of K, $\mathrm{co}(A) \subset \bigcup_{x\in A} G(x)$, where co denotes the convex hull;*

(ii) *$G(x)$ is closed in E for every $x \in K$;*

(iii) *$G(x_0)$ is compact in E for some $x_0 \in K$.*

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

3. Existence of Solutions and Coercivity Conditions

Definition 3.1. F is said to have *variational inequality property on* K if for every nonempty bounded closed convex subset D of K, $\mathrm{GVIP}(F, D)$ has a solution.

Proposition 3.2. *The following classes of mappings have the variational inequality property:*

(i) *every upper semicontinuous set-valued mapping with nonempty compact convex values,*

(ii) *every upper hemicontinuous quasimonotone set-valued mapping with nonempty compact convex values;*

(iii) *if F is a single-valued continuous mapping and T is upper hemicontinuous and monotone with nonempty compact convex values, then $F + T$ has the variational inequality property.*

Proof. (i) is well known in the literature. (ii) is verified in [18]. (iii) is a consequence of the Debrunner-Flor lemma [19] and [20, Theorem 41.1]. Indeed, let D be a bounded closed convex subset of K, and let

$$N_D(x) := \left\{\xi \in \mathbb{R}^n : \sup_{y\in D}\langle \xi, y - x\rangle \leq 0\right\}, \tag{3.1}$$

the normal cone of D at $x \in D$. By [20, Theorem 41.1], $M(x) := T(x) + N_D(x)$ is a maximal monotone mapping. By the Debrunner-Flor lemma [19], there is $u \in D$ such that

$$\langle \xi + F(u), x - u \rangle \geq 0, \quad \forall x \in D, \ \forall \xi \in M(x). \tag{3.2}$$

Since M is maximal monotone, $-F(u) \in M(u) \equiv T(u) + N_D(u)$. By the definition of N_D, $u \in \mathrm{SOL}(F + T, D)$. □

Proposition 3.3 below shows that Proposition 3.2(iii) can be extended to the case where F is a set-valued mapping.

Proposition 3.3. *If $F : K \to 2^{\mathbb{R}^n}$ is upper semicontinuous with nonempty compact convex values and $T : K \to 2^{\mathbb{R}^n}$ is monotone and upper hemicontinuous on K with nonempty compact convex values, then $F + T$ has the variational inequality property on K.*

Proof. Let D be a bounded closed convex subset of K. Define $G, H : D \to 2^D$ by

$$\begin{aligned} G(y) &:= \left\{ x \in D : \sup_{\xi \in T(x)} \langle \xi, y - x \rangle + \sup_{\zeta \in F(x)} \langle \zeta, y - x \rangle \geq 0 \right\}, \\ H(y) &:= \left\{ x \in D : \inf_{\xi \in T(y)} \langle \xi, y - x \rangle + \sup_{\zeta \in F(x)} \langle \zeta, y - x \rangle \geq 0 \right\}. \end{aligned} \tag{3.3}$$

Since T is monotone, $G(y) \subset H(y)$. Since F is upper semicontinuous, $H(y)$ is closed and hence compact in D for every $y \in D$.

Now we prove that G is a KKM map. If not, there is $\{y_1, \ldots, y_n\} \subset D$ and $x_0 = \sum \lambda_i y_i$ such that $x_0 \notin \bigcup_{i=1}^n G(y_i)$

$$\begin{aligned} 0 &= \sup_{\xi \in T(x_0)} \left\langle \xi, \sum \lambda_i y_i - x_0 \right\rangle + \sup_{\zeta \in F(x_0)} \left\langle \zeta, \sum \lambda_i y_i - x_0 \right\rangle \\ &\leq \sum \lambda_i \left[\sup_{\xi \in T(x_0)} \langle \xi, y_i - x_0 \rangle + \sup_{\zeta \in F(x_0)} \langle \zeta, y_i - x_0 \rangle \right] < 0. \end{aligned} \tag{3.4}$$

This contradiction shows that G is a KKM map, so is H. By Lemma 2.2, there is $\overline{x} \in \cap_{y \in D} H(y)$. Fix any $y \in D$. Let $y_t := \overline{x} + t(y - \overline{x})$. Then for small $t \in (0, 1)$, $y_t \in D$ and hence

$$\begin{aligned} 0 &\leq \inf_{\xi \in T(y_t)} \langle \xi, y_t - \overline{x} \rangle + \sup_{\zeta \in F(\overline{x})} \langle \zeta, y_t - \overline{x} \rangle \\ &= t \inf_{\xi \in T(y_t)} \langle \xi, y - \overline{x} \rangle + t \sup_{\zeta \in F(\overline{x})} \langle \zeta, y - \overline{x} \rangle. \end{aligned} \tag{3.5}$$

Dividing $t > 0$ on both sides, we have

$$\sup_{\xi \in T(y_t)} \langle \xi, y - \overline{x} \rangle + \sup_{\zeta \in F(\overline{x})} \langle \zeta, y - \overline{x} \rangle \geq 0. \tag{3.6}$$

Letting $t \to 0+$ yields that $\overline{x} \in G(y)$, as T is upper hemicontinuous. Since $y \in D$ is arbitrary, $\overline{x} \in \bigcap_{y\in D} G(y)$:

$$\sup_{\xi\in T(\overline{x})} \sup_{\zeta\in F(\overline{x})} \langle \xi + \zeta, y - \overline{x} \rangle \geq 0, \quad \forall y \in D. \tag{3.7}$$

Since $F(\overline{x})$ and $T(\overline{x})$ are compact and convex, the Sion minimax theorem implies the existence of $\overline{u} \in F(\overline{x})$ and $\overline{v} \in T(\overline{x})$ such that

$$\langle \overline{u} + \overline{v}, y - \overline{x} \rangle \geq 0, \quad \forall y \in D. \tag{3.8}$$

Thus $\overline{x}$ solves GVIP$(F + T, D)$. □

Before making further discussion, we need to state some coercivity conditions. The relationships of these coercivity conditions are well known in the literature; however, we provide the proof for completeness.

Consider the following coercivity conditions.

(A) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying $\inf_{x^*\in F(x)} \langle x^*, x - y \rangle \geq 0$.

(B) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there is $y \in K_r$ satisfying $\inf_{x^*\in F(x)} \langle x^*, x - y \rangle \geq 0$.

(C) There exists $r > 0$ such that for every $x \in K \setminus K_r$ and every $x^* \in F(x)$, there exists some $y \in K_r$ such that $\langle x^*, x - y \rangle > 0$.

(D) There exists $r > 0$ such that for every $x \in K \setminus K_r$, there exists some $y \in K_r$ such that $\sup_{y^*\in F(y)} \langle y^*, x - y \rangle > 0$.

(E) There exists $y_0 \in K$ such that the set

$$L(y_0) := \left\{ x \in K : \inf_{x^*\in F(x)} \langle x^*, x - y_0 \rangle < 0 \right\} \tag{3.9}$$

is bounded, if nonempty.

Proposition 3.4. *The following statements hold.*

(i) (C)⇒(B) *if F is of convex values.*

(ii) (D)⇒(B) *if F is quasimonotone.*

(iii) (E)⇒(B)⇒(A).

Proof. (i) By (C), for every $x \in K \setminus K_r$, $\inf_{x^*\in F(x)} \sup_{y\in K_r} \langle x^*, x - y \rangle \geq 0$. Since $F(x)$ is convex and K_r is compact convex, the Kneser minimax theorem implies that

$$\sup_{y\in K_r} \inf_{x^*\in F(x)} \langle x^*, x - y \rangle = \inf_{x^*\in F(x)} \sup_{y\in K_r} \langle x^*, x - y \rangle \geq 0. \tag{3.10}$$

Since $y \mapsto \inf_{x^* \in F(x)} \langle x^*, x - y \rangle$ is upper semicontinuous and since K_r is compact, there is $y(x) \in K_r$ such that

$$\inf_{x^* \in F(x)} \langle x^*, x - y(x) \rangle = \sup_{y \in K_r} \inf_{x^* \in F(x)} \langle x^*, x - y \rangle \geq 0. \tag{3.11}$$

This verifies (B).

(ii) The implication of (D) $\Rightarrow$ (B) is an immediate consequence of F being quasimonotone.

(iii) (E) $\Rightarrow$ (B). If $L(y_0) = \emptyset$, then for any $x \in K$, $\inf_{x^* \in F(x)} \langle x^*, x - y_0 \rangle \geq 0$; thus (B) holds. If $L(y_0) \neq \emptyset$, then by (E), there is $r > 0$ such that $L(y_0) \cup \{y_0\} \subset K_r$. Thus (B) is still true.

(B) $\Rightarrow$ (A). Let $r > 0$ be such that (B) holds. Then $x \in K \setminus K_{r+1}$, $x \notin K_r$. By (B), there is $y \in K_r$ such that $\inf_{x^* \in F(x)} \langle x^*, x - y \rangle \geq 0$. Obviously, $\|y\| \leq r < r + 1 < \|x\|$. Thus (A) is verified with r replaced by $r + 1$. □

Remark 3.5. The coercivity condition (A) is actually (C′) in [3] where it is shown that if F is quasimonotone and upper hemicontinuous with nonempty compact convex values, then (A) implies that GVIP(F, K) has a solution. However, it seems unknown whether this assertion still holds if one replaces "quasimonotone and upper hemicontinuous" by "upper semicontinuous." An affirmative answer is given by Corollary 3.9.

Remark 3.6. The coercivity condition (B) is the condition (C) in [3]. The condition (D) appears in [21]. The condition (E) appears essentially in Corollary 3.1 in [22]; see also Proposition 2.2.3 in [8]. If F is single valued, then (E) reduces to Proposition 2.2.3(a) in [8].

Remark 3.7. Example 3.1 in [9] shows that (A) does not necessarily imply (B), even if F is single-valued and continuous.

From the above discussion, (A) is the weakest coercivity condition among them. [9] proved if F is single valued and continuous, then (A) implies that GVIP(F, K) has a solution. Corollary 3.9 shows that this assertion still holds even if F is a set-valued mapping.

Theorem 3.8. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, and let $F : K \to 2^{\mathbb{R}^n}$ be a mapping with nonempty compact convex values. Suppose that (A) holds. If F has the variational inequality property on K, then GVIP(F, K) has a solution.*

Proof. Let $m > r$. Since K_m is bounded closed convex and F has the variational inequality property, there is $x_m \in K_m$ such that

$$\sup_{x^* \in F(x_m)} \langle x^*, y - x_m \rangle \geq 0, \quad \forall y \in K_m. \tag{3.12}$$

(i) If $\|x_m\| = m$, then $\|x_m\| > r$, and by assumption, there is $y_0 \in K$ with $\|y_0\| < \|x_m\|$ such that

$$\sup_{x^* \in F(x_m)} \langle x^*, y_0 - x_m \rangle \leq 0. \tag{3.13}$$

Fix any $y \in K$. Since $\|y_0\| < \|x_m\| \leq m$, there is $t \in (0, 1)$ such that $z_t := y_0 + t(y - y_0) \in K_m$.

It follows that

$$0 \leq \sup_{x^* \in F(x_m)} \langle x^*, z_t - x_m \rangle \leq t \sup_{x^* \in F(x_m)} \langle x^*, y - x_m \rangle + (1-t) \sup_{x^* \in F(x_m)} \langle x^*, y_0 - x_m \rangle$$
$$\leq t \sup_{x^* \in F(x_m)} \langle x^*, y - x_m \rangle. \tag{3.14}$$

Therefore, $\sup_{x^* \in F(x_m)} \langle x^*, y - x_m \rangle \geq 0$. Since $y \in K$ is arbitrary, the conclusion is verified.

(ii) If $\|x_m\| < m$, then for any $y \in K$, there is $t \in (0,1)$ such that

$$z_t := x_m + t(y - x_m) \in K_m. \tag{3.15}$$

It follows that

$$0 \leq \sup_{x^* \in F(x_m)} \langle x^*, z_t - x_m \rangle = t \sup_{x^* \in F(x_m)} \langle x^*, y - x_m \rangle. \tag{3.16}$$

Since $y \in K$ is arbitrary, x_m solves GVIP(F, K). □

Corollary 3.9. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set, and let $F : K \to 2^{\mathbb{R}^n}$ be upper semicontinuous set-valued mapping with nonempty compact convex values. Suppose that (A) holds. Then GVIP(F, K) has a solution.*

Proof. By Proposition 3.2, F has the variational inequality property. The conclusion follows immediately from Theorem 3.8. □

Remark 3.10. Assuming that the mapping F is single valued and continuous, Proposition 2.2.3 in [8] and Theorem 3.2 in [9] show that SOL(F, K) is nonempty if the conditions (E) and (A) hold, respectively. Therefore, Corollary 3.9 improves Proposition 2.2.3 in [8] and Theorem 3.2 in [9]: the mapping F is set valued instead of single valued.

4. The Tikhonov Regularization

Theorem 4.1. *Let K be a nonempty closed convex set in $\mathbb{R}^n$, and let $F : K \to 2^{\mathbb{R}^n}$ be upper semicontinuous with nonempty compact convex values. If assumption (A) holds, then for any $\varepsilon > 0$,*

(i) *GVIP$(F + \varepsilon I, K)$ has a solution;*

(ii) *the set $\{SOL(F + tI, K) : t \in (0, \varepsilon]\}$ is bounded.*

Proof. (i) Let r be as in assumption (A). We claim that for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying

$$\inf_{x^* \in F(x)} \langle x^* + \varepsilon x, x - y \rangle \geq 0. \tag{4.1}$$

Granting this, we obtain that assumption (A) is satisfied with the mapping F replaced by $F + \varepsilon I$. Since F is upper semicontinuous with nonempty compact convex values and I is continuous and monotone, Theorem 3.8 and Proposition 3.3 imply that GVIP$(F+\varepsilon I, K)$ has a solution.

Now we prove the claim. Since for every $x \in K \setminus K_r$, there is $y \in K$ with $\|y\| < \|x\|$ satisfying $\inf_{x^* \in F(x)} \langle x^*, x-y \rangle \geq 0$, we obtain

$$\begin{aligned}
\inf_{x^* \in F(x)} \langle x^* + \varepsilon x, x - y \rangle &= \inf_{x^* \in F(x)} \langle x^*, x-y \rangle + \varepsilon \langle x, x-y \rangle \\
&= \inf_{x^* \in F(x)} \langle x^*, x-y \rangle + \varepsilon \|x\|^2 - \varepsilon \langle x, y \rangle \\
&\geq \varepsilon \|x\| (\|x\| - \|y\|) \geq 0.
\end{aligned} \tag{4.2}$$

(ii) Let $t \in (0, \varepsilon]$ and $x(t) \in \mathrm{SOL}(F+tI, K)$. Then $x(t) \in K_n$. If not, by assumption (A), there is $y(t) \in K$ with $\|y(t)\| < \|x(t)\|$ such that

$$\inf_{x^* \in F(x(t))} \langle x^*, x(t) - y(t) \rangle \geq 0. \tag{4.3}$$

Since $x(t) \in \mathrm{SOL}(F+tI, K)$ and $y(t) \in K$,

$$\begin{aligned}
0 &\geq \inf_{x^* \in F(x(t))} \langle x^* + t x(t), x(t) - y(t) \rangle \\
&= \inf_{x^* \in F(x(t))} \langle x^*, x(t) - y(t) \rangle + t \|x(t)\|^2 - t \langle x(t), y(t) \rangle \\
&\geq t \|x(t)\|^2 - t \|x(t)\| \|y(t)\|.
\end{aligned} \tag{4.4}$$

Therefore $\|x(t)\| \leq \|y(t)\|$, a contradiction. □

Remark 4.2. Theorem 4.1 improves in two ways: the mapping F is set valued instead of single valued; Theorem 1 in [12] used the coercivity condition (B) while our Theorem 4.1 uses the weaker coercivity condition (A).

Theorem 4.3. *Let K be a nonempty closed convex set in $\mathbb{R}^n$, and let $F : K \to 2^{\mathbb{R}^n}$ be upper semicontinuous with nonempty compact convex values. Assume that there is a nonempty bounded closed convex set $D \subset K$ such that*

$$\left\{ x \in K : \sup_{y \in D} \inf_{x^* \in F(x)} \langle x^*, x-y \rangle < 0 \right\} \text{ is bounded, if nonempty.} \tag{4.5}$$

Then for any $\varepsilon > 0$, GVIP$(F+\varepsilon I, K)$ has a solution and the set $\{SOL(F+tJ, K) : t \in (0, \varepsilon]\}$ is bounded.

Proof. Let $r > 0$ be such that

$$D \bigcup \left\{ x \in K : \sup_{y \in D} \inf_{x^* \in F(x)} \langle x^*, x-y \rangle < 0 \right\} \subset K_r. \tag{4.6}$$

Then for every $x \in K \setminus K_r$,

$$\sup_{y \in D} \inf_{x^* \in F(x)} \langle x^*, x - y \rangle \geq 0. \tag{4.7}$$

Since D is bounded closed convex and since $y \mapsto \inf_{x^* \in F(x)} \langle x^*, x - y \rangle$ is upper semicontinuous, there is $y_0 \in D$ such that

$$\inf_{x^* \in F(x)} \langle x^*, x - y_0 \rangle \geq 0. \tag{4.8}$$

Since $y_0 \in D \subset K_r$, $\|y_0\| < r \leq \|x\|$. Therefore, assumption (A) is satisfied. Then the conclusion follows from Theorem 4.1. □

Remark 4.4. If F is a single-valued mapping, and $D = \{x_0\}$ for some $x_0 \in K$, then Theorem 4.3 reduces to Theorem 2.1 in [11].

For every $\varepsilon > 0$, let $A_\varepsilon \subset \mathbb{R}^n$, and we define

$$\limsup_{\varepsilon \to 0+} A_\varepsilon := \{x \in \mathbb{R}^n : \exists \varepsilon_n \longrightarrow 0+ \text{ and } x_n \in A_{\varepsilon_n} \text{ such that } x_n \longrightarrow x\}. \tag{4.9}$$

Theorem 4.5. *Let K be a nonempty closed convex set in $\mathbb{R}^n$, and let $F : K \to 2^{\mathbb{R}^n}$ be upper semicontinuous with nonempty compact convex values. If assumption (A) holds, then*

$$\emptyset \neq \limsup_{\varepsilon \to 0+} SOL(F + \varepsilon I, K) \subset SOL(F, K). \tag{4.10}$$

Proof. The set $\limsup_{\varepsilon \to 0+} \mathrm{SOL}(F + \varepsilon I, K)$ being nonempty follows from Theorem 4.1. Let $x \in \limsup_{\varepsilon \to 0+} \mathrm{SOL}(F + \varepsilon I, K)$. Then there are a sequence $\varepsilon_n \to 0+$ and $x_n \in \mathrm{SOL}(F + \varepsilon_n I, K)$ such that $x_n \to x$. This means that for some $x_n^* \in F(x_n)$,

$$\langle x_n^* + \varepsilon_n x_n, y - x_n \rangle \geq 0, \quad \forall y \in K. \tag{4.11}$$

Since F is upper semicontinuous with nonempty compact values, $\{x_n^*\}$ is compact. Without loss of generality, assume $\lim_{n \to \infty} x_n^* = x^*$ for some $x^* \in F(x)$. Thus for every $y \in K$,

$$\begin{aligned} \langle x_n^* + \varepsilon_n x_n, y - x_n \rangle &= \langle x_n^*, y - x_n \rangle + \varepsilon_n \langle x_n, y \rangle - \varepsilon_n \|x_n\|^2 \\ &\longrightarrow \langle x^*, y - x \rangle \text{ as } n \longrightarrow \infty. \end{aligned} \tag{4.12}$$

It follows from (4.11) that $x \in \mathrm{SOL}(F, K)$. □

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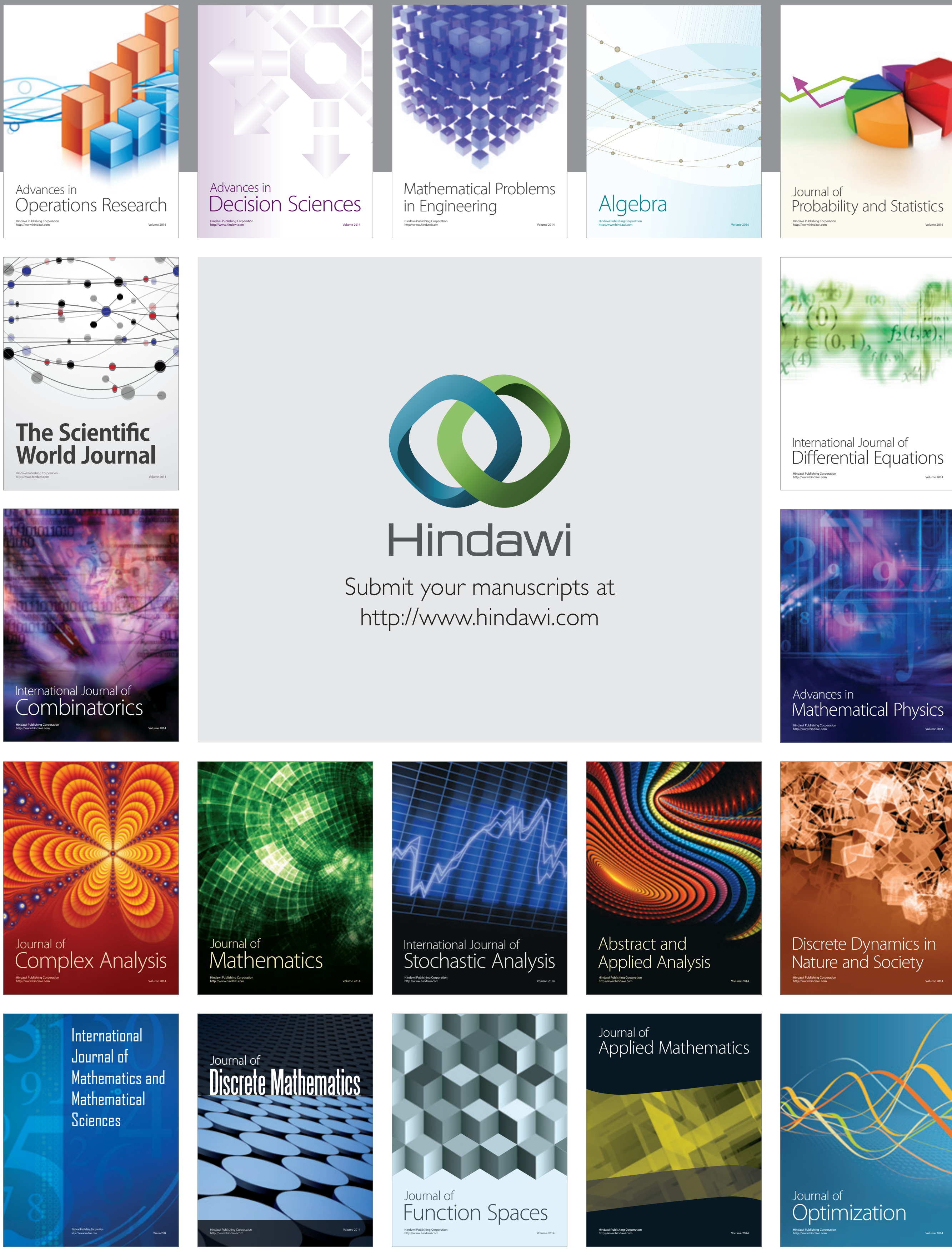

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