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Research Article

Optimal Error Estimate of Chebyshev-Legendre Spectral Method for the Generalised Benjamin-Bona-Mahony-Burgers Equations

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Combining with the Crank-Nicolson/leapfrog scheme in time discretization, Chebyshev-Legendre spectral method is applied to space discretization for numerically solving the Benjamin-Bona-Mahony-Burgers (gBBM-B) equations. The proposed approach is based on Legendre Galerkin formulation while the Chebyshev-Gauss-Lobatto (CGL) nodes are used in the computation. By using the proposed method, the computational complexity is reduced and both accuracy and efficiency are achieved. The stability and convergence are rigorously set up. Optimal error estimate of the Chebyshev-Legendre method is proved for the problem with Dirichlet boundary condition. The convergence rate shows "spectral accuracy." Numerical experiments are presented to demonstrate the effectiveness of the method and to confirm the theoretical results.

1. Introduction

We are interested in numerically solving initial boundary value problem of the generalised Benjamin-Bona-Mahony-Burgers (gBBM-B) equations in the following form:

$$u_t - \mu u_{xxt} - \alpha u_{xx} + \beta u_x + \left(g(u)\right)_x = 0, \tag{1.1}$$

where $u = u(x,t)$ represents the fluid velocity in horizontal direction x, parameters $\mu \in [0,1]$, $\alpha > 0$ and β are any given constants, and $g(u)$ is a nonlinear function with certain smoothness. Notation u_t denotes the first derivative of u with respect to temporal t and u_x, u_{xx} are the first and second derivatives with respect to space variable x.

When $g(u) = (1/2)u^2$ with $\alpha = 0$ and $\beta = 1$, (1.1) is the alternative regularized long-wave equation proposed by Peregrine [1] and Benjamin et al. [2]. Equation (1.1) features a balance between nonlinear and dispersive effects but takes no account of dissipation for the case $\alpha = 0$. In the physical sense, (1.1) with the dissipative term αu_{xx} is proposed if the good predictive power is desired, such problem arises in the phenomena for both the bore propagation and the water waves.

The well posedness and some asymptotic properties of solutions to this problem were discussed by Karch [3], Zhang [4], and so on. For more knowledge about BBM-B equations, please consult with [5, 6] and so forth.

In [7], Al-Khaled et al. implemented the Adomian decomposition method for obtaining explicit and numerical solutions of the BBM-B equation (1.1). By applying the classical Lie method of infinitesimals, Bruzón et al. [8–10] obtained, for a generalization of a family of BBM equations, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic functions. Tari and Ganji [11] have applied two methods known as "variational iteration" and "homotopy perturbation" methods to derive approximate explicit solution for (1.1) with $g(u) = (1/2)u^2$. El-Wakil et al. [12] and Kazeminia et al. [13] used the "exp-function" method with the aid of symbolic computational system to obtain the solitary solutions and periodic solutions for (1.1) with $g(u) = (1/2)u^2$. Variational iteration combining with the exp-function method to solve the generalized BBM equation with variable coefficients was conducted by Gómez and Salas [14]. In [15], Fakhari et al. solved the BBM-B equation by the homotopy analysis method to evaluate the nonlinear equation (1.1) with $g(u) = (1/2)u^2$, $\alpha = 0$, and $\beta = 1$. A tanh method and sinc-Galerkin procedure were used by Alqruan and Al-Khaled [16] to solve gBBM-B equations. Omrani and Ayadi [17] used Crank-Nicolson-type finite difference method for numerical solutions of the BBM-B equation in one space dimension. In [18], a local discontinuous Galerkin finite element method was used and an optimal error estimate was derived by the authors for numerical solution to BBM-B equation. To our knowledge, there is little work on spectral method for numerical solution of the gBBM-B equations.

For completeness, (1.1) may be complemented with certain initial and boundary conditions. Here in this paper we will deal with the initial boundary value problem

$$u_t - \mu u_{xxt} - \alpha u_{xx} + \beta u_x + \left(g(u)\right)_x = f, \quad x \in \Lambda := [-1, 1],\ t > 0, \tag{1.2}$$

with Dirichlet-type boundary conditions

$$u(1, t) = u(-1, t) = 0, \quad t \geq 0, \tag{1.3}$$

and initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Lambda, \tag{1.4}$$

The appearance of the right-hand side term in (1.2) is just for convenience in order to test the numerical efficiency of the proposed method. For other kinds of boundary conditions, a penalty method may be used as in [19–21].

Let $L^2(\Lambda)$ be a square integrable function space with inner product and norm as follows:

$$(u,v) = \int_{\Lambda} u(x)v(x)dx, \qquad \|u\| = \sqrt{(u,u)}. \tag{1.5}$$

For any positive integer r, $H^r(\Lambda)$ denotes the Sobolev-type space defined as

$$H^r(\Lambda) = \left\{ u \in L^2(\Lambda) : \frac{\partial^k u}{\partial x^k} \in L^2(\Lambda),\ 0 \le k \le r \right\}. \tag{1.6}$$

We denote the norm and seminorm of $H^r(\Lambda)$ by $\|\cdot\|_r$ and $|\cdot|_r$, respectively. For any real r, $H^r(\Lambda)$ is defined by space interpolation [22]. $H_0^1(\Lambda)$ is a subspace of $H^1(\Lambda)$ in which the function vanishes at ± 1.

We reformulate the problem (1.2)–(1.4) in weak form: find $u(t) \in H_0^1(\Lambda)$, such that, for all $\phi \in H_0^1(\Lambda)$,

$$\begin{gathered} (u_t,\phi) + \mu(u_{xt},\phi_x) + \alpha(u_x,\phi_x) = \beta(u,\phi_x) + (g(u),\phi_x) + (f,\phi), \quad t \in (0,T], \\ (u,\phi) = (u_0,\phi), \quad t = 0. \end{gathered} \tag{1.7}$$

Due to the high accuracy, spectral/pseudospectral methods are increasingly popular during the last two decades [23–27]. In the context of spectral methods, Legendre and Chebyshev orthogonal polynomials are extensively used for the nonperiodic cases. The Legendre method is natural in the theoretical analysis due to the unity weight function. However, it is well known that the applications of Legendre method are limited by the lack of fast transform between the physical space and the spectral space and by the lack of explicit evaluation formulation of the Legendre-Gauss-Lobatto (LGL) nodes. A Chebyshev-Legendre method that implements the Legendre method at Chebyshev nodes was proposed by Don and Gottlieb [19] for the parabolic and hyperbolic equations. The approach enjoys advantages of both the Legendre and Chebyshev methods. The Chebyshev-Legendre methods were also applied to elliptic problems [28], nonlinear conservative laws in [29, 30], Burgers-like equations in [21], KdV equations in [31], generalised Burgers-Fisher equation [32], nonclassical parabolic equation [33], and optimal control problems [34]. In this paper, Chebyshev-Legendre spectral method will be applied to solving the initial boundary value problem (1.2)–(1.4). The proposed approach is based on Legendre-Galerkin formulation while the Chebyshev-Gauss-Lobatto (CGL) nodes are used in the computation. The computational complexity is reduced and both accuracy and efficiency are achieved by using the proposed method. Compared with the generalised Burgers equation, the gBBM-B equation has an additional term $-\mu u_{xxt}$, which serves as a stabilizer. Due to this reason, numerical analysis of the Chebyshev-Legendre schemes for gBBM-B equations is actually much easier than that for generalised Burgers equations in [20, 21].

This paper is organized as follows. In Section 2, we set up Chebyshev-Legendre spectral method for gBBM-B equations and show how to implement the scheme. Section 3 is preliminary that involves some lemmas which will be used later. In Section 4, error analysis is performed for both semidiscretization and fully discretization schemes. We obtain an optimal convergence rate in sense of H^1-norm. In Section 5, numerical experiments are presented to support the theoretical results. Conclusion is given in Section 6.

2. Chebyshev-Legendre Spectral Method

In this section, we first set up semidiscretization and fully discretization Chebyshev-Legendre spectral method for problem (1.2) and then give how to implement the proposed scheme.

2.1. Chebyshev-Legendre Spectral Method

Let $\mathbb{P}_N$ be the set of all algebraic polynomials of degree at most N. We introduce the operator of interpolation at the CGL nodes $\{\eta_i = \cos(i\pi/N)\}_{0\le i\le N}$, denoted by Π_N^C, which satisfies $\Pi_N^C f \in \mathbb{P}_N$ and

$$\Pi_N^C f(\eta_i) = f(\eta_i), \quad 0 \le i \le N. \tag{2.1}$$

Denote

$$V_N = \mathbb{P}_N(\Lambda) \cap H_0^1(\Lambda). \tag{2.2}$$

The weak form (1.7) leads to the following semidiscretization Legendre-Galerkin scheme: find $u_N(t) \in V_N$, such that, for all $\phi_N \in V_N$,

$$\begin{aligned}
&(u_{Nt}, \phi_N) + \mu(u_{Nxt}, \phi_{Nx}) + \alpha(u_{Nx}, \phi_{Nx}) \\
&\quad = \beta(u_N, \phi_{Nx}) + (g(u_N), \phi_{Nx}) + (f, \phi_N), \quad t \in (0, T], \\
&(u_N, \phi_N) = (u_0, \phi_N), \quad t = 0.
\end{aligned} \tag{2.3}$$

Now we give a Chebyshev-Legendre spectral scheme for the problem (1.2).

The semidiscretization Chebyshev-Legendre spectral method for (1.2) is to find $u_N(t) \in V_N$, such that, for all $\phi_N \in V_N$,

$$\begin{aligned}
&(u_{Nt}, \phi_N) + \mu(u_{Nxt}, \phi_{Nx}) + \alpha(u_{Nx}, \phi_{Nx}) \\
&\quad = \beta(u_N, \phi_{Nx}) + \left(\Pi_N^C g(u_N), \phi_{Nx}\right) + \left(\Pi_N^C f, \phi_N\right), \quad t \in (0, T], \\
&(u_N, \phi_N) = \left(\Pi_N^C u_0, \phi_N\right), \quad t = 0.
\end{aligned} \tag{2.4}$$

Remark 2.1. The difference between Legendre method (2.3) and Chebyshev-Legendre method (2.4) lies in the treatment of three terms: nonlinear term, source term and initial data. But such treatment leads to two advantages: one is improvement of convergence rate in theoretical analysis, the other is free from computing the LGL nodes. We only need Chebyshev transform and Chebyshev-Legendre transform in computing, and the former can perform through FFT.

For the time advance, we adopt the second-order Crank-Nicolson/leapfrog. For a given time step τ, let $S_t = \{k\tau : k = 1, 2, \ldots, M_t, t = M_t\tau\}$; the notations $v_{\hat{t}}(t)$ and $\hat{v}(t)$ are used as

$$v_{\hat{t}}(t) = \frac{v(t+\tau) - v(t-\tau)}{2\tau}, \qquad \hat{v}(t) = \frac{v(t+\tau) + v(t-\tau)}{2}. \tag{2.5}$$

Let $t_k = k\tau$ and $u^k = u(t_k)$. The fully discretization Chebyshev-Legendre spectral method for (1.2) is to find $u_N^k (k = 0, 1, \ldots, M_T = [T/\tau]) \in V_N$, such that, for all $\phi_N \in V_N$,

$$\begin{gathered}
\left(u^k_{N\hat{t}}, \phi_N\right) + \mu\left(u^k_{Nx\hat{t}}, \phi_{Nx}\right) + \alpha\left(\hat{u}^k_{Nx}, \phi_{Nx}\right) \\
= \beta\left(\hat{u}^k_N, \phi_{Nx}\right) + \left(\Pi^C_N g\left(u^k_N\right), \phi_{Nx}\right) + \left(\Pi^C_N \hat{f}^k, \phi_N\right), \quad k = 1, \ldots, M_T - 1, \\
\left(\frac{u^1_N - u^0_N}{\tau}, \phi_N\right) + \mu\left(\frac{u^1_{Nx} - u^0_{Nx}}{\tau}, \phi_{Nx}\right) + \alpha\left(\frac{u^1_{Nx} + u^0_{Nx}}{2}, \phi_{Nx}\right) \\
= \beta\left(\frac{u^1_N + u^0_N}{2}, \phi_{Nx}\right) + \left(\Pi^C_N g\left(u^0_N\right), \phi_{Nx}\right) + \left(\Pi^C_N \frac{f^1 + f^0}{2}, \phi_N\right), \\
u^0_N = \Pi^C_N u_0.
\end{gathered} \tag{2.6}$$

2.2. Implementation of the Chebyshev-Legendre Spectral Method

Let L_k be the kth degree Legendre polynomial that is mutually orthogonal in $L^2(\Lambda)$, that is,

$$\left(L_k, L_j\right) = \int_\Lambda L_k(x) L_j(x) dx = \frac{2}{2k+1}\delta_{kj}. \tag{2.7}$$

We define (in [35])

$$\Phi_k(x) = L_k(x) - L_{k+2}(x). \tag{2.8}$$

One useful property of the Legendre polynomials is

$$(2n+1)L_n(x) = L'_{n+1}(x) - L'_{n-1}(x), \tag{2.9}$$

which gives the following relation:

$$\Phi'_k(x) = -(2k+3)L_{k+1}(x). \tag{2.10}$$

It is easy to verify that, for $N \geq 2$,

$$V_N = \operatorname{span}\{\Phi_0, \Phi_1, \ldots, \Phi_{N-2}\}. \tag{2.11}$$

Let us denote

$$
\begin{aligned}
u_N^i &= \sum_{k=0}^{N-2} \widehat{u}_k^i \Phi_k, \qquad \widehat{\mathbf{u}}^i = \left(\widehat{u}_0^i, \widehat{u}_1^i, \ldots, \widehat{u}_{N-2}^i\right)^T, \\
\widehat{g}_k^i &= \left(\Pi_N^C g\left(u_N^i\right), \Phi_k'\right), \qquad \widehat{\mathbf{g}}^i = \left(\widehat{g}_0^i, \widehat{g}_1^i, \ldots, \widehat{g}_{N-2}^i\right)^T, \\
\widehat{f}_k^i &= \left(\Pi_N^C f^i, \Phi_k\right), \qquad \widehat{\mathbf{f}}^i = \left(\widehat{f}_0^i, \widehat{f}_1^i, \ldots, \widehat{f}_{N-2}^i\right)^T, \\
\widehat{u}_{0,k} &= \left(\Pi_N^C u_0, \Phi_k\right), \qquad \widehat{\mathbf{u}}_0 = \left(\widehat{u}_{0,0}, \widehat{u}_{0,1}, \ldots, \widehat{u}_{0,N-2}\right)^T, \\
m_{ij} &= \left(\Phi_j, \Phi_i\right), \qquad \mathbf{M} = \left(m_{ij}\right)_{i,j=0,1,\ldots,N-2}, \\
s_{ij} &= \left(\Phi_j, \Phi_i'\right), \qquad \mathbf{S} = \left(s_{ij}\right)_{i,j=0,1,\ldots,N-2}, \\
p_{ij} &= \left(\Phi_j', \Phi_i'\right), \qquad \mathbf{P} = \left(p_{ij}\right)_{i,j=0,1,\ldots,N-2}.
\end{aligned}
\tag{2.12}
$$

Then, scheme (2.6) leads to the following linear system series: For $k = 1, 2, \ldots, M_T$

$$
\left[\mathbf{M} + (\mu + \alpha\tau)\mathbf{P} - \beta\tau\mathbf{S}\right]\widehat{\mathbf{u}}^{k+1} = \left[\mathbf{M} + (\mu - \alpha\tau)\mathbf{P} + \beta\tau\mathbf{S}\right]\widehat{\mathbf{u}}^{k-1} + 2\tau\widehat{\mathbf{g}}^k + \tau\left(\widehat{\mathbf{f}}^{k+1} + \widehat{\mathbf{f}}^{k-1}\right), \tag{2.13}
$$

starting with

$$
\begin{gathered}
\left[\mathbf{M} + \left(\mu + \frac{\alpha\tau}{2}\right)\mathbf{P} - \frac{\beta\tau}{2}\mathbf{S}\right]\widehat{\mathbf{u}}^1 = \left[\mathbf{M} + \left(\mu - \frac{\alpha\tau}{2}\right)\mathbf{P} + \frac{\beta\tau}{2}\mathbf{S}\right]\widehat{\mathbf{u}}^0 + \tau\widehat{\mathbf{g}}^0 + \frac{\tau}{2}\left(\widehat{\mathbf{f}}^1 + \widehat{\mathbf{f}}^0\right), \\
\mathbf{M}\widehat{\mathbf{u}}^0 = \widehat{\mathbf{u}}_0.
\end{gathered}
\tag{2.14}
$$

By using integration by parts and orthogonality of Legendre polynomials, one can easily determine that the matrix **M** is pentadiagonal and **P** diagonal. Moreover, the nonzero entries of the matrices **M**, **P** can be determined by using the orthogonal relation (2.7) and (2.10) as follows:

$$
m_{jk} = m_{kj} = \begin{cases} \dfrac{2}{2k+1} + \dfrac{2}{2k+5}, & j = k, \\ -\dfrac{2}{2k+5}, & j = k \pm 2, \\ 0, & \text{otherwise}, \end{cases}
$$

$$s_{jk} = \begin{cases} -2, & j = k-1, \\ 2, & j = k+1, \\ 0, & \text{otherwise}, \end{cases}$$
$$p_{jk} = p_{kj} = \begin{cases} 4k+6, & j = k; \\ 0, & \text{otherwise}. \end{cases} \tag{2.15}$$

Hence, the matrices in the left-hand side of (2.13) are banded and the linear system (2.13) can be easily solved.

For evaluation of the last three terms on the right-side hand in (2.13), it can be split into two steps: one step is the transform from its values at CGL nodes to the coefficients of its Chebyshev expansion, which can be done by using fast Chebyshev transform or fast Fourier transform. The other step involves the fast Legendre transform between the coefficients of the Chebyshev expansion and of the Legendre expansion, which has been addressed by Alpert and Rokhlin [36]. It is worthy to note that Shen suggested an algorithm [28] that seems more attractive for moderate N.

3. Preliminaries

In this section, we introduce a suitable comparison function and give some lemmas needed in error analysis. We denote by C a generic positive constant independent of N or any function. Now we introduce two projection operators and their approximation properties. One is $P_N^L : L^2(\Lambda) \to \mathbb{P}_N$ such that

$$\left(P_N^L v, \phi\right) = (v, \phi), \quad \forall \phi \in \mathbb{P}_N. \tag{3.1}$$

The approximation property (Theorem 6.1 and (6.8) in [24], page 258, 261) is the following.

Lemma 3.1. *For any $u \in H^r(\Lambda)(r > 1)$, there exists a positive constant C independent of N such that*

$$\left\| u - P_N^L u \right\| \le C N^{-r} \|u\|_r, \tag{3.2}$$

$$\left\| u - P_N^L u \right\|_s \le \begin{cases} C N^{3s/2-r} \|u\|_r, & s \le 1, \\ C N^{2s-1/2-r} \|u\|_r, & s \ge 1. \end{cases} \tag{3.3}$$

The other projection operator is defined as follows:

$$P_N u(x) = \int_{-1}^{x} P_{N-1}^L u_y(y)\, dy. \tag{3.4}$$

It is easy to know that $P_N : H_0^1(\Lambda) \to V_N$ such that

$$((P_N v)_x, \phi_x) = (v_x, \phi_x), \quad \forall \phi \in V_N. \tag{3.5}$$

Its approximation property (Theorem 6.2 in [24], page 262) is the following.

Lemma 3.2. *For any $u \in H^r(\Lambda) \cap H_0^1(\Lambda)$, there exists a positive constant C independent of N such that*

$$\|u - P_N u\|_s \le C N^{s-r} \|u\|_r, \quad 0 \le s \le 1 \le r. \tag{3.6}$$

For the interpolation operator Π_N^C at the Chebyshev-Gauss-Lobatto points, we cite the following approximation result [29].

Lemma 3.3. *If $u \in H^1(\Lambda)$, then*

$$N \left\| \Pi_N^C u - u \right\| + \left| \Pi_N^C u \right|_1 \le C \|u\|_1, \tag{3.7}$$

moreover, if $u \in H^s(\Lambda)(s \ge 1)$, then

$$\left\| \Pi_N^C u - u \right\|_l \le C N^{l-s} \|u\|_s, \quad 0 \le l \le 1. \tag{3.8}$$

In general, the discretization inner product and norm are defined as follows:

$$(u, v)_N = \sum_{j=0}^{N} u(y_j) v(y_j) \omega_j, \qquad \|u\|_N = \sqrt{(u, u)_N}, \tag{3.9}$$

where y_j and ω_j $(j = 0, 1, \ldots, N)$ are the Legendre-Gauss-Lobatto points and the corresponding quadrature weights. Associating with this quadrature rule, we denote by Π_N^L the Legendre interpolation operator. The following result is cited from [26].

Lemma 3.4. *If $u \in H^\sigma(\Lambda)(\sigma \ge 1)$, then*

$$\begin{gathered} \left\| \Pi_N^L u - u \right\|_l \le C N^{l-\sigma} \|u\|_\sigma, \quad 0 \le l \le 1, \\ |(u, v) - (u, v)_N| \le C N^{-\sigma} \|u\|_\sigma \|v\|, \quad \forall v \in P_N. \end{gathered} \tag{3.10}$$

Further, if $u \in \mathbb{P}_N$, then

$$\|u\| \le \|u\|_N \le \sqrt{2 + \frac{1}{N}} \|u\|, \tag{3.11}$$

$$\|u\|_{L^\infty(\Lambda)} \le \frac{N+1}{\sqrt{2}} \|u\|. \tag{3.12}$$

4. Error Analysis of the Schemes

In this section, we set up the stability and convergence results first for the semidiscretization Chebyshev-Legendre spectral method (2.4) and then for the fully discretization scheme (2.6).

4.1. Stability and Convergence of Scheme (2.4)

Since the initial value and the right-hand side term cannot be exactly evaluated, we consider here how stable the numerical solution of (2.4) depending on the initial value and the right-hand side term. As in [21], we suppose that u_N and f are computed with errors $\widetilde{u}$ and $\widetilde{f}$ respectively. That is,

$$(\widetilde{u}_t, \phi_N) + \mu(\widetilde{u}_{xt}, \phi_{Nx}) + \alpha(\widetilde{u}_x, \phi_{Nx}) = \beta(\widetilde{u}, \phi_{Nx}) + \left(\Pi_N^C G, \phi_{Nx}\right) + \left(\Pi_N^C \widetilde{f}, \phi_N\right), \quad \forall \phi_N \in V_N, \tag{4.1}$$

where $G := [g(u_N + \widetilde{u}) - g(u_N)]$. Since $\widetilde{u} \in V_N$ leads to $(\widetilde{u}, \widetilde{u}_x) = 0$, we come out to, by taking $\phi_N = \widetilde{u}$ in (4.1),

$$\frac{1}{2}\frac{d}{dt}\left[\|\widetilde{u}\|^2 + \mu|\widetilde{u}|_1^2\right] + \alpha|\widetilde{u}|_1^2 = \left(\Pi_N^C G, \widetilde{u}_x\right) + \left(\Pi_N^C \widetilde{f}, \widetilde{u}\right). \tag{4.2}$$

Now we need to estimate the nonlinear term $\Pi_N^C G$ in the right-hand side (4.2). Let C_0 be a positive constant and

$$\begin{aligned} u_M &= \max_{0\le s\le T}\left\{\|u(s)\|_{L^\infty(\Lambda)} + \|u_x(s)\|_{L^\infty(\Lambda)}\right\}, \\ C_g(z_1, z_2) &= \max_{|z|\le|z_1|+|z_2|}\left|g'(z)\right| + (|z_1| + |z_2|)\max_{|z|\le|z_1|+|z_2|}\left|g''(z)\right|. \end{aligned} \tag{4.3}$$

For any given $t \in (0, T]$, if

$$\|\widetilde{u}(s)\|_{L^\infty(\Lambda)} \le C_0, \quad \forall s \in (0, t), \tag{4.4}$$

then

$$\begin{aligned} |G| &= \left|g'(u_N + \theta\widetilde{u})\widetilde{u}\right| \le C_g(u_M, C_0)|\widetilde{u}|, \\ \left|G'\right| &\le \left|g''(u_N + \theta\widetilde{u})u_{Nx}\widetilde{u} + g'(u_N + \widetilde{u})\widetilde{u}_x\right| \\ &\le C_g(u_M, C_0)(|\widetilde{u}_x(s)| + |\widetilde{u}(s)|). \end{aligned} \tag{4.5}$$

As an immediate consequence of the above estimates and inequality (3.8), it is true that, for large N,

$$\begin{aligned} \left\|\Pi_N^C G\right\| &\leq \left\|\Pi_N^C G - G\right\| + \|G\| \leq CN^{-1}|G|_1 + \|G\| \\ &\leq CN^{-1}C_g(u_M, C_0)(|\widetilde{u}|_1 + \|\widetilde{u}\|) + C_g(u_M, C_0)\|\widetilde{u}\| \\ &\leq CC_g(u_M, C_0)\left(N^{-1} + 1\right)\|\widetilde{u}\| + \frac{CC_g(u_M, C_0)}{N}|\widetilde{u}|_1. \end{aligned} \tag{4.6}$$

Hence, it is true that, for large N,

$$\left|\Pi_N^C G, \widetilde{u}_x\right| \leq C^* \|\widetilde{u}\|^2 + \frac{\alpha}{2}|\widetilde{u}|_1^2. \tag{4.7}$$

Therefore, integrating (4.2) in time leads to

$$\begin{aligned} \|\widetilde{u}(s)\|^2 + \mu|\widetilde{u}(s)|_1^2 + \alpha\int_0^s |\widetilde{u}(t)|_1^2 dt &\leq \|\widetilde{u}(0)\|^2 + \mu|\widetilde{u}(0)|_1^2 + C\int_0^s \|\widetilde{u}(t)\|^2 dt \\ &\quad + 2\int_0^s \left\|\Pi_N^C \widetilde{f}(t)\right\|^2 dt. \end{aligned} \tag{4.8}$$

Denoting

$$\begin{aligned} E(\widetilde{u}, s) &= \|\widetilde{u}(s)\|^2 + \mu|\widetilde{u}(s)|_1^2 + \alpha\int_0^s |\widetilde{u}(t)|_1^2 dt, \\ \rho(\widetilde{u}, s) &= \|\widetilde{u}(0)\|^2 + \mu|\widetilde{u}(0)|_1^2 + 2\int_0^s \left\|\Pi_N^C \widetilde{f}(t)\right\|^2 dt, \end{aligned} \tag{4.9}$$

then, we have

$$E(\widetilde{u}, t) \leq \rho(\widetilde{u}, t) + C\int_0^t E(\widetilde{u}(s), s) ds. \tag{4.10}$$

We have the following stability result.

Theorem 4.1. *If $g \in C^2(\mathbb{R})$ and*

$$\rho(\widetilde{u}, T) \leq 2C_0^2 \frac{e^{-CT}}{(N+1)^2}, \tag{4.11}$$

then there holds the following inequality:

$$E(\widetilde{u}, t) \leq \rho(\widetilde{u}, t)e^{Ct}, \quad \forall t \in (0, T]. \tag{4.12}$$

Proof. We need to verify that

$$\|\widetilde{u}(s)\|_{L^\infty(\Lambda)} \leq C_0, \quad \forall s \in (0,t). \tag{4.13}$$

Otherwise, there must exist $t_1 < T$ such that

$$\max_{0\leq s\leq t_1}\|\widetilde{u}(s)\|_{L^\infty(\Lambda)} \leq C_0, \qquad \|\widetilde{u}(t_1)\|_{L^\infty(\Lambda)} = C_0, \tag{4.14}$$

while by (4.10), (4.11), and the Gronwall inequality, we have

$$E(\widetilde{u},t_1) \leq \rho\left(\widetilde{u},\widetilde{f},t_1\right)e^{Ct_1} < \rho\left(\widetilde{u},\widetilde{f},T\right)e^{CT} \leq 2C_0^2(N+1)^{-2}. \tag{4.15}$$

Thus, from Lemma 3.4,

$$\|\widetilde{u}(t_1)\|_{L^\infty(\Lambda)} \leq \frac{N+1}{\sqrt{2}}\|\widetilde{u}(t_1)\| \leq \frac{N+1}{\sqrt{2}}\sqrt{E(\widetilde{u},t_1)} < C_0, \tag{4.16}$$

which is contradictory with (4.14). Thus, (4.35) holds, and we derive (4.12) from the Gronwall inequality. □

Remark 4.2. Condition (4.11) in Theorem 4.1 means that errors occurrs during evaluation of the initial value and the right-hand side term should not be larger than $2C_0^2(e^{-CT}/(N+1)^2)$, which becomes small when N goes large. This condition may be caused by the nonlinearity of the problem. Also, the condition could be satisfied because the orthogonal polynomial approximation goes faster than $1/(N+1)^2$ in general. As for the result (4.12) in Theorem 4.1, it is analogous to those in [29–33] just for different energy norm $E(u,t)$.

Next we turn to convergence of the semidiscretization scheme (2.4). Let $u(t)$ and $u_N(t)$ be the solutions to (1.7) and (2.4), respectively. We denote $u^* = P_N u,\ e(t) = u_N(t) - u^*(t)$. Then we have the following result.

Theorem 4.3. *If $u \in C^1(0,T;H^r(\Lambda)\cap H_0^1(\Lambda))$, $f \in C(0,T;H^r(\Lambda)\cap H_0^1(\Lambda))(r>1)$, and $g(z) \in C^1(\mathbb{R})$ which satisfies the assumption of Theorem 4.1, then the following error estimate*

$$\|e(t)\|^2 + \mu|e(t)|_1^2 + \alpha\int_0^t |e(s)|_1^2 ds \leq C\left(1+\mu N^2\right)N^{-2r}, \quad 0 \leq t \leq T, \tag{4.17}$$

holds.

Proof. From (1.7), (2.4), and (3.5), we know that

$$\begin{aligned}\left[(e_t,\phi) + \mu(e_{xt},\phi_x) + \alpha(e_x,\phi_x)\right] - \beta(e,\phi_x) &= \left[\left(\Pi_N^C g(u_N) - g(u),\phi_x\right)\right] + \left[((u-u^*)_t,\phi)\right] \\ &\quad + \beta\left[(u-u^*,\phi_x)\right] + \left[\left(f - \Pi_N^C f,\phi\right)\right].\end{aligned} \tag{4.18}$$

Taking $\phi = e$ in (4.18), we have

$$\frac{1}{2}\frac{d}{dt}\left[\|e\|^2 + \mu|e|_1^2\right] + \alpha|e|_1^2 \le I_1 + I_2 + I_3 + I_4, \tag{4.19}$$

$$\begin{aligned}
I_1 &:= \left|\left(\Pi_N^C g(u_N) - g(u), e_x\right)\right| \le \left\|\Pi_N^C g(u_N) - g(u)\right\| |e|_1 \le \frac{1}{\alpha}\left\|\Pi_N^C g(u_N) - g(u)\right\|^2 + \frac{\alpha}{4}|e|_1^2, \\
I_2 &:= |((u-u^*)_t, e)| \le \|u_t - P_N u_t\|\|e\| \le CN^{-2r}\|u_t\|_r^2 + \|e\|^2, \\
I_3 &:= \beta|((u-u^*), e_x)| \le \|u - P_N u\||e|_1 \le \frac{1}{\alpha}CN^{-2r}\|u\|_r^2 + \frac{\alpha}{4}|e|_1^2, \\
I_4 &:= \left|\left(f - \Pi_N^C f, e\right)\right| \le \left\|f - \Pi_N^C f\right\|^2 + \|e\|^2 \le CN^{-2r}\|f\|_r^2 + \|e\|^2.
\end{aligned} \tag{4.20}$$

Next, we estimate $\|\Pi_N^C g(u_N) - g(u)\|$,

$$\begin{aligned}
\left\|\Pi_N^C g(u_N) - g(u)\right\| &\le \left\|\Pi_N^C g(u_N) - g(u_N)\right\| + \|g(u_N) - g(u^*)\| + \|g(u^*) - g(u)\| \\
&\le CN^{-r}\|g(u_N)\|_r + \|g'(u_N + \theta u^*)\|\|e\| + \|g'(u^* + \theta u)\|\|u^* - u\| \\
&\le CN^{-r}\|u\|_r + C'\|e\|.
\end{aligned} \tag{4.21}$$

Combining with the estimates above, inequality (4.19) becomes

$$\frac{d}{dt}\left[\|e\|^2 + \mu|e|_1^2\right] + \alpha|e|_1^2 \le CN^{-2r}\left(\|u\|_r^2 + \|u_t\|_r^2 + \|f\|_r^2\right) + C'\|e\|^2. \tag{4.22}$$

Denoting

$$\begin{aligned}
E(t) &= \|e(t)\|^2 + \mu|e(t)|_1^2 + \alpha\int_0^t |e(s)|_1^2 ds, \\
\rho(t) &= \|e(0)\|^2 + \mu|e(0)|_1^2 + CN^{-2r}\left(\int_0^t \|u(s)\|_r^2 + \|u_t(s)\|_r^2 + \|f(t)\|_r^2 ds\right),
\end{aligned} \tag{4.23}$$

we have

$$E(t) \le \rho(t) + C\int_0^t E(s)ds. \tag{4.24}$$

Thanks to the Gronwall inequality, we get

$$E(t) \le \rho(t)e^{Ct}, \quad 0 < t \le T. \tag{4.25}$$

Because of $e(0) = P_N u_0 - \Pi_N^C u_0 = (u_0 - P_N u_0) - (u_0 - \Pi_N^C u_0)$, we combine with the approximation properties (3.6)–(3.3) to obtain

$$\|e(0)\| \le \|u_0 - P_N u_0\| + \left\|u_0 - \Pi_N^C u_0\right\| \le CN^{-r}\|u\|_r,$$
$$|e(0)|_1 \le |u_0 - P_N u_0|_1 + \left|u_0 - \Pi_N^C u_0\right|_1 \le CN^{1-r}\|u\|_r. \tag{4.26}$$

Then it is easy to show the desired result. □

Remark 4.4. The result in Theorem 4.3 is optimal in sense of H^1 *estimation.*

4.2. Stability and Convergence of the Fully Discretization Scheme

In this section we consider the stability and convergence of the fully discretization scheme of (2.6). We assume that all functions below are valued at time s unless otherwise specified. Suppose that u_N and f in (2.6) have the errors $\widetilde{u}$ and $\widetilde{f}$, respectively. Then, we have

$$(\widetilde{u}_{\hat{t}}, \phi) + \mu(\widetilde{u}_{x\hat{t}}, \phi_x) + \alpha\left(\widehat{\widetilde{u}}_x, \phi_x\right) = \beta\left(\widehat{\widetilde{u}}, \phi_x\right) + \left(\Pi_N^C \widetilde{g}, \phi_x\right) + \left(\Pi_N^C \widehat{\widetilde{f}}, \phi\right), \tag{4.27}$$

where $\widetilde{g} = g(u_N^k + \widetilde{u}) - g(u_N^k)$. Taking $\phi = \widehat{\widetilde{u}}$ in (4.27), we get

$$\frac{1}{2}\left[\|\widetilde{u}\|_{\hat{t}}^2 + \mu\left(|\widetilde{u}|_1^2\right)_{\hat{t}}\right] + \alpha\left|\widehat{\widetilde{u}}\right|_1^2 = \left(\Pi_N^C \widetilde{g}, \widehat{\widetilde{u}}_x\right) + \left(\Pi_N^C \widehat{\widetilde{f}}, \widehat{\widetilde{u}}\right). \tag{4.28}$$

Now we need to estimate $\|\Pi_N^C \widetilde{g}\|$, which would be easy to do, provided that Π_N^C is replaced by Legendre interpolation operator or the norm is in the weighted Chebyshev one. But here we cannot deal with this directly. Just like in [21], by taking $\phi = \widetilde{u}_{\hat{t}}$ in (4.27), we again get

$$\left[\left\|\widetilde{u}_{\hat{t}}\right\|^2 + \mu|\widetilde{u}_{\hat{t}}|_1^2\right] + \frac{1}{2}\alpha\left(|\widetilde{u}|_1^2\right)_{\hat{t}} = \beta\left(\widehat{\widetilde{u}}, \widetilde{u}_{x\hat{t}}\right) + \left(\Pi_N^C \widetilde{g}, \widetilde{u}_{x\hat{t}}\right) + \left(\Pi_N^C \widehat{\widetilde{f}}, \widetilde{u}_{\hat{t}}\right). \tag{4.29}$$

Combining (4.28) and (4.29) through the factor N^{-2}, we arrive at

$$\begin{aligned}
&\left(\|\widetilde{u}\|^2 + \mu|\widetilde{u}|_1^2 + N^{-2}\alpha|\widetilde{u}|_1^2\right)_{\hat{t}} + 2\left[N^{-2}\left\|\widetilde{u}_{\hat{t}}\right\|^2 + N^{-2}\mu|\widetilde{u}_{\hat{t}}|_1^2 + \alpha\left|\widehat{\widetilde{u}}\right|_1^2\right] \\
&\quad = 2\left(\Pi_N^C \widetilde{g}, \widehat{\widetilde{u}}_x\right) + 2\left(\Pi_N^C \widehat{\widetilde{f}}, \widehat{\widetilde{u}}\right) + N^{-2}2\beta\left(\widehat{\widetilde{u}}, \widetilde{u}_{x\hat{t}}\right) \\
&\qquad + N^{-2}2\left(\Pi_N^C \widetilde{g}, \widetilde{u}_{x\hat{t}}\right) + N^{-2}2\left(\Pi_N^C \widehat{\widetilde{f}}, \widetilde{u}_{\hat{t}}\right).
\end{aligned} \tag{4.30}$$

By using the Hölder inequality, we have

$$\begin{aligned}
\left|2\left(\Pi_N^C \tilde{g}, \hat{\tilde{u}}_x\right)\right| &\le 2\left\|\Pi_N^C \tilde{g}\right\|\left|\hat{\tilde{u}}\right|_1 \le \frac{3}{\alpha}\left\|\Pi_N^C \tilde{g}\right\|^2 + \frac{\alpha}{3}\left|\hat{\tilde{u}}\right|_1^2, \\
\left|2\left(\Pi_N^C \hat{\tilde{f}}, \hat{\tilde{u}}\right)\right| &\le 2\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}\left|\hat{\tilde{u}}\right|_1 \le \frac{3}{\alpha}\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2 + \frac{\alpha}{3}\left|\hat{\tilde{u}}\right|_1^2, \\
\left|N^{-2} 2\beta\left(\hat{\tilde{u}}, \tilde{u}_{x\hat{t}}\right)\right| &\le N^{-2} 2\beta\left|\hat{\tilde{u}}\right|_1 \|\tilde{u}_{\hat{t}}\| \le N^{-2} 2\beta^2\left|\hat{\tilde{u}}\right|_1^2 + \frac{1}{2} N^{-2}\|\tilde{u}_{\hat{t}}\|^2, \\
\left|N^{-2} 2\left(\Pi_N^C \tilde{g}, \tilde{u}_{x\hat{t}}\right)\right| &\le N^{-2} 2\left|\Pi_N^C \tilde{g}\right|_1 \|\tilde{u}_{\hat{t}}\| \le N^{-2} 2\left|\Pi_N^C \tilde{g}\right|_1^2 + N^{-2}\frac{1}{2}\|\tilde{u}_{\hat{t}}\|^2, \\
\left|N^{-2} 2\left(\Pi_N^C \hat{\tilde{f}}, \tilde{u}_{\hat{t}}\right)\right| &\le N^{-2} 2\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}|\tilde{u}_{\hat{t}}|_1 \le N^{-2}\mu^{-1}\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2 + N^{-2}\mu|\tilde{u}_{\hat{t}}|_1^2.
\end{aligned} \tag{4.31}$$

Thus, if $N \ge \beta\sqrt{6/\alpha}$, we have

$$\begin{aligned}
&\left(\|\tilde{u}\|^2 + \mu|\tilde{u}|_1^2 + N^{-2}\alpha|\tilde{u}|_1^2\right)_{\hat{t}} + \left[N^{-2}\|\tilde{u}_{\hat{t}}\|^2 + N^{-2}\mu|\tilde{u}_{\hat{t}}|_1^2 + \alpha\left|\hat{\tilde{u}}\right|_1^2\right] \\
&\quad \le \frac{3}{\alpha}\left\|\Pi_N^C \tilde{g}\right\|^2 + N^{-2} 2\left|\Pi_N^C \tilde{g}\right|_1^2 + \frac{3}{\alpha}\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2 + N^{-2}\mu^{-1}\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2 \\
&\quad \le C\left(\left\|\Pi_N^C \tilde{g}\right\|^2 + N^{-2}\left|\Pi_N^C \tilde{g}\right|_1^2 + \left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2 + N^{-2}\left\|\Pi_N^C \hat{\tilde{f}}\right\|_{-1}^2\right),
\end{aligned} \tag{4.32}$$

where C is a positive constant dependent on α^{-1}, μ^{-1}. Summing (4.32) for $s \in S_{t-\tau}$ gives

$$E(\tilde{u}, t) \le \rho\left(\tilde{u}, \tilde{f}, t\right) + C\tau \sum_{s \in S_{t-\tau}}\left(\left\|\Pi_N^C \tilde{g}(s)\right\|^2 + N^{-2}\left|\Pi_N^C \tilde{g}\right|_1^2\right), \tag{4.33}$$

where

$$\begin{aligned}
E(\tilde{u}, t) &= \|\tilde{u}(t)\|^2 + \mu|\tilde{u}(t)|_1^2 + N^{-2}\alpha|\tilde{u}(t)|_1^2 \\
&\quad + 2\tau \sum_{s \in S_{t-\tau}}\left(N^{-2}\|\tilde{u}_{\hat{t}}(s)\|^2 + N^{-2}\mu|\tilde{u}_{\hat{t}}(s)|_1^2 + \alpha\left|\hat{\tilde{u}}(s)\right|_1^2\right), \\
\rho\left(\tilde{u}, \tilde{f}, t\right) &= \|\tilde{u}(0)\|^2 + \mu|\tilde{u}(0)|_1^2 + N^{-2}\alpha|\tilde{u}(0)|_1^2 + \|\tilde{u}(\tau)\|^2 + \mu|\tilde{u}(\tau)|_1^2 \\
&\quad + N^{-2}\alpha|\tilde{u}(\tau)|_1^2 + C\tau \sum_{s \in S_{t-\tau}}\left(\left\|\Pi_N^C \hat{\tilde{f}}(s)\right\|_{-1}^2 + N^{-2}\left\|\Pi_N^C \hat{\tilde{f}}(s)\right\|_{-1}^2\right).
\end{aligned} \tag{4.34}$$

For any given $t \in S_T$, if

$$\|\tilde{u}(s)\|_{L^\infty(\Lambda)} \le C_0, \quad \forall s \in S_{t-\tau}, \tag{4.35}$$

then, by (3.7) and (3.8),

$$\begin{aligned}\left\|\Pi_N^C \tilde{g}\right\| + N^{-1}\left|\Pi_N^C \tilde{g}\right|_1 &\leq \|\tilde{g}\| + \left\|\Pi_N^C \tilde{g} - \tilde{g}\right\| + N^{-1}\left|\Pi_N^C \tilde{g}\right|_1 \\ &\leq \|\tilde{g}\| + CN^{-1}|\tilde{g}|_1 \leq C_g(u_M, C_0)\left(\|\tilde{u}\| + N^{-1}|\tilde{u}|_1\right), \quad \forall s \in S_{t-\tau}.\end{aligned} \tag{4.36}$$

Thus, we have shown that for any $t \in S_T$, if (4.35) holds, then

$$E(\tilde{u}, t) \leq \rho\left(\tilde{u}, \tilde{f}, t\right) + C^*\tau \sum_{s\in S_{t-\tau}} E(\tilde{u}, s), \tag{4.37}$$

where C^* is a positive constant dependent on $C_g(u_M, C_0)$, α^{-1} and μ^{-1}.

Theorem 4.5. *Let τ be suitably small and $N \geq \beta\sqrt{6/\alpha}$. If $g \in C^2(\mathbb{R})$ and $\rho(\tilde{u}, \tilde{f}, T) \leq 2C_0^2(e^{-C^*T}/(N+1)^2)$, then*

$$E(\tilde{u}, t) \leq \rho\left(\tilde{u}, \tilde{f}, t\right)e^{C^*t}, \quad \forall t \in S_T. \tag{4.38}$$

Proof. We verify the result by induction over $t \in S_T$. It is easy to see that the result is true for $t = \tau$. Assume that it is true for all $s \in S_{t-\tau}$ that

$$E(\tilde{u}, s) \leq \rho\left(\tilde{u}, \tilde{f}, s\right)e^{C^*s}. \tag{4.39}$$

Then, from the inverse inequality (3.12) we have

$$\|\tilde{u}(s)\|^2_{L^\infty(\Lambda)} \leq \frac{(N+1)^2}{2}\|\tilde{u}(s)\|^2 \leq \frac{(N+1)^2}{2}\rho\left(\tilde{u}, \tilde{f}, s\right)e^{C^*s} \leq C_0^2, \tag{4.40}$$

which means that (4.35) holds. Therefore, we have, from (4.37) and (4.39), that

$$\begin{aligned}E(\tilde{u}, t) &\leq \rho\left(\tilde{u}, \tilde{f}, t\right) + C^*\tau \sum_{s\in S_{T-\tau}} E(\tilde{u}, s) \leq \rho\left(\tilde{u}, \tilde{f}, t\right) + C^*\tau \sum_{s\in S_{T-\tau}} \rho\left(\tilde{u}, \tilde{f}, s\right)e^{C^*s} \\ &\leq \rho\left(\tilde{u}, \tilde{f}, t\right)\left(1 + C^*\tau \sum_{s\in S_{T-\tau}} e^{C^*s}\right) \leq \rho\left(\tilde{u}, \tilde{f}, t\right)e^{C^*t}.\end{aligned} \tag{4.41}$$

Thus, the proof is completed. □

Next we consider the convergence of scheme (2.6). Let $u(t)$ and u_N^k be the solutions to (1.7) and (2.6), respectively. Setting $v(t) = P_N u(t), e^k = v^k - u_N^k$, we have

$$\begin{aligned}\left(e_{\hat{t}}^k, \phi\right) + \mu\left(e_{x\hat{t}}^k, \phi_x\right) + \alpha\left(\hat{e}_x^k, \phi_x\right) = \beta\left(\hat{e}^k, \phi_x\right) &+ \left(\Pi_N^C \widetilde{G}, \phi_x\right) + \left(\Pi_N^C \hat{f}^k - \hat{f}^k, \phi\right) \\ &+ \left((P_N u)_{\hat{t}}^k - \hat{u}_t^k, \phi\right) + \mu\left((P_N u)_{x\hat{t}}^k - \hat{u}_{xt}, \phi_x\right) \\ &+ \beta\left(\hat{u}^k - P_N \hat{u}^k, \phi_x\right) + \left(\hat{g}\left(u^k\right) - \Pi_N^C g\left(u_N^k + e^k\right), \phi_x\right),\end{aligned} \tag{4.42}$$

where $\widetilde{G} = g(u_N^k + e^k) - g(u_N^k)$ and $\hat{g}(u^k) = (g(u(t_k + \tau)) + g(u(t_k - \tau)))/2$. It is easy to obtain the following estimate:

$$\begin{aligned}\left\|\Pi_N^C \hat{f}^k - \hat{f}^k\right\| &\le CN^{-r}\|f\|_r, \\ \left\|(P_N u)_{\hat{t}}^k - \hat{u}_t^k\right\| &\le \left\|P_N u_{\hat{t}}^k - u_{\hat{t}}^k\right\| + \left\|u_{\hat{t}}^k - \hat{u}_t^k\right\| \le CN^{-r}\|u\|_r + \left\|u_{\hat{t}}^k - u_t(t_k)\right\| \\ &\quad + \left\|u_t(t_k) - \hat{u}_t^k\right\| \le CN^{-r}\|u\|_r + C'\tau^2\|u_{ttt}(t_k)\| + C''\tau^2\|u_{tt}(t_k)\|, \\ \left\|(P_N u)_{x\hat{t}}^k - \hat{u}_{xt}^k\right\| &\le \left\|P_N u_{x\hat{t}}^k - u_{x\hat{t}}^k\right\| + \left\|u_{x\hat{t}}^k - \hat{u}_{xt}^k\right\| \le CN^{1-r}\|u\|_r + \left\|u_{x\hat{t}}^k - u_{xt}(t_k)\right\| \\ &\quad + \left\|u_{xt}(t_k) - \hat{u}_{xt}^k\right\| \le CN^{1-r}\|u\|_r + C'\tau^2\|u_{xttt}(t_k)\| + C''\tau^2\|u_{xtt}(t_k)\|, \\ \left\|\hat{u}^k - P_N \hat{u}^k\right\| &\le CN^{-r}\|u\|_r, \\ \left\|\hat{g}\left(u^k\right) - \Pi_N^C g\left(u_N^k + e^k\right)\right\| &\le \left\|\hat{g}\left(u^k\right) - \Pi_N^C \hat{g}\left(u^k\right)\right\| + \left\|\Pi_N^C\left(\hat{g}\left(u^k\right) - g\left(u_N^k + e^k\right)\right)\right\| \\ &\le CN^{-r}\|u\|_r + C'\left\|\hat{g}\left(u^k\right) - g(u(t_k))\right\| + C''\left\|g\left(u^k\right) - g\left(u_N^k + e^k\right)\right\| \\ &\le CN^{-r}\|u\|_r + C'\tau^2\|u\| + C''N^{-r}\|u\|_r.\end{aligned} \tag{4.43}$$

Combining with the stability results of Theorem 4.5, we can obtain the following convergence result.

Theorem 4.6. *Let u and u_N^k be the solutions of* (1.7) *and* (2.6), *respectively. If $u(x,t) \in C^3(0,T; H^r(\Lambda) \cap H_0^1(\Lambda))(r \ge 2)$ and $g \in C^2(\mathbb{R})$ and τ is suitably small, then one has*

$$\left\|u(t_k) - u_N^k\right\| + \mu\left|u(t_k) - u_N^k\right|_1 \le C\left(\tau^2 + N^{1-r}\right). \tag{4.44}$$

5. Numerical Experiments

In this section, we will present some numerical experiments to confirm the effectiveness and robustness of scheme (2.6) or (2.13).

Table 1: Convergence rates at $t = 1$ with $\mu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $n = 2$.

N	L_∞-error	Order	L_2-error	Order
4	$2.875E-01$	—	$3.504E-02$	—
8	$1.279E-03$	$N^{-7.81}$	$1.278E-04$	$N^{-8.10}$
12	$2.499E-06$	$N^{-15.4}$	$2.771E-07$	$N^{-15.1}$
16	$2.573E-09$	$N^{-23.9}$	$2.816E-10$	$N^{-24.1}$

Table 2: Convergence rates at $t = 1$ with $\mu = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $n = 2$.

N	L_∞-error	Order	L_2-error	Order
4	$2.474E-01$	—	$2.962E-02$	—
8	$4.189E-04$	$N^{-9.21}$	$4.912E-05$	$N^{-9.24}$
12	$1.491E-07$	$N^{-19.6}$	$1.781E-08$	$N^{-19.5}$
16	$4.408E-09$	$N^{-12.2}$	$5.606E-10$	$N^{-12.0}$

Table 3: Convergence rates at $t = 1$ with $\mu = 1$, $\alpha = \beta = 1$, $\gamma = 3$, and $n = 2$.

τ	L_∞-error	Order	L_2-error	Order
$2 * 10^{-2}$	$8.242E-05$	—	$1.252E-05$	—
$1 * 10^{-2}$	$2.061E-05$	$\tau^{2.00}$	$3.129E-06$	$\tau^{2.00}$
$2 * 10^{-3}$	$8.249E-07$	$\tau^{2.00}$	$1.252E-07$	$\tau^{2.00}$
$1 * 10^{-3}$	$2.063E-07$	$\tau^{2.00}$	$3.133E-08$	$\tau^{2.00}$
$2 * 10^{-4}$	$8.279E-09$	$\tau^{2.00}$	$1.258E-09$	$\tau^{2.00}$
$1 * 10^{-4}$	$2.076E-09$	$\tau^{2.00}$	$3.155E-10$	$\tau^{2.00}$

Example 5.1. We consider the initial boundary problem (1.2)–(1.4) with the exact solution as follows:

$$u(x,t) = t^\gamma \sin(\pi x). \tag{5.1}$$

The data $u_0(x)$ is zero function, and the source term f is determined by (1.2) and $g(u) = u^n/n$, that is,

$$f(x,t) = t^{\gamma-1}\sin(\pi x)\left[\gamma + \mu\pi^2\gamma + \alpha\pi^2 t\right] + \beta\pi t^\gamma\cos(\pi x) + \pi t^n \sin^{n-1}(\pi x)\cos(\pi x). \tag{5.2}$$

Because the exact solution is known, we can compute the error between the numerical solution and the exact solution. In Table 1, the convergent rate in L_∞ and L_2 errors for spatial discretization is computed by fixing time step $\tau = 2\times\ 10^{-5}$ and changing N from 2^2 to 2^4 with the parameters $\mu = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, and $n = 2$ at $t = 1$. In Table 2, the convergent rate in L_∞ and L_2 errors for spatial discretization is computed by fixing time step $\tau = 2\times\ 10^{-5}$ and changing N from 2^2 to 2^4 with the parameters $\mu = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, and $n = 2$ at $t = 1$. The results show clearly "spectral accuracy."

Table 4: Convergence rates at $t = 1$ with $\mu = 0.1$, $\alpha = 1$, $\beta = 0.01$, $\gamma = 3$, and $n = 2$.

N	τ	L_∞-error	Order	L_2-error	Order
4		$2.986E-01$	—	$4.722E-01$	—
8		$2.064E-03$	$N^{-7.18}$	$1.403E-03$	$N^{-8.39}$
12	$2E-5$	$5.166E-06$	$N^{-14.77}$	$2.429E-06$	$N^{-15.68}$
16		$5.213E-09$	$N^{-23.98}$	$1.899E-09$	$N^{-24.88}$
20		$2.750E-10$	$N^{-13.19}$	$7.859E-11$	$N^{-14.27}$
	$2*10^{-3}$	$2.734E-06$	—	$4.368E-07$	—
	$1*10^{-3}$	$6.836E-07$	$\tau^{2.00}$	$1.092E-07$	$\tau^{2.00}$
36	$2*10^{-4}$	$2.734E-08$	$\tau^{2.00}$	$4.368E-09$	$\tau^{2.00}$
	$1*10^{-4}$	$6.836E-09$	$\tau^{2.00}$	$1.092E-09$	$\tau^{2.00}$
	$2*10^{-5}$	$2.733E-10$	$\tau^{2.00}$	$4.366E-11$	$\tau^{2.00}$

Table 5: Convergence rates at $t = 1$ with $\mu = 0.1$, $\alpha = 1$, $\beta = 0.01$, $\gamma = 3$, and $n = 2$.

N	τ	L_∞-error	Order	L_2-error	Order
4		$1.722E-02$	—	$2.112E-02$	—
8		$2.049E-05$	$N^{-9.71}$	$1.437E-05$	$N^{-10.52}$
12	$1E-5$	$3.901E-09$	$N^{-21.13}$	$1.845E-09$	$N^{-22.10}$
16		$4.639E-12$	$N^{-23.41}$	$1.963E-12$	$N^{-23.80}$
20		$4.718E-12$	—	$1.570E-12$	—
	$2*10^{-3}$	$2.048E-07$	—	$3.511E-08$	—
	$1*10^{-3}$	$5.119E-08$	$\tau^{2.00}$	$8.777E-09$	$\tau^{2.00}$
36	$2*10^{-4}$	$2.047E-09$	$\tau^{2.00}$	$3.511E-10$	$\tau^{2.00}$
	$1*10^{-4}$	$5.117E-10$	$\tau^{2.00}$	$8.778E-11$	$\tau^{2.00}$
	$2*10^{-5}$	$2.028E-11$	$\tau^{2.01}$	$3.535E-12$	$\tau^{2.00}$

For temporal discretization, the convergent rate in L_∞ and L_2 errors is computed by fixing $N = 32$ and changing time step τ from 2×10^{-2} to 1×10^{-4} with the parameters $\mu = 0$, $\alpha = \beta = 1$, $\gamma = 3$, and $n = 2$ at $t = 1$. In Table 3, the results show clearly second-order convergence for time discretization.

In Table 4, the results show clearly second-order convergence for time discretization and spectral accuracy for space discretization again.

Example 5.2. We consider the initial boundary problem (1.2)–(1.4) with the exact solution as follows:

$$u(x,t) = \left(1 - x^2\right)e^{x-t}. \tag{5.3}$$

The data $u_0(x) = (1 - x^2)e^x$ and the source term f is determined by (1.2) and $g(u) = u^n/n$.

In Table 5, the results show clearly second-order convergence for time discretization and spectral accuracy for space discretization again.

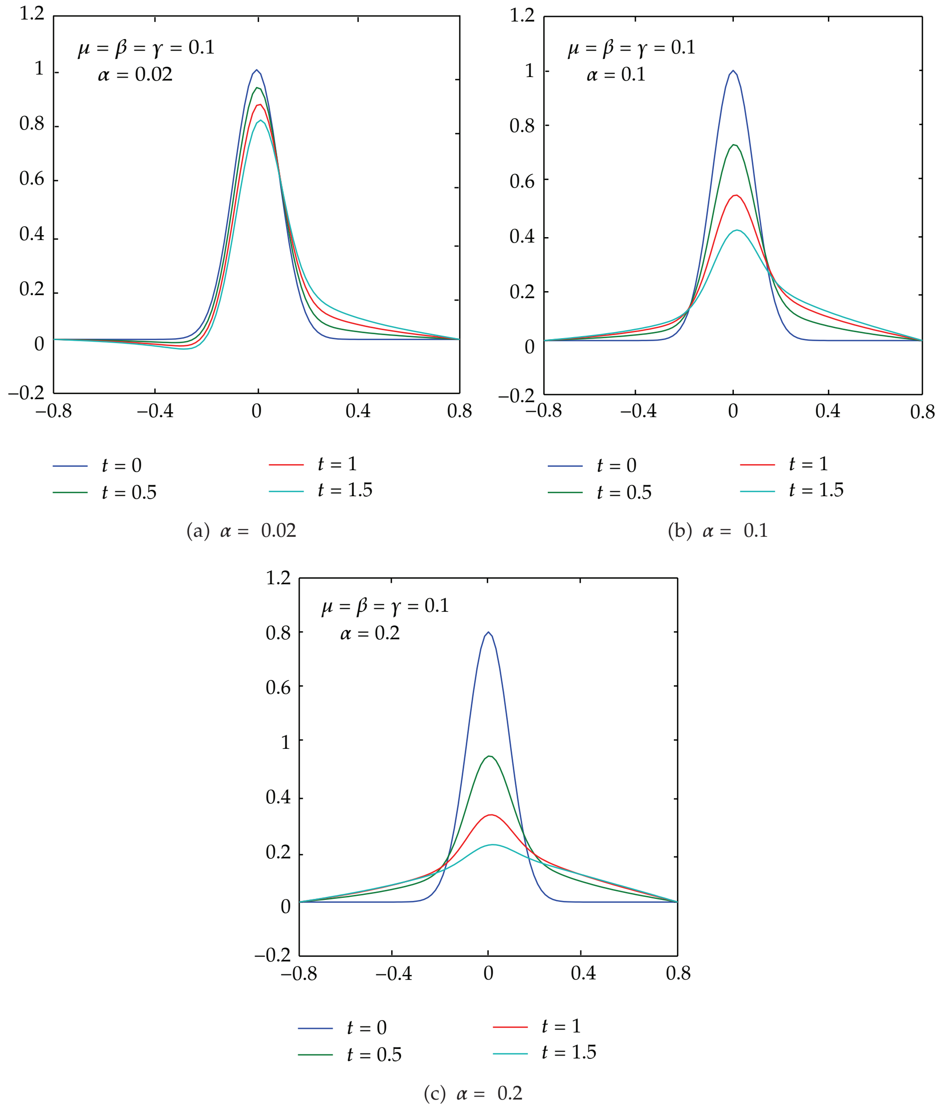


Figure 1: The evolution of $u(x,t)$ with different α.

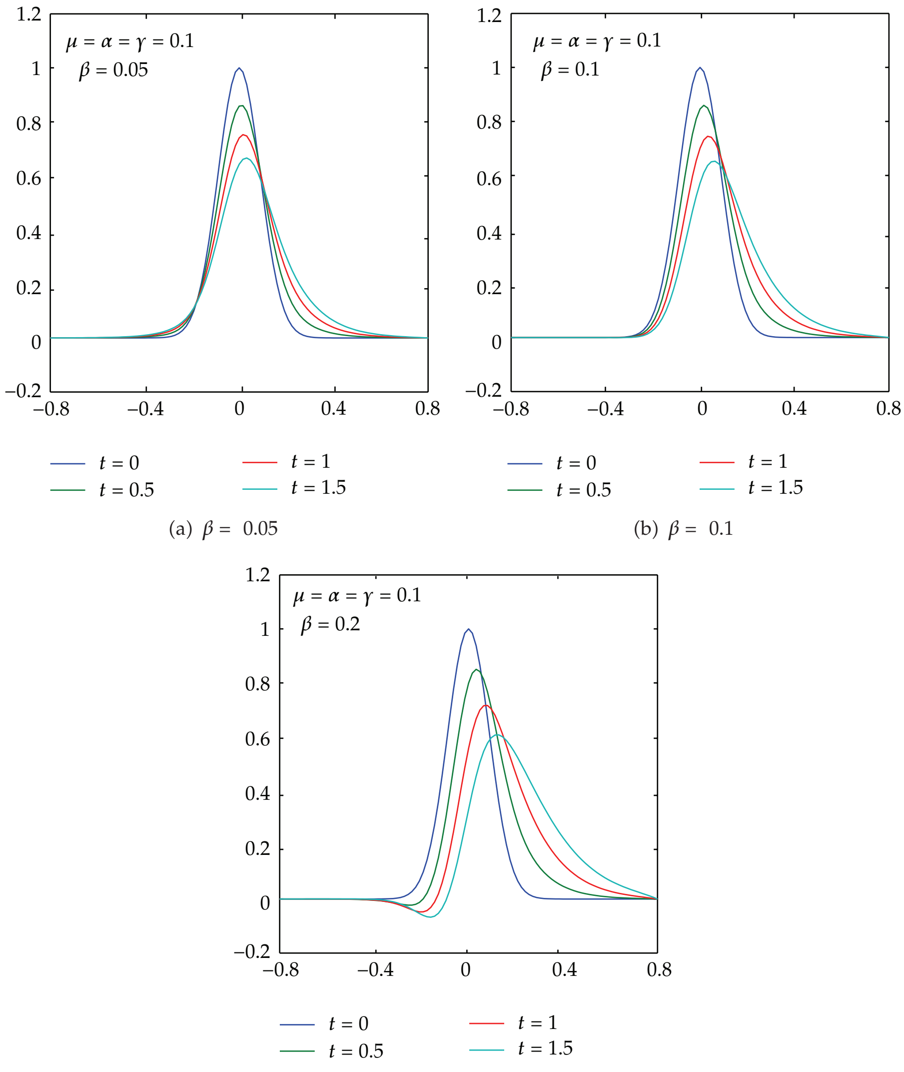
Example 5.3. We consider the gBBM-B equation

$$u_t - \mu u_{xxt} - \alpha u_{xx} + \beta u_x + \gamma u u_x = 0, \tag{5.4}$$

with initial profile as

$$u_0(x) = e^{-40x^2}. \tag{5.5}$$

In Figures 1 and 2, the results show how the numerical diffusion does with the increasing diffusion α and reaction coefficient β.



(a) $\beta = 0.05$

(b) $\beta = 0.1$

(c) $\beta = 0.2$

Figure 2: The evolution of $u(x,t)$ with different β.

6. Conclusion

In this paper, we have developed the Chebyshev-Legendre spectral method to the generalised Benjamin-Bona-Mahony-Burgers (gBBM-B) equations. As is well known, Chebyshev method is popular for its explicit formulae and fast Chebyshev transform. But the Chebyshev weight makes much trouble in error analysis. Legendre method is natural in the theoretical analysis due to the unity weight function. However, it is well known that the applications of Legendre method are limited by the lack of fast transform between the physical space and the spectral space and by the lack of explicit evaluation formulation of the LGL nodes. Chebyshev-Legendre method adopts advantages of both Chebyshev method and Legendre method. Hence, it gives a better scheme. Also, error analysis shows that an optimal convergence rate can be obtained by using the Chebyshev-Legendre method. The numerical results show that this scheme is an efficient one. For the problems of high dimension and with complex geometry boundary, Chebyshev-Legendre method could perform well. This will be left for our further research.

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