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Research Article

Bäcklund Transformation and New Exact Solutions of the Sharma-Tasso-Olver Equation

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The Sharma-Tasso-Olver (STO) equation is investigated. The Painlevé analysis is efficiently used for analytic study of this equation. The Bäcklund transformations and some new exact solutions are formally derived.

1. Introduction

Let α be a constant. We consider the Sharma-Tasso-Olver (STO) equation

$$u_t + \alpha\left(u^3\right)_x + \frac{3}{2}\alpha\left(u^2\right)_{xx} + \alpha u_{xxx} = 0. \tag{1.1}$$

This paper is concerned with the STO equation [1] and [2–17].

Attention has been focused on STO equation (1.1) in [1] and [2–4] and the references therein due to its appearance in scientific applications. In [1], the tanh method, the extended tanh method, and other ansatz involving hyperbolic and exponential functions are efficiently used for the analytic study of this equation. And some new solitons and kinks solutions are formally derived. The proposed schemes are reliable and manageable. In [2], this equation was handled by using the Cole-Hopf transformations method. The simple symmetry reduction procedure is repeatedly used in [3] to obtain exact solutions where soliton fission and fusion were examined. However, in [4], the soliton fission and fusion were examined thoroughly by using Hirota's direct method and the Bäcklund transformations method. In

[6], the authors show that symmetry constraints do not always yield exact solutions through analyzing the STO equation. The generalized Kaup-Newell-type hierarchy of nonlinear evolution equations is explicitly related to STO equation from [9]. Using the improved tanh function method in [10], the STO equation with its fission and fusion has some exact solutions. In [11] some exact solution of the STO equation are given by implying a generalized tanh function method for approximating some solutions which have been known. In [12], the method of third-order mode coupling is applied to the STO equation. In [13], the same exact explicit solutions of the STO equation, as this work are derived by using an extension of the homogeneous balance method and the Bäcklund transformation. Recently in [7, 8], the authors have introduced some other discussions about new solution techniques, a multiple exp-function method and linear superposition principle applying to nonlinear equations, which rely on a close observation on relations between linear objects and nonlinear objects.

Many types of travelling waves are of particular interest in solitary wave theory. The solitons, which are localized travelling waves, asymptotically zero at large distances, the periodic solutions, the kink waves which rise or descend from one asymptotic state to another, are well-known types of travelling waves.

The objectives of this work are twofold. First, we seek to establish new solitons and kink solutions of distinct physical structures for the nonlinear equation (1.1). Second, we aim to implement many strategies to achieve our goal, namely, the Painlevé test [18, 19], hyperbolic functions ansatze, and exponential functions ansatze to obtain new exact solutions. In what follows, the Painlevé test will be reviewed briefly.

We give the three steps of the test for a single PDE,

$$\mathcal{F}(x, t, u(x, t)) = 0, \tag{1.2}$$

in two independent variables, x and t. Following [18, 20], the Laurent expansion of the solution $u(x, t)$,

$$u(x, t) = \phi^{\alpha}(x, t) \sum_{k=0}^{\infty} u_k(x, t) \phi^{k}(x, t), \tag{1.3}$$

should be single valued in the neighborhood of a noncharacteristic, movable and singular manifold $\phi(x, t)$, which can be viewed as the surface of the movable poles in the complex plane. In (1.3), $u_0(x, t) \neq 0$, α is a negative integer and $u_k(x, t)$ are analytic functions in a neighborhood of $\phi(x, t)$.

The Painlevé test have the following steps

Step 1 (leading order analysis). Determine the (negative) integer α and u_0 by balancing the minimal power terms after the substitution of $u = u_0 \phi^{\alpha}$ into the given PDE. There may be several branches for u_0, and for each the next two steps must be performed.

Step 2 (determination of the resonances). For selected α and u_0, calculate the nonnegative integers r, called the *resonances*, at which arbitrary functions u_r enter the series (1.3). To do so, substitute (1.3) into (1.1), and normalize all orders of ϕ with the minimal power term. Then we have the cycle formula on u_j, through reducing the formula, and the resonance r can be obtained.

Step 3 (verification of the compatibility conditions). Substituting $j = k$ at which k is not equal r into the cycle formula, we have nonlinear equation $f_k(u_k \cdots u_0, \phi \cdots) = 0$ $(1 < k < r)$.

At resonance levels, u_r should be arbitrary, and then we are deducing a nonlinear equation $g(u_{r-1}\cdots u_0,\phi\cdots)=0$. If the equation f implies g, then the compatibility condition is unconditionally satisfied.

An equation for which these three steps can be carried out consistently and unambiguously passes the Painlevé test. Equation (1.3) is called the truncated Painlevé expansion.

2. The Compatibility Conditions and Bäcklund Transform

Let

$$u=\phi^{\alpha}\sum_{j=0}^{\infty}u_j\phi^{j}, \tag{2.1}$$

where $\phi=\phi(x,t)$, $u_j=u_j(x,t)$ are analytic functions on (x,t) in the neighborhood of the manifold $M=\{(x,t):\phi(x,t)=0\}$.

Substituting (2.1) into (1.1), we can get the $\alpha=-1$. Thus, (2.1) becomes

$$u=\sum_{j=0}^{\infty}u_j\phi^{j-1}. \tag{2.2}$$

Expressing (1.1), we have that

$$u_t+3\alpha u^2u_x+3\alpha u_x^2+3\alpha uu_{xx}+\alpha u_{xxx}=0. \tag{2.3}$$

Differentiating (2.2), we obtain that

$$\begin{aligned}
u_t&=(j-1)\sum_{j=0}^{\infty}u_j\phi^{j-2}\phi_t+\sum_{j=0}^{\infty}u_{j,t}\phi^{j-1},\\
u_x&=(j-1)\sum_{j=0}^{\infty}u_j\phi^{j-2}\phi_x+\sum_{j=0}^{\infty}u_{j,x}\phi^{j-1},\\
u_{xx}&=(j-1)\sum_{j=0}^{\infty}u_j\phi^{j-2}\phi_{xx}+2(j-1)\sum_{j=0}^{\infty}u_{j,x}\phi^{j-2}\phi_x+(j-1)(j-2)\sum_{j=0}^{\infty}u_j\phi^{j-3}\phi_x^2+\sum_{j=0}^{\infty}u_{j,xx}\phi^{j-1},\\
u_{xxx}&=3(j-1)\sum_{j=0}^{\infty}u_{j,x}\phi^{j-2}\phi_{xx}+(j-1)\sum_{j=0}^{\infty}u_j\phi^{j-2}\phi_{xxx}+3(j-1)(j-2)\sum_{j=0}^{\infty}u_j\phi^{j-3}\phi_x\phi_{xx}\\
&\quad+3(j-1)\sum_{j=0}^{\infty}u_{j,xx}\phi^{j-2}\phi_x+\sum_{j=0}^{\infty}u_{j,xxx}\phi^{j-1}+3(j-1)(j-2)\sum_{j=0}^{\infty}u_{j,x}\phi^{j-3}\phi_x^2\\
&\quad+(j-1)(j-2)(j-3)\sum_{j=0}^{\infty}u_j\phi^{j-4}\phi_x^3.
\end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.3), we get the cycle formula on u_j

$$\begin{aligned}
&(j-3)u_{j-2}\phi_t + u_{j-3,t} + 3\alpha\sum_{m=0}^{j-n}\sum_{n=0}^{j} u_{j-m-n}u_m\left[(n-1)u_n\phi_x + u_{n-1,x}\right] \\
&\quad + 3\alpha\sum_{m=0}^{j}\left[(j-m-1)u_{j-m}\phi_x + u_{j-m-1,x}\right]\left[(m-1)u_m\phi_x + u_{m-1,x}\right] \\
&\quad + 3\alpha\sum_{m=0}^{j}u_{j-m}\left[(m-2)u_{m-1}\phi_{xx} + 2(m-2)u_{m-1,x}\phi_x + (m-1)(m-2)u_m\phi_x^2 + u_{m-2,xx}\right] \\
&\quad + \alpha\left[3(j-3)u_{j-2,x}\phi_{xx} + (j-3)u_{j-2}\phi_{xxx} + 3(j-2)(j-3)u_{j-1}\phi_x\phi_{xx}\right. \\
&\qquad \left. + 3(j-2)(j-3)u_{j-1,x}\phi_x^2 + u_{j-3,xxx} + 3(j-3)u_{j-2,xx}\phi_x + (j-1)(j-2)(j-3)u_j\phi_x^3\right] = 0.
\end{aligned} \tag{2.5}$$

Taking $j = 0$ in (2.5), we deduce that $u_0 = \phi_x$ or $u_0 = 2\phi_x$.

We will get the Bäcklund transformations, according to the above two cases, respectively.

Case 1 ($u_0 = \phi_x$). Substituting $u_0 = \phi_x$ into (2.5), we have the following equation on u_j:

$$(j+1)(j-1)(j-3)\alpha u_j\phi_x^3 = F_j\left(u_{j-1}\cdots u_0, \phi_t, \phi_x, \phi_{xx}, \phi_{xxx}\cdots\right) \quad (j = 1, 2, \ldots). \tag{2.6}$$

By (2.6) we see that u_{-1}, u_1 and u_3 are arbitrary. Hence $j = 3$ are resonances. Moreover the compatibility conditions will be deduced by (2.5) or (2.6).

Setting $j = 2$ and $j = 3$, we infer that

$$\phi_t + 3\alpha\left(u_1^2\phi_x + u_2\phi_x^2 + u_{1,x}\phi_x + u_1\phi_{xx}\right) + \alpha\phi_{xxx} = 0, \tag{2.7}$$

$$\frac{\partial}{\partial x}\left[\phi_t + 3\alpha\left(u_1^2\phi_x + u_2\phi_x^2 + u_{1,x}\phi_x + u_1\phi_{xx}\right) + \alpha\phi_{xxx}\right] = 0. \tag{2.8}$$

It is easy to see that (2.8) does so if (2.7) holds. So u_3 is arbitrary.

Taking $j = 4$, $u_3 = 0$ and letting $u_2 = u_4 = 0$ in (2.6), we have that

$$u_{1,t} + 3\alpha u_1^2 u_{1,x} + 3\alpha u_{1,x}^2 + 3\alpha u_1 u_{1,xx} + \alpha u_{1,xxx} = 0. \tag{2.9}$$

By (2.6), noting that $u_2 = 0$, $u_3 = 0$ and $u_4 = 0$, we can infer that

$$u_j = 0 \quad (j \geq 2). \tag{2.10}$$

Thus, we obtain from the Bäckland transformation of the STO equation that

$$u = \frac{\phi_x}{\phi} + u_1, \tag{2.11}$$

where u_1 is the seed solution of (1.1), the u_1 and ϕ satisfy the following condition:

$$\phi_t + 3\alpha\left(u_1^2\phi_x + u_{1,x}\phi_x + u_1\phi_{xx}\right) + \alpha\phi_{xxx} = 0. \tag{2.12}$$

Case 2 ($u_0 = 2\phi_x$). Rewrite (2.5) as the following form on u_j

$$(j+1)(j+2)(j-3)\alpha u_j\phi_x^3 = G_j(u_{j-1}\cdots u_0, \phi_t, \phi_x, \phi_{xx}, \phi_{xxx}\cdots), \quad (j = 1, 2, \dots). \tag{2.13}$$

From (2.13) we know that u_3 is arbitrary. That is, $j = 3$ are resonances.

Taking $j = 1$, $j = 2$, and $j = 3$ in (2.13), we get that

$$u_1 = -\frac{\phi_{xx}}{\phi_x}, \tag{2.14}$$

$$\phi_t + 6\alpha u_2\phi_x^2 - 3\alpha u_1\phi_{xx} - 2\alpha\phi_{xxx} = 0, \tag{2.15}$$

$$\frac{\partial}{\partial x}\left[\phi_t + 6\alpha u_2\phi_x^2 - 3\alpha u_1\phi_{xx} - 2\alpha\phi_{xxx}\right] = 0. \tag{2.16}$$

It is easy to see that (2.15) implies (2.16). So u_3 is arbitrary. Equation (1.1) satisfies Painlevé property.

Taking $j = 4$, $u_3 = 0$ and letting $u_2 = u_4 = 0$ in (2.5) or (2.13), we have that

$$u_{1,t} + 3\alpha u_1^2 u_{1,x} + 3\alpha u_{1,x}^2 + 3\alpha u_1 u_{1,xx} + \alpha u_{1,xxx} = 0. \tag{2.17}$$

Substituting $u_1 = -\phi_{xx}/\phi_x$ into the above equation, we deduce the second condition below

$$-\phi_x^3\phi_{xxt} + \phi_x^2\phi_{xx}\phi_{xt} + 18\alpha\phi_{xx}^4 - 30\alpha\phi_x\phi_{xx}^2\phi_{xxx} + 7\alpha\phi_x^2\phi_{xx}\phi_{xxxx} + 6\alpha\phi_x^2\phi_{xxx}^2 - \alpha\phi_x^3\phi_{xxxxx} = 0 \tag{2.18}$$

By (2.5) or (2.13), noting that $u_2 = 0$, $u_3 = 0$, and $u_4 = 0$, we can easy to have that

$$u_j = 0 \quad (j \geq 2). \tag{2.19}$$

Thus, we obtain from the other Bäckland transformation of the STO equation that

$$u = \frac{2\phi_x}{\phi} - \frac{\phi_{xx}}{\phi_x}, \tag{2.20}$$

where $\phi(x,t)$ satisfies the following conditions:

$$\phi_x\phi_t + 3\alpha\phi_{xx}^2 - 2\alpha\phi_x\phi_{xxx} = 0,$$
$$-\phi_x^3\phi_{xxt} + \phi_x^2\phi_{xx}\phi_{xt} + 18\alpha\phi_{xx}^4 - 30\alpha\phi_x\phi_{xx}^2\phi_{xxx} + 7\alpha\phi_x^2\phi_{xx}\phi_{xxxx} + 6\alpha\phi_x^2\phi_{xxx}^2 - \alpha\phi_x^3\phi_{xxxxx} = 0. \tag{2.21}$$

3. New Exact Solutions for STO Equation

As it is well known that the Bäcklund transformation is one of the most effective methods for finding exact solutions of nonlinear partial differential equations. By (2.11) and (2.20), choose some seed solutions, and then we can get some new single travelling solitary wave solutions below, respectively.

Case 1 (for Bäcklund transformation (2.11)). Taking the seed solution $u_1 = 0$ in the Bäcklund transformation (2.11), (2.12) becomes

$$\phi_t + \alpha\phi_{xxx} = 0. \tag{3.1}$$

The arbitrary multiple solitary wave solution of (1.1) can be easily written down. For the sake of simplicity, we only give some new single travelling solitary wave solutions here. Substituting the ansatz $\phi_1 = k_1x^2 + k_2x + r$ which satisfies (3.1) into (2.11), and leads the new exact solutions of (1.1)

$$u_1(x,t) = \frac{2k_1x + k_2}{k_1x^2 + k_2x + r}. \tag{3.2}$$

Substituting the ansatz $\phi_2 = a\cosh(kx + \omega t + r) + b\sinh(kx + \omega t + r) + c$ into (3.1) will produce the dispersion relation between ω and k,

$$\omega + \alpha k^3 = 0, \tag{3.3}$$

and the new exact solutions are that

$$u_2(x,t) = \frac{ak\sinh(kx - \alpha k^3t + r) + bk\cosh(kx - \alpha k^3t + r)}{a\cosh(kx - \alpha k^3t + r) + b\sinh(kx - \alpha k^3t + r) + c}. \tag{3.4}$$

Taking the seed solution $u_1 = c$ for (1.1) will get

$$\phi_t + 3\alpha\left(c^2\phi_x + c\phi_{xx}\right) + \alpha\phi_{xxx} = 0. \tag{3.5}$$

Observing (3.5), we have $\phi_3 = kx - 3\alpha c^2kt + r$, $\phi_4 = a + e^{kx-(3\alpha c^2k+3\alpha ck^2+\alpha k^3)t+r}$ and $\phi_5 = \sinh[kx - (3\alpha c^2k + 3\alpha ck^2 + \alpha k^3)t + r] + \cosh[kx - (3\alpha c^2k + 3\alpha ck^2 + \alpha k^3)t + r] + b$.

Substituting ϕ_3, ϕ_4, ϕ_5 and $u_1 = c$ into the Bäcklund transformation (2.11), we obtain the exact solutions for STO equation (1.1)

$$u_3(x,t) = \frac{k}{kx - 3\alpha c^2 kt + r} + c,$$

$$u_4(x,t) = \frac{ke^{kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r}}{a + e^{kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r}} + c,$$

$$u_5(x,t) = \frac{k\sinh\left[kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r\right]+k\cosh\left[kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r\right]}{\sinh[kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r]+\cosh[kx-(3\alpha c^2 k+3\alpha ck^2+\alpha k^3)t+r]+b}+c. \tag{3.6}$$

Taking the seed solution $u_1 = 1/x$ for STO equation (1.1), (2.12) becomes

$$\phi_t + 3\alpha\frac{\phi_{xx}}{x} + \alpha\phi_{xxx} = 0. \tag{3.7}$$

Substitute the ansatz $\phi_6 = k_1x^3 + k_3x - 24\alpha k_1 t + b$ into (2.11), yielding the solutions for the STO equation (1.1)

$$u_6(x,t) = \frac{3k_1x^2 + k_3}{k_1x^3 + k_3x - 24\alpha k_1 t + b} + \frac{1}{x}. \tag{3.8}$$

Remark 3.1. When $a = 0$ or $b = 0$, the solution $u_2(x,t)$ is obtained in [1] by tanh method. In [4], the authors have obtained the $u_4(x,t)$. As for the case $u_0 = 2\phi_x$, the authors of [4, page 238] said that no new meaningful results can be obtained; we could here obtain some meaningful results.

Case 2 (for Bäcklund transformation (2.20)). Observe, (2.21), we have ansatz $\phi_1 = \sinh(kx + \omega t + r) + \cosh(kx + \omega t + r)$. Substituting ϕ_1 into (2.21) and seting the coefficients to be equal to zero, we have the dispersion relation between ω and k,

$$\omega = -\alpha k^3. \tag{3.9}$$

Thus, (2.20) yields the new exact solutions for the STO equation (1.1)

$$u_7 = 2\frac{k\sinh\left[kx - \alpha k^3 t + r\right] + k\cosh\left[kx - \alpha k^3 t + r\right]}{\sinh[kx - \alpha k^3 t + r] + \cosh[kx - \alpha k^3 t + r] + b} - k. \tag{3.10}$$

Substituting ansatz $\phi_2 = e^{kx+\omega t+r} + b$ into (2.21) and balancing the coefficients, we have the new exact solutions for the STO equation (1.1)

$$u_7(x,t) = \frac{2ke^{kx-\alpha k^3 t+r}}{e^{kx-\alpha k^3 t+r} + b} - k. \tag{3.11}$$

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