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Research Article

On Asymptotic Behaviour of Solutions to n-Dimensional Systems of Neutral Differential Equations

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This paper presents the properties and behaviour of solutions to a class of n-dimensional functional differential systems of neutral type. Sufficient conditions for solutions to be either oscillatory, or $\lim_{t\to\infty} y_i(t) = 0$, or $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1, 2, \ldots, n$, are established. One example is given.

1. Introduction

The authors have investigated some properties of solutions to n-dimensional functional differential systems

$$\begin{aligned} \left[y_1(t) - a(t)y_1\left(g(t)\right)\right]' &= p_1(t)y_2(t), \\ y_i'(t) &= p_i(t)y_{i+1}(t), \quad i = 2, 3, \ldots, n-1, \\ y_n'(t) &= \sigma p_n(t) f\left(y_1(h(t))\right), \quad t \geq t_0, \end{aligned} \tag{1.1}$$

in [1]. We studied the properties of solutions presupposing that both functions $a(t)$ and $y_1(t)$ were bounded and there were presented theorems where sufficient conditions to every solution with the first component of the solution $y_1(t)$ to be either oscillatory, or $\lim_{t\to\infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$.

The goal of this paper is to enquire about the behaviour of the solution to n-dimensional functional differential system of neutral type (1.1) under no restriction to $a(t)$ and to the first component $y_1(t)$ of solution $y(t)$. Results are given in theorems where sufficient conditions are stated to every solution to have the next properties: a solution to be either oscillatory, or $\lim_{t\to\infty} y_i(t) = 0$, or $\lim_{t\to\infty}|y_i(t)| = \infty$, $i = 1, 2, \ldots, n$.

The system (1.1) is investigated under the assumptions $\sigma \in \{-1, 1\}$, $n \geq 3$, and throughout this paper, the next conditions are considered:

(a) $a : [t_0, \infty) \to (0, \infty]$ is a continuous function;

(b) $g : [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function, $\lim_{t\to\infty} g(t) = \infty$;

(c) $p_i : [t_0, \infty) \to [0, \infty)$, $i = 1, 2, \ldots, n$, are continuous functions; p_n not identically equal to zero in any neighbourhood of infinity, $\int^{\infty} p_j(t)\mathrm{d}t = \infty$, $j = 1, 2, \ldots, n-1$;

(d) $h : [t_0, \infty) \to \mathbb{R}$ is a continuous and increasing function, $\lim_{t\to\infty} h(t) = \infty$;

(e) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function; moreover, for $u \neq 0$, $uf(u) > 0$ and $|f(u)| \geq K|u|$ hold, where K is a positive constant.

For a function $y_1(t)$,

$$z_1(t) = y_1(t) - a(t)y_1(g(t)) \tag{1.2}$$

is defined, and for $t_1 \geq t_0$, we introduce

$$\widetilde{t}_1 = \min\{t_1, g(t_1), h(t_1)\}. \tag{1.3}$$

A vector function $y = (y_1, \ldots, y_n)$ is a solution to the system (1.1) if there is a $t_1 \geq t_0$ such that y is continuous on $[\widetilde{t}_1, \infty)$; functions $z_1(t)$, $y_i(t)$, $i = 2, 3, \ldots, n$ are continuously differentiable on $[t_1, \infty)$ and y satisfies (1.1) on $[t_1, \infty)$.

W denotes the set of all solutions $y = (y_1, \ldots, y_n)$ to the system (1.1) that exist on some interval $[T_y, \infty) \subset [t_0, \infty)$ and satisfy the condition

$$\sup\left\{\sum_{i=1}^{n}|y_i(t)| : t \geq T\right\} > 0 \quad \text{for any } T \geq T_y. \tag{1.4}$$

A solution $y \in W$ is considered nonoscillatory if there exists a $T_y \geq t_0$ such that every component is different from zero for $t \geq T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

Properties of solutions to similar differential equations and systems like system (1.1) have been studied in [1–6] and in the references cited therein. Problems of existence of solutions to neutral differential systems were analysed, for example, in [7, 8].

It will be useful to define two types of recursion formulae. Let $i_k \in \{1,2,\dots,n\}$, $k = 1,2,\dots,n$, and $t,u \in [t_0,\infty)$. One has

$$I_0(u,t) \equiv 1,$$
$$I_k(u,t;p_{i_1},p_{i_2},\dots,p_{i_k}) = \int_t^u p_{i_1}(x) I_{k-1}(x,t;p_{i_2},p_{i_3},\dots,p_{i_k})\mathrm{d}x, \tag{1.5}$$

$$J_0(u,t) \equiv 1,$$
$$J_k(u,t;p_{i_1},p_{i_2},\dots,p_{i_k}) = \int_t^u p_{i_k}(x) J_{k-1}(u,x;p_{i_1},p_{i_2},\dots,p_{i_{k-1}})\mathrm{d}x. \tag{1.6}$$

It is easy to prove that the following identities hold:

$$I_k(u,t;p_{i_1},p_{i_2},\dots,p_{i_k}) = J_k(u,t;p_{i_1},p_{i_2},\dots,p_{i_k}) \tag{1.7}$$

for $k = 1,2,\dots,n$.

Functions $g^{-1}(t)$, $h^{-1}(t)$ denote the inverse functions to $g(t)$, $h(t)$.

2. Preliminaries

Lemma 2.1 (see [9, Lemma 1]). *Let $y \in W$ be a solution of* (1.1) *with $y_1(t) \neq 0$ on $[t_1,\infty)$, $t_1 \geq t_0$. Then y is nonoscillatory and $z_1(t), y_2(t),\dots,y_n(t)$ are monotone on some ray $[T,\infty)$, $T \geq t_1$.*

Let $y \in W$ be a non-oscillatory solution of (1.1). By (1.1) and (c), it follows that the function $z_1(t)$ from (1.2) has to be eventually of constant sign, so that either

$$y_1(t) z_1(t) > 0 \tag{2.1}$$

or

$$y_1(t) z_1(t) < 0 \tag{2.2}$$

for sufficiently large t.

We mention for the comfort of proofs a classification of non-oscillatory solutions of the system (1.1) which was introduced by the authors in [1].

Assume first that (2.1) holds.

By [9, Lemma 4], the statement in Lemma 2.2 follows.

Lemma 2.2. *Let $y = (y_1,y_2,\dots,y_n) \in W$ be a non-oscillatory solution to* (1.1) *on $[t_1,\infty)$, and assume that* (2.1) *holds. Then there exists an integer $l \in \{1,2,\dots,n\}$ such that $\sigma \cdot (-1)^{n+l+1} = 1$ or $l = n$, and $t_2 \geq t_1$ such that for $t \geq t_2$*

$$y_i(t) z_1(t) > 0, \quad i = 1,2,\dots,l,$$
$$(-1)^{i+l} y_i(t) z_1(t) > 0, \quad i = l+1,\dots,n. \tag{2.3}$$

Denote by N_l^+ the set of non-oscillatory solutions to (1.1) satisfying (2.3). Now assume that (2.2) holds.

By the aid of Kiguradze's lemma, it is easy to prove Lemma 2.3.

Lemma 2.3. *Let $y = (y_1, y_2, \dots, y_n) \in W$ be a non-oscillatory solution to* (1.1) *on* $[t_1, \infty)$*, and assume that* (2.2) *holds. Then there exists an integer $l \in \{1, 2, \dots, n\}$ and $\sigma \cdot (-1)^{n+l} = 1$ or $l = n$, and $t_2 \geq t_1$ such that for $t \geq t_2$ either*

$$y_1(t)z_1(t) < 0,$$
$$(-1)^i y_i(t)z_1(t) < 0, \quad i = 2, \dots, n, \tag{2.4}$$

or

$$y_1(t)z_1(t) < 0,$$
$$y_i(t)z_1(t) > 0, \quad i = 2, 3, \dots, l, \tag{2.5}$$
$$(-1)^{i+l} y_i(t)z_1(t) > 0, \quad i = l + 1, \dots, n.$$

Denote by N_1^- the set of nonoscillatory solutions to (1.1) satisfying (2.4), and by N_l^- the set of non-oscillatory solutions to (1.1) satisfying (2.5). Denote by N the set of all non-oscillatory solutions to (1.1). Obviously by Lemmas 2.2 and 2.3, we have the classification of non-oscillatory solutions to the system (1.1):

n odd, $\sigma = 1$:

$$N = N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ \cup N_1^- \cup N_3^- \cup \cdots \cup N_n^-, \tag{2.6}$$

n odd, $\sigma = -1$:

$$N = N_1^+ \cup N_3^+ \cup \cdots \cup N_n^+ \cup N_2^- \cup N_4^- \cup \cdots \cup N_{n-1}^- \cup N_n^-, \tag{2.7}$$

n even, $\sigma = 1$:

$$N = N_1^+ \cup N_3^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ \cup N_2^- \cup N_4^- \cup \cdots \cup N_n^-, \tag{2.8}$$

n even, $\sigma = -1$:

$$N = N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ \cup N_1^- \cup N_3^- \cup \cdots \cup N_{n-1}^- \cup N_n^-. \tag{2.9}$$

The next lemma can be proved similarly as Lemma 2 in [9].

Lemma 2.4. *Let $y = (y_1, y_2, \ldots, y_n) \in W$ be a non-oscillatory solution to* (1.1) *on* $[t_1, \infty)$, $t_1 \geq t_0$, *and let* $\lim_{t\to\infty}|z_1(t)| = L_1$, $\lim_{t\to\infty}|y_k(t)| = L_k$, $k = 2, \ldots, n$. *Then*

$$\begin{gathered} k \geq 2, \quad L_k > 0 \Longrightarrow L_i = \infty, \quad i = 1, \ldots, k-1, \\ 1 \leq k < n, \quad L_k < \infty \Longrightarrow L_i = 0, \quad i = k+1, \ldots, n. \end{gathered} \tag{2.10}$$

Remark 2.5. If $g(t) < t$, and $0 < a(t) \leq \lambda^* < 1$, ($\lambda^*$ is a constant), then from [9], we have $N_k^- = \emptyset$, $k \in \{2, 3, \ldots, n\}$.

Lemma 2.6 (see [10, Lemma 2.2]). *In addition to conditions (a) and (b) suppose that*

$$1 \leq a(t), \quad t \geq t_0. \tag{2.11}$$

Let $y_1(t)$ be a continuous non-oscillatory solution to the functional inequality

$$y_1(t)\left[y_1(t) - a(t)y_1(g(t))\right] > 0 \tag{2.12}$$

defined in a neighbourhood of infinity. Suppose that $g(t) > t$ for $t \geq t_0$. Then $y_1(t)$ is bounded. If, moreover,

$$1 < \lambda_* \leq a(t), \quad t \geq t_0 \tag{2.13}$$

for some positive constant λ_*, then $\lim_{t\to\infty} y_1(t) = 0$.

3. Main Results

Theorem 3.1. *Suppose that*

$$0 < a(t) \leq \lambda^* < 1, \quad \text{for some constant } \lambda^*, \quad t \geq t_0, \tag{3.1}$$

$$g(t) < h(t) < t \quad \text{for } t \geq t_0, \tag{3.2}$$

$$\alpha : [t_0, \infty) \longrightarrow \mathbb{R} \text{ is a continuous function}, \quad \alpha(t) < t, \quad \lim_{t\to\infty} \alpha(t) = \infty, \tag{3.3}$$

$$\int^{\infty} p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n \cdots \mathrm{d}x_1 = \infty, \tag{3.4}$$

$$\begin{aligned} &\limsup_{t\to\infty} K I_{l-2}(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}((*), \alpha(t);\ p_{n-1}, p_{n-2}, \ldots, p_{l-1})) \\ &\qquad \times \int_{h^{-1}(t)}^{\infty} p_n(x_n) \mathrm{d}x_n > 1 \end{aligned} \tag{3.5}$$

for $l = 3, 5, \ldots, n-2$,

$$\limsup_{t\to\infty} K I_{n-1}(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}) \int_{h^{-1}(t)}^{\infty} p_n(x_n) \mathrm{d}x_n > 1. \tag{3.6}$$

If n is odd and $\sigma = -1$, then every solution $y \in W$ to (1.1) *is oscillatory or* $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$.

Proof. Let $y \in W$ be a non-oscillatory solution to (1.1). The Expression (2.7) holds. Taking into account Remark 2.5, one may write

$$N = N_1^+ \cup N_3^+ \cup \cdots \cup N_n^+. \tag{3.7}$$

Without loss of generality we may suppose that $y_1(t)$ is positive for $t \geq t_2$.

(I) Let $y \in N_1^+$ on $[t_2, \infty)$. In this case, we can write for $t \geq t_2$

$$y_1(t) > 0, z_1(t) > 0, y_2(t) < 0, y_3(t) > 0, \ldots, y_n(t) > 0, \tag{3.8}$$

and $\lim_{t\to\infty} z_1(t) = L_1 \geq 0$. We claim that $L_1 = 0$. Otherwise $L_1 > 0$. Then

$$L_1 \leq z_1(h(t)) \leq y_1(h(t)) \quad \text{for } t \geq t_3, \tag{3.9}$$

where $t_3 \geq t_2$ is sufficiently large.

Integrating the last equation of (1.1) from x_{n-1} to x_{n-1}^*, we get for $x_{n-1} \geq t_3$

$$y_n(x_{n-1}) - y_n(x_{n-1}^*) = \int_{x_{n-1}}^{x_{n-1}^*} p_n(x_n) f(y_1(h(x_n))) \mathrm{d}x_n. \tag{3.10}$$

From (3.10) with regard to (e), (3.8), and (3.9), we have for $x_{n-1}^* \to \infty$

$$y_n(x_{n-1}) \geq KL_1 \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n, \quad x_{n-1} \geq t_3. \tag{3.11}$$

Multiplying (3.11) by $p_{n-1}(x_{n-1})$ and then using the $(n-1)$th equation of the system (1.1), we get for $x_{n-1} \geq t_3$

$$y_{n-1}'(x_{n-1}) \geq KL_1 p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n. \tag{3.12}$$

Integrating (3.12) from x_{n-2} to $x_{n-2}^* \to \infty$, and then using (3.8), we get for $x_{n-2} \geq t_3$

$$-y_{n-1}(x_{n-2}) \geq KL_1 \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n \, \mathrm{d}x_{n-1}. \tag{3.13}$$

Multiplying (3.13) by $p_{n-2}(x_{n-2})$ and then using the $(n-2)$th equation of the system (1.1), and the new inequality we integrate from x_{n-3} to $x_{n-3}^* \to \infty$ we employ (3.8) and for $x_{n-3} \geq t_3$

$$y_{n-2}(x_{n-3}) \geq KL_1 \int_{x_{n-3}}^{\infty} p_{n-2}(x_{n-2}) \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \, \mathrm{d}x_{n-2}. \tag{3.14}$$

Similarly for $x_1 \ge t_3$, we have

$$-z_1'(t) \ge KL_1 p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots p_{n-1}(x_{n-1}) \times \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n \mathrm{d}x_{n-1} \cdots \mathrm{d}x_2. \tag{3.15}$$

Integrating (3.15) from T to $T^* \to \infty$ and then using (3.8), we get for $T \ge t_3$

$$z_1(T) \ge KL_1 \int_{T}^{\infty} p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \cdots p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{\infty} p_n(x_n) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1, \tag{3.16}$$

which a contradiction to (3.4). Hence $\lim_{t\to\infty} z_1(t) = 0$.

Then $z_1(t) \le 1, t \ge t_4$, where $t_4 \ge t_3$ is sufficiently large and

$$y_1(t) \le a(t) y_1(g(t)) + 1 \le \lambda^* y_1(g(t)) + 1, \quad t \ge t_4. \tag{3.17}$$

We prove that $y_1(t)$ is bounded indirectly. Let $y_1(t)$ be unbounded. Then there exists a sequence $\{\bar{t}_n\}_{n=1}^{\infty}$, $\bar{t}_n \ge t_4$, where $n = 1, 2, \dots,$ $\bar{t}_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} y_1\left(\bar{t}_n\right) = \infty, \qquad y_1\left(\bar{t}_n\right) = \max_{t_4 \le s \le \bar{t}_n} y_1(s). \tag{3.18}$$

It follows from (3.1), (3.2), and (3.17),

$$y_1\left(\bar{t}_n\right) \le \lambda^* y_1\left(g\left(\bar{t}_n\right)\right) + 1 \le \lambda^* y_1\left(\bar{t}_n\right) + 1, \qquad y_1\left(\bar{t}_n\right) \le \frac{1}{1-\lambda^*}, \quad n = 1, 2, \dots. \tag{3.19}$$

That is a contradiction to $\lim_{n\to\infty} y_1(\bar{t}_n) = \infty$, and the function $y_1(t)$ is bounded. We claim that $\lim_{t\to\infty} y_1(t) = 0$ and prove it indirectly. Let $\limsup_{t\to\infty} y_1(t) = s > 0$. Let $\{t_n^*\}_{n=1}^{\infty}, t_n^* \ge t_4$, $n = 1, 2, \dots,$ be such a kind of sequence, that $t_n^* \to \infty$ as $n \to \infty$, and $\limsup_{n\to\infty} y_1(t_n^*) = s$. Then $\limsup_{n\to\infty} y_1(g(t_n^*)) \le s$. From (1.2) and (3.1),

$$z_1(t_n^*) \ge y_1(t_n^*) - \lambda^* y_1(g(t_n^*)), \quad n = 1, 2, \dots, \qquad y_1(g(t_n^*)) \ge \frac{y_1(t_n^*) - z_1(t_n^*)}{\lambda^*}, \quad n = 1, 2, \dots \tag{3.20}$$

follow.

From the last inequality, we have

$$s \ge \frac{s}{\lambda^*}, \quad \lambda^* \ge 1. \tag{3.21}$$

That is a contradiction to condition (3.1) and $\limsup_{t\to\infty} y_1(t) = 0 = \lim_{t\to\infty} y_1(t)$. Since $\lim_{t\to\infty} z_1(t) = L_1 = 0$ and from Lemma 2.4, implie $\lim_{t\to\infty} y_i(t) = 0$, $i = 2, 3, \ldots, n$.

(II) Let $y \in N_l^+$, for some $l = 3, 5, \ldots, n-2$, on $[t_2, \infty)$. In this case, one can consider for $t \geq t_2$

$$y_1(t) > 0, z_1(t) > 0,\ y_2(t) > 0,\ \ldots, y_l(t) > 0, y_{l+1}(t) < 0, \ldots, y_n(t) > 0. \tag{3.22}$$

Integrating the first equation of the system (1.1) from $\alpha(t)$ to t and using (3.22) above, we get

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) y_2(x_1) \mathrm{d}x_1, \quad t \geq t_3, \tag{3.23}$$

where $t_3 \geq t_2$ is sufficiently large. Integrating step by step 2nd, 3rd, $\ldots$, $(l-1)$th equations of the system (1.1) and subsequently substituting into (3.23) for $t \geq t_3$, we obtain

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) \int_{\alpha(t)}^{x_1} p_2(x_2) \cdots \int_{\alpha(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) y_l(x_{l-1}) \mathrm{d}x_{l-1}\, \mathrm{d}x_{l-2} \cdots \mathrm{d}x_1. \tag{3.24}$$

Integrating lth, $(l+1)$th, $\ldots$, $(n-1)$th equation of the system (1.1) and using (3.22), we have

$$\begin{aligned} y_l(x_{l-1}) &\geq -\int_{x_{l-1}}^{x_{l-2}} p_l(x_l) y_{l+1}(x_l) \mathrm{d}x_l, \\ -y_{l+1}(x_l) &\geq \int_{x_l}^{x_{l-2}} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) \mathrm{d}x_{l+1}, \\ y_{l+2}(x_{l+1}) &\geq -\int_{x_{l+1}}^{x_{l-2}} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) \mathrm{d}x_{l+2}, \\ &\vdots \\ -y_{n-1}(x_{n-2}) &\geq \int_{x_{n-2}}^{x_{l-2}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \mathrm{d}x_{n-1}. \end{aligned} \tag{3.25}$$

Combining expressions (3.24) and (3.25) and using (3.22), we get for $t \geq t_3$

$$\begin{aligned} z_1(t) \geq y_n(t) &\int_{\alpha(t)}^{t} p_1(x_1) \int_{\alpha(t)}^{x_1} p_2(x_2) \cdots \int_{\alpha(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) \int_{x_{l-1}}^{x_{l-2}} p_l(x_l) \\ &\times \int_{x_l}^{x_{l-2}} p_{l+1}(x_{l+1}) \cdots \int_{x_{n-2}}^{x_{l-2}} p_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-2} \cdots \mathrm{d}x_1. \end{aligned} \tag{3.26}$$

The formula above may be rewritten by (1.5) and (1.6) for $t \geq t_3$ to

$$z_1(t) \geq y_n(t) I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}\big)\big), \tag{3.27}$$

Integrating the last equation of (1.1) from $t \to t^* \to \infty$ and using (e), (1.2), and (3.22), we obtain for $t \geq t_4$ where $t_4 \geq t_3$ is sufficiently large,

$$y_n(t) \geq K \int_t^\infty p_n(x_n) z_1(h(x_n)) \mathrm{d}x_n. \tag{3.28}$$

From (3.2), (3.27), and (3.28) and the monotonicity of $z_1(h)$, we have

$$\begin{aligned}
z_1(t) &\geq K I_{l-2}\big(t, \alpha(t); p_1, p_2, \dots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, p_{n-2}, \dots, p_{l-1}\big)\big) \\
&\quad \times \int_t^\infty p_n(x_n) z_1(h(x_n)) \mathrm{d}x_n \\
&\geq z_1(t) K I_{l-2}\big(t, \alpha(t); p_1, p_2, \dots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, p_{n-2}, \dots, p_{l-1}\big)\big) \\
&\quad \times \int_{h^{-1}(t)}^\infty p_n(x_n) \mathrm{d}x_n, \\
1 &\geq K I_{l-2}\big(t, \alpha(t); p_1, p_2, \dots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, p_{n-2}, \dots, p_{l-1}\big)\big) \\
&\quad \times \int_{h^{-1}(t)}^\infty p_n(x_n) \mathrm{d}x_n
\end{aligned} \tag{3.29}$$

for $t \geq t_4$, which is a contradiction to (3.5), and it gives

$$N_3^+ \cup N_5^+ \cup \cdots \cup N_{n-2}^+ = \emptyset. \tag{3.30}$$

(III) Let $y \in N_n^+$ on $[t_2, \infty)$. In this case we consider for the components of solution $y(t)$ and for function z_1

$$z_1(t) > 0, \quad y_i(t) > 0, \quad i = 1, 2, \dots, n, \quad t \geq t_2. \tag{3.31}$$

Analogically as in the previous part of the proof,

$$z_1(t) \geq y_n(t) I_{n-1}\big(t, \alpha(t); p_1, p_2, \dots, p_{n-1}\big), \quad t \geq t_3, \tag{3.32}$$

holds and also (3.28), and for $t \geq t_3$

$$1 \geq K I_{n-1}\big(t, \alpha(t); p_1, p_2, \dots, p_{n-1}\big) \int_{h^{-1}(t)}^\infty p_n(x_n) \mathrm{d}x_n, \tag{3.33}$$

which is a contradiction to (3.6) and $N_n^+ = \emptyset$. □

Theorem 3.2. *Suppose that* (3.1)–(3.4) *are employed and* (3.5) *holds for* $l = 3, 5, \ldots, n-1$ *and*

$$\int_s^\infty p_n(x_n) \int_{h(s)}^{h(x_n)} p_1(x_1) \int_{h(s)}^{x_1} p_2(x_2) \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \mathrm{d}x_n = \infty \tag{3.34}$$

for s *sufficiently large.*

If n *is even and* $\sigma = 1$, *then every solution* $y \in W$ *to the system* (1.1) *is either oscillatory, or* $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$, *or* $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1, 2, \ldots, n$.

Proof. Let $y \in W$ be a non-oscillatory solution to (1.1). Expression (2.8) holds. Taking into account Remark 2.5,

$$N = N_1^+ \cup N_3^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+. \tag{3.35}$$

Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_2$.

(I) Let $y \in N_1^+$ on $[t_2, \infty)$. In this case, for $t \ge t_2$

$$y_1(t) > 0, \;\; z_1(t) > 0, y_2(t) < 0, \;\; y_3(t) > 0, \;\; y_4(t) < 0, \;\; \ldots, \;\; y_n(t) < 0. \tag{3.36}$$

We may choose analogical approach as in Theorem 3.1 part (I). Equation (3.9) holds and we replace (3.11) by inequality

$$-y_n(x_{n-1}) \ge K L_1 \int_{x_{n-1}}^\infty p_n(x_n) \mathrm{d}x_n, \qquad x_{n-1} \ge t_3. \tag{3.37}$$

Moreover (3.15) holds and similarly as in the proof of Theorem 3.1 case (I). We prove that $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$.

(II) Let $y \in N_l^+$ on $[t_2, \infty)$, for some $l = 3, 5, \ldots, n-1$. In this case, for $t \ge t_2$,

$$y_1(t) > 0, z_1(t) > 0, \;\; y_2(t) > 0, \;\; \ldots, \;\; y_l(t) > 0, y_{l+1}(t) < 0, \;\; \ldots, y_n(t) < 0. \tag{3.38}$$

The analogical approach as in Theorem 3.1 part (II) follows out.

Instead of inequality (3.27), we get for $t \ge t_3$

$$z_1(t) \ge -y_n(t) I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}\big)\big) \tag{3.39}$$

and instead of (3.28) for $t \ge t_4$

$$-y_n(t) \ge K \int_t^\infty p_n(x_n) z_1(h(x_n)) \mathrm{d}x_n, \tag{3.40}$$

and in the end we gain the contradiction to (3.5).

(III) Let $y \in N_n^+$ on $[t_2, \infty)$. In this case (3.31) holds. Integrating the last equation of the system (1.1) and on the basis of (3.31), (3.2), (e), and (1.2), we have

$$y_n(t) \geq K \int_s^t p_n(x_n) z_1(h(x_n)) \mathrm{d}x_n, \quad t \geq s \geq t_3, \tag{3.41}$$

where $t_3 \geq t_2$ is sufficiently large.

Integrating the first equation of the system (1.1) from $h(s)$ to $h(x_n)$ and employing (3.31), we obtain

$$z_1(h(x_n)) \geq \int_{h(s)}^{h(x_n)} p_1(x_1) y_2(x_1) \mathrm{d}x_1, \quad s \geq t_3. \tag{3.42}$$

Combining (3.41) and (3.42), we have for $t \geq s \geq t_3$

$$y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(t)} p_1(x_1) y_2(x_1) \mathrm{d}x_1 \, \mathrm{d}x_n. \tag{3.43}$$

Further consequently integrating the 2nd, 3 rd, ..., $(l-1)$th equations of the system (1.1) and step by step substituting into (3.43), we get for $t \geq s \geq t_3$

$$\begin{aligned} y_n(t) \geq K \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} p_1(x_1) \int_{h(s)}^{x_1} p_2(x_2) \\ \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \mathrm{d}x_{n-1} \, \mathrm{d}x_{n-2} \cdots \, \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}x_n. \end{aligned} \tag{3.44}$$

On basis of (3.31), for $x_{n-1} \geq t_3$

$$y_n(x_{n-1}) \geq C, \quad 0 < C = \text{const.}, \quad \text{for } x_{n-1} \geq t_3, \tag{3.45}$$

hold.

Combining (3.44) and (3.45) for $t \geq s \geq t_3$, we have

$$\begin{aligned} y_n(t) \geq KC \int_s^t p_n(x_n) \int_{h(s)}^{h(x_n)} p_1(x_1) \int_{h(s)}^{x_1} p_2(x_2) \\ \cdots \int_{h(s)}^{x_{n-2}} p_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \, \mathrm{d}x_{n-2} \cdots \mathrm{d}x_2 \, \mathrm{d}x_1 \, \mathrm{d}x_n. \end{aligned} \tag{3.46}$$

From the inequality above and relation (3.34), we obtain $\lim_{t\to\infty} y_n(t) = \infty$. Lemma 2.4 implies $\lim_{t\to\infty} z_1(t) = \infty$ and $\lim_{t\to\infty} y_i(t) = \infty$, $i = 2, 3, \ldots, n-1$. Since $z_1(t) < y_1(t)$ for $t \geq t_2$, so $\lim_{t\to\infty} y_1(t) = \infty$ and the final conclusion is $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 1, 2, \ldots, n$. □

Theorem 3.3. *Suppose that* (3.3) *holds and*

$$1 < \lambda^* \le a(t) \quad \text{for some constant } \lambda^*, \quad t \ge t_0, \tag{3.47}$$

$$t < g(t) < h(t) \quad \text{for } t \ge t_0, \tag{3.48}$$

$$\int^{\infty} p_1(x_1) \int_{x_1}^{\infty} p_2(x_2) \int_{x_2}^{\infty} p_3(x_3) \cdots \int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \times \int_{x_{n-1}}^{\infty} \frac{p_n(x_n)\mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1}{a\left(g^{-1}(h(x_n))\right)} = \infty, \tag{3.49}$$

$$\limsup_{t\to\infty} KI_{l-2}\left(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\left((*), \alpha(t); p_{n-1}, p_{n-2}, \ldots, p_{l-1}\right)\right) \times \int_t^{\infty} \frac{p_n x_n \mathrm{d}x_n}{a\left(g^{-1}(h(x_n))\right)} > 1, \tag{3.50}$$

for $l = 3, 5, \ldots, n-2$,

$$\limsup_{t\to\infty} KI_{n-1}\left(t, \alpha(t); p_1, p_2, \ldots, p_{n-1}\right) \int_t^{\infty} \frac{p_n(x_n)\mathrm{d}x_n}{a\left(g^{-1}(h(x_n))\right)} > 1. \tag{3.51}$$

If n is odd and $\sigma = 1$ then every solution $y \in W$ to (1.1) *is either oscillatory, or* $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$.

Proof. Let $y \in W$ be a non-oscillatory solution to (1.1). Expression (2.6) holds. Without loss of generality we may suppose that $y_1(t)$ is positive for $t \ge t_2$.

(I) Let $y \in N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+$ on $[t_2, \infty)$. Lemma 2.6 implies $\lim_{t\to\infty} y_1(t) = 0$. In this case, for $t \ge t_2$,

$$0 < z_1(t) < y_1(t), \tag{3.52}$$

and so $\lim_{t\to\infty} z_1(t) = 0$ which is a contradiction to the fact that the $z_1(t)$ is positive and a nondecreasing function on the interval $[t_2, \infty)$ and

$$N_2^+ \cup N_4^+ \cup \cdots \cup N_{n-1}^+ \cup N_n^+ = \emptyset. \tag{3.53}$$

(II) Let $y \in N_1^-$ on $[t_2, \infty)$. In this case, we can write for $t \ge t_2$

$$y_1(t) > 0, z_1(t) < 0, \ y_2(t) > 0, \ y_3(t) < 0, \ldots, \ y_n(t) < 0. \tag{3.54}$$

We indirectly prove $\lim_{t\to\infty} z_1(t) = 0$.

Since $z_1(t)$ is nondecreasing $\lim_{t\to\infty} z_1(t) = -L_1$, $L_1 > 0$, $L_1 = \text{const.}$, and

$$z_1(t) \le -L_1 \quad \text{for } t \ge t_2. \tag{3.55}$$

Because $z_1(t) > -a(t)y_1(g(t))$,

$$z_1\left(g^{-1}(h(t))\right) > -a\left(g^{-1}(h(t))\right)y_1(h(t)), \tag{3.56}$$

$$-y_1(h(t)) < \frac{z_1(g^{-1}(h(t)))}{a(g^{-1}(h(t)))}, \quad t \geq t_2 \tag{3.57}$$

follows.

From (3.55) and (3.57), we get

$$-L_1 \geq z_1\left(g^{-1}(h(x_n))\right) \geq -a\left(g^{-1}(h(x_n))\right)y_1(h(x_n)), \quad x_n > t_2. \tag{3.58}$$

By (c), (e), the last equation of (1.1), and (3.58), we get for $x_n > t_2$

$$\frac{KL_1 p_n(x_n)}{a(g^{-1}(h(x_n)))} \leq Kp_n(x_n)y_1(h(x_n)) \leq p_n(x_n)f\left(y_1(h(x_n))\right) = y_n'(x_n). \tag{3.59}$$

Integrating (3.59) from x_{n-1} to $x_{n-1}^* \to \infty$, we get

$$KL_1\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)\mathrm{d}x_n}{a(g^{-1}(h(x_n)))} \leq -y_n(x_{n-1}) \quad \text{for } x_{n-1} \geq t_2. \tag{3.60}$$

Multiplying (3.60) by $p_{n-1}(x_{n-1})$ and then using the $(n-1)$th equation of system (1.1), we get for $x_{n-1} \geq t_2$

$$KL_1 p_{n-1}(x_{n-1})\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)\mathrm{d}x_n}{a(g^{-1}(h(x_n)))} \leq -y_{n-1}(x_{n-1}). \tag{3.61}$$

Integrating (3.61) from x_{n-2} to $x_{n-2}^* \to \infty$, we get for $x_{n-2} \geq t_2$

$$KL_1\int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1})\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)\mathrm{d}x_n\mathrm{d}x_{n-1}}{a(g^{-1}(h(x_n)))} \leq y_{n-1}(x_{n-2}). \tag{3.62}$$

Similarly we continue by the same way until we derive for $x_1 \geq t_2$

$$\begin{aligned} &KL_1 p_1(x_1)\int_{x_1}^{\infty} p_2(x_2)\int_{x_2}^{\infty} p_3(x_3)\cdots\int_{x_{n-2}}^{\infty} p_{n-1}(x_{n-1}) \\ &\quad\times\int_{x_{n-1}}^{\infty} \frac{p_n(x_n)\mathrm{d}x_n\mathrm{d}x_{n-1}\cdots\mathrm{d}x_2}{a(g^{-1}(h(x_n)))} \leq z_1'(x_1). \end{aligned} \tag{3.63}$$

Integrating (3.63) from T to $T^* \longrightarrow \infty$, we get for $T \geq t_2$

$$\begin{aligned} &KL_1 \int_T^\infty p_1(x_1) \int_{x_1}^\infty p_2(x_2) \int_{x_2}^\infty p_3(x_3) \cdots \int_{x_{n-2}}^\infty p_{n-1}(x_{n-1}) \\ &\quad \times \int_{x_{n-1}}^\infty \frac{p_n(x_n) \mathrm{d}x_n \mathrm{d}x_{n-1} \cdots \mathrm{d}x_2 \mathrm{d}x_1}{a(g^{-1}(h(x_n)))} \leq -z_1(T). \end{aligned} \tag{3.64}$$

That contradicts (3.49), and consequently $\lim_{t\to\infty} z_1(t) = 0$ holds.

We prove that $y_1(t)$ is bounded and $\lim_{t\to\infty} y_1(t) = 0$. There is some positive constant $B > 0$, $z_1(t) \geq -B$ for $t \geq t_2$, and by (1.2) and (3.47), one has for $t \geq t_2$

$$y_1(t) = a(t)y_1(g(t)) + z_1(t) \geq a(t)y_1(g(t)) - B \geq \lambda^* y_1(g(t)) - B. \tag{3.65}$$

We prove indirectly that $y_1(t)$ is bounded. Let us suppose that $y_1(t)$ is unbounded. Then $y_1(g(t))$ is unbounded, and there is a sequence

$$\begin{gathered} \left\{\bar{t}_n\right\}_{n=1}^\infty, \quad \bar{t}_n \geq t_2, \quad n = 1, 2, \ldots, \quad \bar{t}_n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \\ \lim_{n\to\infty} y_1\left(\bar{t}_n\right) = \infty, \qquad y_1\left(g\left(\bar{t}_n\right)\right) = \max_{t_2 \leq s \leq g(\bar{t}_n)} y_1(s). \end{gathered} \tag{3.66}$$

By (3.65)

$$\begin{gathered} \lambda^* y_1\left(g\left(\bar{t}_n\right)\right) \leq y_1\left(\bar{t}_n\right) + B \leq y_1\left(g\left(\bar{t}_n\right)\right) + B, \\ y_1\left(g\left(\bar{t}_n\right)\right) \leq \frac{B}{\lambda^* - 1}, \quad n = 1, 2, \ldots. \end{gathered} \tag{3.67}$$

That is a contradiction to $\lim_{n\to\infty} y_1(g(\bar{t}_n)) = \infty$, and the function $y_1(t)$ is bounded. We claim that $\lim_{t\to\infty} y_1(t) = 0$, and we will prove it indirectly.

Let $\limsup_{t\to\infty} y_1(g(t)) = s$, $0 < s$, $s = \text{const}$. Then $\limsup_{t\to\infty} y_1(t) = s$.

Let $\{t_n^*\}_{n=1}^\infty$, $t_n^* \geq t_2$, $n = 1, 2, \ldots,$ be such a kind of sequence that $\lim_{n\to\infty} t_n^* = \infty$ and $\limsup_{n\to\infty} y_1(g(t_n^*)) = s$.

Then, $\limsup_{n\to\infty} y_1(t_n^*) \leq s$.

By (1.2) and (3.47),

$$\begin{gathered} z_1(t_n^*) \leq y_1(t_n^*) - \lambda^* y_1(g(t_n^*)), \quad n = 1, 2, \ldots, \\ y_1(g(t_n^*)) \leq \frac{y_1(t_n^*) - z_1(t_n^*)}{\lambda^*}, \quad n = 1, 2, \ldots, \end{gathered} \tag{3.68}$$

follows.

By the last inequality, we have

$$s = \limsup_{t\to\infty} y_1(g(t_n^*)) \leq \frac{\limsup_{t\to\infty} y_1(t_n^*)}{\lambda^*} \leq \frac{s}{\lambda^*}. \tag{3.69}$$

$1 \geq \lambda^*$ holds. That is a contradiction to (3.47). It means $\limsup_{t\to\infty} y_1(g(t)) = 0$ and also $\limsup_{t\to\infty} y_1(t) = 0$. Moreover, $y_1(t) > 0$ holds, so $\liminf_{t\to\infty} \lim_{t\to\infty} y_1(t) = 0$ and this leads to $\lim_{t\to\infty} y_1(t) = 0$.

By Lemma 2.4 it follows that

$$\lim_{t\to\infty} y_i(t) = 0, \quad i = 2, 3, \ldots, n. \tag{3.70}$$

(III) Let $y \in N_l^-$, $l = 3, 5, \ldots, n-2$, on $[t_2, \infty)$. In this case for, $t \geq t_2$,

$$y_1(t) > 0,\ z_1(t) < 0, y_2(t) < 0,\ \ldots,\ y_l(t) < 0,\ y_{l+1}(t) > 0, \ldots,\ y_n(t) < 0. \tag{3.71}$$

Integrating the first equation of (1.1) from $\alpha(t)$ to t and using (3.71), we get

$$z_1(t) \geq \int_{\alpha(t)}^{t} p_1(x_1) y_2(x_1) \mathrm{d}x_1, \quad t \geq t_3, \tag{3.72}$$

where $t_3 \geq t_2$ is sufficiently large.

Integrating the 2nd, 3 rd, $\ldots$, $(l-1)$th equations of the system (1.1), and substituting into (3.72), we get for $t \geq t_3$

$$z_1(t) \leq \int_{\alpha(t)}^{t} p_1(x_1) \int_{\alpha(t)}^{x_1} p_2(x_2) \cdots \int_{\alpha(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) y_l(x_{l-1}) \mathrm{d}x_{l-1}\, \mathrm{d}x_{l-2} \cdots \mathrm{d}x_1. \tag{3.73}$$

Integrating lth, $(l+1)$th, $\ldots$, $(n-1)$th equations of the system (1.1) we gain the syste

$$\begin{aligned}
y_l(x_{l-1}) &\leq -\int_{x_{l-1}}^{x_{l-2}} p_l(x_l) y_{l+1}(x_l) \mathrm{d}x_l, \\
-y_{l+1}(x_l) &\leq \int_{x_l}^{x_{l-2}} p_{l+1}(x_{l+1}) y_{l+2}(x_{l+1}) \mathrm{d}x_{l+1}, \\
y_{l+2}(x_{l+1}) &\leq -\int_{x_{l+1}}^{x_{l-2}} p_{l+2}(x_{l+2}) y_{l+3}(x_{l+2}) \mathrm{d}x_{l+2}, \\
&\ \vdots \\
-y_{n-1}(x_{n-2}) &\leq \int_{x_{n-2}}^{x_{l-2}} p_{n-1}(x_{n-1}) y_n(x_{n-1}) \mathrm{d}x_{n-1}.
\end{aligned} \tag{3.74}$$

We combine the formulae (3.73) and (3.74), and with regard to (3.71), we get for $t \geq t_3$

$$\begin{aligned} z_1(t) \leq y_n(t) \int_{\alpha(t)}^{t} p_1(x_1) \int_{\alpha(t)}^{x_1} p_2(x_2) \cdots \int_{\alpha(t)}^{x_{l-2}} p_{l-1}(x_{l-1}) \int_{x_{l-1}}^{x_{l-2}} p_l(x_l) \\ \times \int_{x_l}^{x_{l-2}} p_{l+1}(x_{l+1}) \cdots \int_{x_{n-2}}^{x_{l-2}} p_{l-1}(x_{l-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-2} \cdots \mathrm{d}x_1. \end{aligned} \tag{3.75}$$

Employing (1.5) and (1.6) the equation above may be rewritten to

$$z_1(t) \leq y_n(t) I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, \ldots, p_{l-1}\big)\big) \tag{3.76}$$

for $t \geq t_3$.

Integrating the last equation of (1.1) from t to $t^* \to \infty$ and using (e) and (3.71),

$$y_n(t) \leq -K \int_t^{\infty} p_n(x_n) y_1(h(x_n)) \mathrm{d}x_n, \quad t \geq t_3. \tag{3.77}$$

From (3.2), (3.57) in regard to (3.76), (3.77) and monotonicity of $z_1(g^{-1}(h))$, we get for $t \geq t_3$

$$\begin{aligned} z_1(t) &\leq K I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, \ldots, p_{l-1}\big)\big) \\ &\quad \times \int_t^{\infty} \frac{p_n(x_n) z_1\big(g^{-1}(h(x_n))\big) \mathrm{d}x_n}{a\big(g^{-1}(h(x_n))\big)} \\ &\leq z_1(t) K I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, \ldots, p_{l-1}\big)\big) \\ &\quad \times \int_t^{\infty} \frac{p_n(x_n) \mathrm{d}x_n}{a\big(g^{-1}(h(x_n))\big)}, \end{aligned} \tag{3.78}$$

which means for $t \geq t_3$

$$\begin{aligned} 1 \geq K I_{l-2}\big(t, \alpha(t); p_1, p_2, \ldots, p_{l-2}(*) \times J_{n-l+1}\big((*), \alpha(t); p_{n-1}, \ldots, p_{l-1}\big)\big) \\ \times \int_t^{\infty} \frac{p_n(x_n) \mathrm{d}x_n}{a\big(g^{-1}(h(x_n))\big)}. \end{aligned} \tag{3.79}$$

This is a contradiction to (3.50) and

$$N_3^- \cup N_5^- \cup \cdots \cup N_{n-2}^- = \emptyset. \tag{3.80}$$

(IV) Let $y \in N_n^-$, on $[t_2, \infty)$.

In this case, we can write for $t \geq t_2$

$$y_1(t) > 0, \quad z_1(t) < 0, \quad y_i(t) < 0, \quad i = 2, 3, \ldots, n. \tag{3.81}$$

We may lead the proof analogically as in the previous part of the proof and we will prove that (3.77), (3.57), and

$$z_1(t) \le y_n(t) I_{n-1}\big(t, \alpha(t); p_1, p_2, \dots, p_{n-1}\big) \tag{3.82}$$

hold and also

$$1 \ge K I_{n-1}\big(t, \alpha(t); p_1, p_2, \dots, p_{n-1}\big) \int_t^{\infty} \frac{p_n(x_n)\,\mathrm{d}x_n}{a\big(g^{-1}(h(x_n))\big)}, \qquad t \ge t_3 \tag{3.83}$$

which is a contradiction to (3.51) and $N_n^- = \emptyset$. □

Theorem 3.4. *Suppose that* (3.3), (3.47)–(3.49) *hold and condition* (3.50) *is fulfilled for* $l = 3, 5, \dots, n-1$, *and*

$$\begin{aligned} &\int_s^{\infty} \frac{p_n(x_n)}{a\big(g^{-1}(h(x_n))\big)} \int_{g^{-1}(h(s))}^{g^{-1}(h(x_n))} p_1(x_1) \int_{g^{-1}(h(s))}^{x_1} p_2(x_2) \\ &\quad \cdots \int_{g^{-1}(h(s))}^{x_{n-2}} p_{n-1}(x_{n-1})\,\mathrm{d}x_{n-1}\,\mathrm{d}x_{n-2} \cdots \mathrm{d}x_1\,\mathrm{d}x_n = \infty \end{aligned} \tag{3.84}$$

for $s \ge t_0$.

If n *is even and* $\sigma = -1$, *then every solution* $y \in W$ *to* (1.1) *is either oscillatory, or* $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \dots, n$, *or* $\lim_{t\to\infty} |z_1(t)| = \infty$ *and* $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 2, \dots, n$.

Proof. Let $y \in W$ be a non-oscillatory solution to (1.1). Expression (2.9) holds.

(I) Let $y \in N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+$. Analogically as in the proof of Theorem 3.3 (I), we prove that

$$N_2^+ \cup N_4^+ \cup \cdots \cup N_n^+ = \emptyset. \tag{3.85}$$

(II) Let $y \in N_1^-$ on $[t_2, \infty)$. Similarly to the proof of Theorem 3.3 (II), we prove $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2, \dots, n$.

(III) Let $y \in N_l^-$, for some $l = 3, 5, \dots, n-1$, for $t \in [t_2, \infty)$. Likewise as proof of Theorem 3.3 (III), for sets N_l^- we prove that $N_3^- \cup N_5^-, \dots, N_{n-1}^- = \emptyset$.

(IV) Let $y \in N_n^-$ for $t \in [t_2, \infty)$. Analogically to the proof of case (III) of Theorem 3.2, we claim $\lim_{t\to\infty} |z_1(t)| = \infty$, $\lim_{t\to\infty} |y_i(t)| = \infty$, $i = 2, \dots, n$. □

Example 3.5. We consider system (1.1) as follows:

$$\begin{aligned}
\left(y_1(t) - \frac{1}{2}y_1\left(\frac{t}{4}\right)\right)' &= e^{\frac{t}{2}} y_2(t), \\
y_2'(t) &= \frac{1}{2}e^{\frac{t}{4}} y_3(t), \\
y_3'(t) &= \frac{1}{2}e^{\frac{t}{8}} y_4(t), \\
y_4'(t) &= \frac{1}{16}\left(e^{-3t/8} + \frac{5}{8}e^{-9t/8}\right) y_1\left(\frac{t}{2}\right), \quad t \geq 1.
\end{aligned} \tag{3.86}$$

All assumptions of Theorem 3.2 are satisfied, and every solution $y \in W$ to (3.86) is either oscillatory or

$$\lim_{t\to\infty} y_i(t) = 0, \quad i = 1,2,3,4, \quad \text{or} \quad \lim_{t\to\infty} |y_i(t)| = \infty, \quad i = 1,2,3,4. \tag{3.87}$$

One of the solutions has particular components as follows:

$$\begin{aligned}
y_1(t) = e^t, \qquad y_2(t) = e^{t/2} - \frac{1}{8}e^{-t/4}, \\
y_3(t) = e^{t/4} + \frac{1}{16}e^{-t/2}, \quad y_4(t) = \frac{1}{2}\left(e^{t/8} - \frac{1}{8}e^{-5t/8}\right), \quad t \geq 1,
\end{aligned} \tag{3.88}$$

and in this case

$$\lim_{t\to\infty} y_i(t) = \infty, \quad i = 1,2,3,4. \tag{3.89}$$

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