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Research Article

Wave Breaking and Propagation Speed for a Class of One-Dimensional Shallow Water Equations

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We investigate a more general family of one-dimensional shallow water equations. Analogous to the Camassa-Holm equation, these new equations admit blow-up phenomenon and infinite propagation speed. First, we establish blow-up results for this family of equations under various classes of initial data. It turns out that it is the shape instead of the size and smoothness of the initial data which influences breakdown in finite time. Then, infinite propagation speed for the shallow water equations is proved in the following sense: the corresponding solution $u(t,x)$ with compactly supported initial datum $u_0(x)$ does not have compact x-support any longer in its lifespan.

1. Introduction

In this paper we consider the equation

$$u_t - u_{xxt} + (a+b)uu_x = au_xu_{xx} + buu_{xxx}, \tag{1.1}$$

where $a > 0$ and $b > 0$ are real constants.

Set $\Lambda = (1-\partial_x^2)^{1/2}$, then we can rewrite (1.1) as

$$u_t + buu_x + \partial_x\Lambda^{-2}\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) = 0. \tag{1.2}$$

Let $y(t,x) = u(t,x) - u_{xx}(t,x)$, then (1.1) can be reformulated in terms of $y(t,x)$:

$$y_t + au_xy + buy_x = 0. \tag{1.3}$$

By comparison with the Camassa-Holm equation [1], the Degasperis-Procesi equation, [2] and the Holm-Staley b-family of equations [3], it is easy to find that (1.1) is more general. The Camassa-Holm equation, the Degasperis-Procesi equation, and the Holm-Staley b-family of equations are the special cases with $a = 2$, $b = 1$; $a = 3$, $b = 1$, and $b = 1$, respectively. The equation of type (1.1) arises in the modeling of shallow water waves; compare with the discussion in the papers [4, 5].

When $a = 2b$, use the scaling $\tilde{u}(t,x) = bu(t,x)$, then (1.1) can be reformulated into

$$\tilde{u}_t - \tilde{u}_{xxt} + 3\tilde{u}\tilde{u}_x = 2\tilde{u}_x\tilde{u}_{xx} + \tilde{u}\tilde{u}_{xxx}, \tag{1.4}$$

which is the well-known Camassa-Holm equation. The Camassa-Holm equation was first written explicitly and derived physically as a water wave equation by Camassa and Holm [1], who also studied its solutions. It has infinitely many conserved integrals including the H^1-norm, but wave breaking also happens in finite time, which coincides with physical phenomenon. Some satisfactory results have been obtained recently [6–8] for strong solutions. Moreover, wave breaking for a large class of initial data has been established in [9–11], and recently, a new and direct proof for McKean's theorem is given in [12]. In [13] some new criterion on blowup is established for the Camassa-Holm equation with weakly dissipative term. In [14] (see also [15]), Xin and Zhang showed global existence and uniqueness for weak solutions with $u_0(x) \in H^s$. The solitary waves of the Camassa-Holm equation are peaked solitons and are orbitally stable [16] (see also [17, 18]). It is worthwhile to point out that the peakons replicate a feature that is characteristic for the waves of great height waves of largest amplitude that are exact solutions of the governing equations for water waves; compare with the papers [19–21]. Similarly, if $a = 3b$, use the some scaling, then (1.1) can be reformulated into the Degasperis-Procesi equation, the existence of global solutions, persistence properties and propagation speed of Degasperis-Procesi equation is given in [22] and the references therein. For the the weakly dissipative Degasperis-Procesi equation, [23] is concerned with some aspects of existence of global solutions, persistence properties, and propagation speed. So, in our following discussion, we will always exclude these two cases.

In general, the family of equations are not always completely integrable systems. However, one can find the following conservation laws:

$$H_0 = \int_{\mathbb{R}} y\,dx, \qquad H_1 = \int_{\mathbb{R}} y^{b/a}\,dx, \qquad H_2 = \int_{\mathbb{R}} y_x^2 y^{-2-(b/a)} + \frac{a^2}{b} y^{-b/a}\,dx, \tag{1.5}$$

which are quite different from the invariants of the Camassa-Holm equation

$$H_0 = \int_{\mathbb{R}} y\,dx, \qquad H_1 = \int_{\mathbb{R}} u^2 + u_x^2\,dx, \qquad H_2 = \int_{\mathbb{R}} u^3 + uu_x^2\,dx. \tag{1.6}$$

Due to the similarity of (1.2) and the Camassa-Holm equation, just by following the argument for the Camassa-Holm equation, it is easy to establish the following well-posedness theorem for (1.2).

Theorem 1.1. *Given* $u_0(x) \in H^s(\mathbb{R})$, $s > 3/2$, *then there exist a* T *and a unique solution* u *to* (1.1) *such that*

$$u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1\Big([0,T); H^{s-1}(\mathbb{R})\Big). \tag{1.7}$$

To make the paper concise we would like to omit the detailed proof, since one can find similar ones for these types of equations in [7, 24].

When we study the Camassa-Holm equation, the most frequently (crucially) used conservation law is that of the H^1-norm of the solution. However, if u is a strong solution (decays rapidly at infinity) to (1.1), direct computation yields

$$\frac{d}{dt}\int_{\mathbb{R}} u^2 + u_x^2\,dx = (2b-a)\int_{\mathbb{R}} u_x^3\,dx. \tag{1.8}$$

Hence, the H^1-norm of the solution is not conserved at all except for $a = 2b$. However, we also have a clear blow-up scenario as follows: the solution blows up if and only if the first-order derivative blows up that is, wave breaking occurs. More precisely, assume that T is the lifespan of the corresponding solution, then

$$\limsup_{t\to T}\|u_x(t,x)\|_{L^\infty(\mathbb{R})} = +\infty. \tag{1.9}$$

Before giving our main results, we list some notations that will be used in our paper. $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $L^2(\mathbb{R})$, $(f,g)_r \doteq (\Lambda^r f, \Lambda^r g)$; use $\|\cdot\|_{L^s(\mathbb{R})}$ for the corresponding $L^s(\mathbb{R})$ norm and $\|\cdot\|_{H^s(\mathbb{R})}$ to denote the corresponding $H^s(\mathbb{R})$ norm.

The remainder of the paper will be organized as follows. In the next section, we establish blow-up results for this family of equations under various classes of initial data. In the last section, infinite propagation speed for the shallow water equations is proved.

2. Blow-Up Phenomenon

After local well posedness of strong solutions is established, the next question is whether this local solution can exist globally. If the solution exists only for a finite time, how about the behavior of the solution when it blows up? What induces the blowup? On the other hand, to find sufficient conditions to guarantee the finite time blowup or global existence is of great interest, particularly for sufficient conditions added on the initial data.

Set $p(t,x)$ as the characteristic evolved by the solution; that is, it satisfies

$$\begin{aligned} p_t(t,x) &= bu(t,p(t,x)), \\ p(0,x) &= x. \end{aligned} \tag{2.1}$$

Taking derivative with respect to x in (2.1), we obtain

$$\frac{dp_x}{dt} = bu_x(p,t)p_x. \tag{2.2}$$

Hence

$$p_x(t,x) = e^{b\int_0^t u_x(p(\tau,x),\tau)d\tau}, \tag{2.3}$$

which is always positive before the blow-up time. Therefore, the function $p(t,x)$ is an increasing diffeomorphism of the line before blowup. In addition, from (1.1), the following identity is proved:

$$y(t,p(t,x))p_x^{a/b}(t,x) = y_0(x). \tag{2.4}$$

The following theorem is proved in [25] it is one of sufficient conditions to guarantee the finite time blowup added on the initial data.

Theorem 2.1. *Let $a > 0$, $b > 0$, $a > b$, $3b - a > 0$, $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ being odd and $u_0'(0) < 0$. Then the corresponding solution of (1.1) blows up in finite time.*

From the above blow-up result we can obtain the following.

Theorem 2.2. *Let $a > 0$, $b > 0$, $a > b$, $3b - a > 0$, $u_0 \in H^3(\mathbb{R})$. Suppose that y_0 is odd and $\int_0^\infty e^{-\xi}y_0(\xi) \le 0$. Then the corresponding solution of (1.1) blows up in finite time.*

Proof. If $y_0(x)$ is odd, then $u_0(x)$ is also odd. In fact

$$\begin{aligned} u_0(x) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}y_0(z)dz \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(-y_0(-z))dz \\ &= -\frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}y_0(z)dz \\ &= -u_0(-x). \end{aligned} \tag{2.5}$$

On the other hand

$$\begin{aligned} u_0'(0) &= -\frac{1}{2}\int_{-\infty}^0 e^{\xi}y_0(\xi)d\xi + \frac{1}{2}\int_{-\infty}^0 e^{-\xi}y_0(\xi)d\xi \\ &= \int_0^\infty e^{-\xi}y_0(\xi)d\xi < 0. \end{aligned} \tag{2.6}$$

So Theorem 2.2, follows from Theorem 2.1. □

Before giving our main result, we first recall the following three lemmas that will be used in our proof.

Lemma 2.3. *If $s > 0$, then $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is algebra. Moreover*

$$\|fg\|_{H^s(\mathbb{R})} \leq C\Big(\|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}\|f\|_{H^s(\mathbb{R})}\Big). \tag{2.7}$$

Lemma 2.4. *If $s > 0$, then*

$$\|[\Lambda^s, f]g\|_{L^2(\mathbb{R})} \leq C\left(\|\partial_x f\|_{L^\infty(\mathbb{R})}\left\|\Lambda^{s-1}g\right\|_{L^2(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}\|\Lambda^s f\|_{L^2(\mathbb{R})}\right). \tag{2.8}$$

Here $[A, B] = AB - BA$ denotes the commutator of the linear operators A and B.

Lemma 2.5 (see [26]). *Suppose that $\Psi(t)$ is twice continuously differentiable satisfying*

$$\begin{gathered} \Psi''(t) \geq C_0\Psi'(t)\Psi(t), \quad t > 0,\ C_0 > 0, \\ \Psi(0) > 0, \qquad \Psi'(0) > 0. \end{gathered} \tag{2.9}$$

Then $\Psi(t)$ blows up in finite time. Moreover the blow-up time T can be estimated in terms of the initial datum as

$$T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}. \tag{2.10}$$

Now, we are ready to give our main results. Firstly, we present the result which improves Theorem 2.1.

Theorem 2.6. *Let $u_0 \in H^r(\mathbb{R})$ with $r > 3/2$, and assume that T is the existence time of the corresponding solution with the initial data u_0. If there exists $M > 0$ such that*

$$\|u_x(t,x)\|_{L^\infty(\mathbb{R})} \leq M, \quad t \in [0, T), \tag{2.11}$$

then the $H^r(\mathbb{R})$ norm of $u(t,x)$ does not blow up on $[0,T)$.

Proof. Let u be the solution to (1.1) with initial data $u_0 \in H^r(\mathbb{R})$ and let T be the maximal existence time of the solution u. Applying the operator Λ^r to (1.2), multiplying by $\Lambda^r u$, and integrating, we obtain

$$\frac{d}{dt}\|u\|^2_{H^r(\mathbb{R})} = -2b(u, uu_x)_r + 2\big(u, f(u)\big)_r + 2\big(u, g(u)\big)_r, \tag{2.12}$$

where $f(u) = -a(1-\partial_x^2)^{-1}(uu_x)$ and $g(u) = -\partial_x(1-\partial_x^2)^{-1}(((3b-a)/2)u_x^2)$. We have

$$\begin{aligned}
|(uu_x, u)_r| &= |(\Lambda^r(uu_x), \Lambda^r u)| \\
&= |([\Lambda^r, u]\partial_x u, \Lambda^r u) + (u\Lambda^r \partial_x u, \Lambda^r u)| \\
&= \|[\Lambda^r, u]\partial_x u\|_{L^2(\mathbb{R})}\|\Lambda^r u\|_{L^2(\mathbb{R})} + \frac{1}{2}|(u_x\Lambda^r u, \Lambda^r u)| \\
&\leq C\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^2_{H^r(\mathbb{R})},
\end{aligned} \tag{2.13}$$

where we applied Lemma 2.4 with $s = r$. Again applying Lemma 2.4 with $s = r-1$, we obtain

$$\begin{aligned}
\left|(f(u), u)_r\right| &= \left|-a\left(\left(1-\partial_x^2\right)^{-1} uu_x, u\right)_r\right| \\
&= |a|\left|\left(\Lambda^{r-1}(uu_x), \Lambda^{r-1}u\right)\right| = |a|\left|\left(\left[\Lambda^{r-1}, u\right]\partial_x u, \Lambda^{r-1}u\right) + \left(u\Lambda^{r-1}\partial_x u, \Lambda^{r-1}u\right)\right| \\
&\leq C\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^2_{H^r(\mathbb{R})},
\end{aligned} \tag{2.14}$$

and, from Lemma 2.3, we have

$$\begin{aligned}
\left|(g(u), u)_r\right| &\leq \left\|g(u)\right\|_{H^r(\mathbb{R})}\|u\|_{H^r(\mathbb{R})} \\
&= \left\|-\partial_x\left(1-\partial_x^2\right)^{-1}\left(\frac{3b-a}{2}u_x^2\right)\right\|_{H^r(\mathbb{R})}\|u\|_{H^r(\mathbb{R})} \\
&\leq C\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^2_{H^r(\mathbb{R})}.
\end{aligned} \tag{2.15}$$

Combining the above estimates, we obtain

$$\frac{d}{dt}\|u\|^2_{H^r(\mathbb{R})} \leq CM\|u\|^2_{H^r(\mathbb{R})}. \tag{2.16}$$

From Gronwall inequality, we obtain

$$\|u(t)\|^2_{H^r(\mathbb{R})} \leq e^{CMt}\|u(0)\|^2_{H^r(\mathbb{R})}. \tag{2.17}$$

This completes the proof of the theorem. □

In order to help the readers to understand the following theorem, we would like to give an intuitive explanation for this result. Neglecting lower-order source terms, (1.2) reduces to the generalized Hunter-Saxton equation:

$$u_t + buu_x = \frac{3a-b}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)u_x^2 dy. \tag{2.18}$$

Differentiating with respect to x, one obtains

$$(u_x)_t + buu_{xx} = (2b - a)u_x^2. \tag{2.19}$$

By looking at the evolution of u_x along characteristics, it is clear that one can have $u_x \to -\infty$ or $u_x \to +\infty$ depending on the sign of $2b - a$.

Theorem 2.7. *Let $u_0 \in H^r(\mathbb{R})$ with $r > 3/2$. If $a = b/2$, then every solution will exist globally in time. If $a > b/2$, then the solution blows up in finite time if and only if the slope of the solution becomes unbounded from below in finite time. If $a < b/2$, then the solution blows up in finite time if and only if the slope of the solution becomes unbounded from above in finite time.*

Proof. It suffices to consider the case $r = 3$. Let $T > 0$ be the maximal time of existence of the solution of (1.1). From [25] we know that $u \in C([0,T); H^3(\mathbb{R})) \cap C^1([0,T); H^2(\mathbb{R}))$. Multiplying (1.2) by $y = u - u_{xx}$ and integrating by parts, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} y^2 dx &= -a\int_{\mathbb{R}} y^2 u_x dx - b\int_{\mathbb{R}} uyy_x dx \\ &= \left(-a + \frac{b}{2}\right)\int_{\mathbb{R}} y^2 u_x dx. \end{aligned} \tag{2.20}$$

Note that

$$\|u\|_{H^2(\mathbb{R})} \le C_1 \|y\|_{L^2(\mathbb{R})} \le C_2 \|u\|_{H^2(\mathbb{R})}. \tag{2.21}$$

From (2.20), for $a = b/2$, we have

$$\|u_x\|_{L^\infty(\mathbb{R})} \le \|u\|_{H^2(\mathbb{R})} \le C_1 \|y(t)\|_{L^2(\mathbb{R})} = C_1 \|y(0)\|_{L^2(\mathbb{R})}. \tag{2.22}$$

From Theorem 2.6, for $a = b/2$ the solution exists globally in time. If $a > b/2$ and the slope of the solution is bounded from below or if $a < b/2$ and the slope of the solution is bounded from above on $[0, T)$, then there exists $M > 0$ such that

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx \le M\int_{\mathbb{R}} y^2 dx. \tag{2.23}$$

By Gronwall inequality, we have

$$\|y(t)\|_{L^2(\mathbb{R})} \le e^{Mt} \|y(0)\|_{L^2(\mathbb{R})}. \tag{2.24}$$

□

There is no doubt that it is the shape of the initial data but not their smoothness or size that influences the lifespan. However, we try to understand what induces wave breaking. Another sufficient condition is given in the following theorem. The approach used to prove the following theorem is rooted in the considerations made in the paper [27].

Theorem 2.8. *Let $a-2b>0$, $b>0$. Suppose that $u_0 \in H^2(\mathbb{R})$ and there exists $x_0 \in \mathbb{R}$ such that $y_0(x_0)=(1-\partial_x^2)u_0(x_0)=0$, and*

$$y_0(x) \geq 0 \quad \text{for } x \in (-\infty, x_0), \qquad y_0(x) \leq 0 \quad \text{for } x \in (x_0, \infty). \tag{2.25}$$

Then the corresponding solution $u(t,x)$ of (1.1) *blows up in finite time with the lifespan*

$$T \leq \max\left\{ \frac{-2}{bu_0(x)}, \frac{-2u_0(x)}{b\left(u_{0x}^2(x)-u_0^2(x)\right)} \right\}. \tag{2.26}$$

Proof. Suppose that the solution exists globally. Due to (2.4) and the initial condition, we have $y_0(t,p(t,x_0))=0$, and

$$\begin{aligned} y(t,p(t,x)) \geq 0 \quad \text{for } x \in (-\infty, x_0), \\ y(t,p(t,x)) \leq 0 \quad \text{for } x \in (x_0, \infty) \end{aligned} \tag{2.27}$$

for all t.

Since $u(x,t)=G*y(t,x),\ x\in\mathbb{R},\ t\geq 0$, one can write $u(t,x)$ and $u_x(t,x)$ as

$$\begin{aligned} u(t,x) &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi, \\ u_x(t,x) &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi. \end{aligned} \tag{2.28}$$

Consequently,

$$u_x^2(t,x)-u^2(t,x) = -\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi \int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi. \tag{2.29}$$

From the expression of $u_x(t,x)$ in terms of $y(t,x)$,

$$\begin{aligned} &\frac{d}{dt}u_x(t,p(t,x_0)) \\ &\quad = bu^2(t,p(t,x_0)) - \frac{1}{2}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}y_t(t,\xi)d\xi + \frac{1}{2}e^{p(t,x_0)}\int_{p(t,x_0)}^{\infty} e^{\xi}y_t(t,\xi)d\xi. \end{aligned} \tag{2.30}$$

Rewrite (1.1) as

$$y_t + buy_x + 2bu_xy + \frac{a-2b}{2}\left(u^2-u_x^2\right)_x = 0. \tag{2.31}$$

Using this identity, we can obtain

$$\begin{aligned}
&\frac{d}{dt}u_x(t,p(t,x_0)) \\
&\quad = bu^2(t,p(t,x_0)) \\
&\qquad + \frac{b}{2}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}(uy_{\xi}+2u_{\xi}y)d\xi - \frac{b}{2}e^{p(t,x_0)}\int_{p(t,x_0)}^{\infty} e^{-\xi}(uy_{\xi}+2u_{\xi}y)d\xi \\
&\qquad + \frac{a-2b}{4}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}\left(u^2-u_{\xi}^2\right)_{\xi}d\xi - \frac{a-2b}{4}e^{p(t,x_0)}\int_{p(t,x_0)}^{\infty} e^{-\xi}\left(u^2-u_{\xi}^2\right)_{\xi}d\xi.
\end{aligned} \tag{2.32}$$

By direct calculation, we have

$$\begin{aligned}
&\int_{-\infty}^{p(t,x_0)} e^{\xi}(uy_{\xi}+2yu_{\xi})(t,\xi)d\xi \\
&\quad = \int_{-\infty}^{p(t,x_0)} e^{\xi}(u(t,\xi)y(t,\xi))_{\xi}d\xi + \int_{-\infty}^{p(t,x_0)} e^{\xi}y(t,\xi)u_{\xi}(t,\xi)d\xi \\
&\quad = -\int_{-\infty}^{p(t,x_0)} e^{\xi}u(t,\xi)y(t,\xi)d\xi + \frac{1}{2}\int_{-\infty}^{p(t,x_0)} e^{\xi}\left(u^2(t,\xi)-u_{\xi}^2(t,\xi)\right)_{\xi}d\xi \\
&\quad = -\int_{-\infty}^{p(t,x_0)} e^{\xi}\left[u^2(t,\xi)+\frac{1}{2}u_x^2(t,\xi)\right]d\xi + \left[e^{\xi}\left(u(t,\xi)u_x(t,\xi)-\frac{1}{2}u_x^2(t,\xi)\right)\right]_{\xi=p(t,x_0)}.
\end{aligned} \tag{2.33}$$

Using the inequality $\int_{-\infty}^{x} e^{\xi}[u^2(t,\xi)+1/2u_x^2(t,\xi)]d\xi \ge e^{x}(u^2(t,x)/2)$ (see [26]), we obtain

$$\begin{aligned}
&\frac{b}{2}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}(uy_x+2yu_x)(t,\xi)d\xi \\
&\qquad \le -\frac{b}{4}u^2(t,p(t,x_0)) - \frac{b}{4}u_x^2(t,p(t,x_0)) + \frac{b}{2}u(t,p(t,x_0))u_x(t,p(t,x_0)).
\end{aligned} \tag{2.34}$$

Similarly, we have

$$\begin{aligned}
&-\frac{b}{2}e^{p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{-\xi}(uy_x+2yu_x)(t,\xi)d\xi \\
&\qquad \le -\frac{b}{4}u^2(t,p(t,x_0)) - \frac{b}{4}u_x^2(t,p(t,x_0)) - \frac{b}{2}u(t,p(t,x_0))u_x(t,p(t,x_0)).
\end{aligned} \tag{2.35}$$

Now using the inequality (see [26])

$$u_x^2(t,x) - u^2(t,x) \le \left(u_x^2-u^2\right)(t,p(t,x_0)), \tag{2.36}$$

we obtain

$$\frac{a-2b}{4}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}\left(u^2-u_{\xi}^2\right)_{\xi}(t,\xi)d\xi \leq 0. \tag{2.37}$$

Similarly, we have

$$\frac{a-2b}{4}e^{p(t,x_0)}\int_{p(t,x_0)}^{\infty} e^{-\xi}\left(u^2-u_x^2\right)(t,\xi)d\xi \geq 0. \tag{2.38}$$

Combining all the above terms together, we have

$$\frac{d}{dt}u_x(t,p(t,x_0)) \leq \frac{b}{2}u^2(t,p(t,x_0)) - \frac{b}{2}u_x^2(t,p(t,x_0)). \tag{2.39}$$

Claim. $u_x(t,p(t,x_0)) < 0$ is decreasing and $u^2(t,p(t,x_0)) < u_x^2(t,p(t,x_0))$ for all $t \geq 0$.

Suppose not that there exists a t_0 such that $u^2(t,p(t,x_0)) < u_x^2(t,p(t,x_0))$ on $[0,t_0)$ and $u^2(t_0,p(t_0,x_0)) = u_x^2(t_0,p(t_0,x_0))$. Now, let

$$\begin{aligned} I(t) &= \frac{1}{2}e^{-p(t,x_0)}\int_{-\infty}^{p(t,x_0)} e^{\xi}y(t,\xi)d\xi > 0, \\ II(t) &= \frac{1}{2}e^{p(t,x_0)}\int_{p(t,x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi < 0. \end{aligned} \tag{2.40}$$

Firstly, by the same trick as above, we obtain

$$\begin{aligned} \frac{dI(t)}{dt} &\geq \frac{b}{4}\left(u_x^2-u^2\right)(t,p(t,x_0)) > 0, \\ \frac{dII(t)}{dt} &\leq -\frac{b}{4}\left(u_x^2-u^2\right)(t,p(t,x_0)) < 0, \\ \left(u_x^2-u^2\right)(t,p(t,x_0)) &= -4I(t)II(t) \geq -4I(0)II(0) > 0. \end{aligned} \tag{2.41}$$

This implies that t_0 can be extended to infinity.

Moreover, due to the above inequality, we have

$$\begin{aligned} \frac{d}{dt}\left(u_x^2-u^2\right)(t,p(t,x_0)) &= 4\frac{d}{dt}I(t)\cdot(-II(t)) + 4I(t)\cdot\frac{d}{dt}(-II(t)) \\ &\geq b\left(u_x^2-u^2\right)(t,p(t,x_0))[I(t)-II(t)] \\ &= -bu_x(t,p(t,x_0))\left(u_x^2-u^2\right)(t,p(t,x_0)). \end{aligned} \tag{2.42}$$

Now substituting (2.39) in (2.42), we get

$$\frac{d}{dt}\left(u_x^2-u^2\right)(t,p(t,x_0)) \geq \frac{b^2}{2}\left(u_x^2-u^2\right)(t,p(t,x_0))\left[\int_0^t\left(u_x^2-u^2\right)(\tau,p(\tau,x_0))d\tau-\frac{2}{b}u_{0x}(x_0)\right]. \tag{2.43}$$

Now the theorem follows from Lemma 2.5 with $\Psi(t) = \int_0^t (u_x^2-u^2)(\tau,p(\tau,x_0))d\tau-(2/b)u_{0x}(x_0)$ and $C_0 = b^2/2$. Then, we complete our proof. □

The final goal for us is to establish a necessary and sufficient condition for guaranteeing singularity formation in finite time, which is in the opposite direction to global existence. So if the necessary and sufficient condition can be found, then the problem can be solved completely.

Now, let us try to find a condition for global existence. Unfortunately, when $a \neq 2b$ like the Degasperis-Procesi equation [22] and the Holm-Staley b-family of equations [26], only the following easy form can be proved at present.

Theorem 2.9. *Suppose that $u_0 \in H^3(\mathbb{R})$, $y_0(x) = (1-\partial_x^2)u_0$ does not change sign. Then the corresponding solution to* (1.1) *exists globally.*

Proof. Without loss of generality we assume $y_0(x) \geq 0$. It is sufficient to prove that $u_x(t,x)$ is bounded for all t.

In fact,

$$\begin{aligned} u_x(t,x) &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &\geq -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi \\ &\geq -\frac{1}{2}\int_{-\infty}^{x} y(t,\xi)d\xi \\ &\geq -\frac{1}{2}\int_{-\infty}^{+\infty} y(t,\xi)d\xi \\ &= -\frac{1}{2}\int_{-\infty}^{+\infty} y_0(\xi)d\xi, \end{aligned} \tag{2.44}$$

while we can also obtain

$$\begin{aligned} u_x(t,x) &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(t,\xi)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &\leq \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &\leq \frac{1}{2}\int_{-\infty}^{+\infty} y_0(\xi)d\xi. \end{aligned} \tag{2.45}$$

Combining the above terms together, we have

$$|u_x(t,x)| \leq \frac{1}{2}\left|\int_{-\infty}^{+\infty} y_0(\xi)d\xi\right|. \tag{2.46}$$

Then, we complete our proof. □

3. Propagation Speed

The purpose of this section is to give a detailed description of the corresponding strong solution $u(t,x)$ in its lifespan with u_0 being compactly supported. The analogous results were obtained in [6] for the Camassa-Holm equation and in the papers [28, 29] for the Degasperis-Procesi equation. The main theorem reads as follows.

Theorem 3.1. *Let $a > 0$ and $3b - a > 0$. Assume that the initial datum $u_0 \in H^3(\mathbb{R})$ is compactly supported in $[c,d]$. Then, the corresponding solution of (1.1) has the following property: for $0 < t < T$*

$$u(t,x) = L(t)e^{-x} \quad \text{as } x > p(t,d), \qquad u(t,x) = l(t)e^{x} \quad \text{as } x < p(t,c), \tag{3.1}$$

with $L(t) > 0$ and $l(t) < 0$, respectively, where $p(t,x)$ is defined by (2.1) and T is its lifespan. Furthermore, $L(t)$ and $l(t)$ denote continuous nonvanishing functions, with $L(t) > 0$ and $l(t) < 0$ for $t \in (0,T]$. And $L(t)$ is a strictly increasing function, while $l(t)$ is a strictly decreasing function.

Proof. Since $u_0(x)$ has a compact support, $y_0(x) = (1-\partial_x^2)u_0(x)$ also does. From (2.4), it follows that $y(t,x) = (1-\partial_x^2)u(t,x)$ is compactly supported in $[p(t,c), p(t,d)]$ in its lifespan. Hence the following functions are well defined:

$$E(t) = \int_{\mathbb{R}} e^{x} y(t,x)dx, \qquad F(t) = \int_{\mathbb{R}} e^{-x} y(t,x)dx \tag{3.2}$$

with $E(0) = 0 = F(0)$. For $x > p(t,d)$, we have

$$u(t,x) = \frac{1}{2}e^{-|x|} * y(t,x) = \frac{1}{2}e^{-x}\int_{p(t,c)}^{p(t,d)} e^{-\xi}y(t,\xi)d\xi = \frac{1}{2}e^{-x}E(t), \tag{3.3}$$

and, for $x < p(t,c)$, we have

$$u(t,x) = \frac{1}{2}e^{-|x|} * y(t,x) = \frac{1}{2}e^{x}\int_{p(t,c)}^{p(t,d)} e^{-\xi}y(t,\xi)d\xi = \frac{1}{2}e^{x}F(t). \tag{3.4}$$

Hence, as consequence of (3.3) and (3.4), we have

$$\begin{aligned} u(t,x) = -u_x(t,x) = u_{xx}(t,x) &= \frac{1}{2}e^{-x}E(t), \quad \text{for } x > p(t,d), \\ u(t,x) = u_x(t,x) = u_{xx}(t,x) &= \frac{1}{2}e^{x}F(t), \quad \text{for } x < p(t,c). \end{aligned} \tag{3.5}$$

On the other hand,

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^{x} y_t(t,x)dx. \tag{3.6}$$

Differentiating (1.1) twice, we get

$$\begin{aligned} 0 &= u_{xxt} + b(uu_x)_{xx} + \Lambda^{-2}\partial_x^3\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) \\ &= u_{xxt} + b(uu_x)_{xx} + \Lambda^{-2}\partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) - \Lambda^{-2}\left(1-\partial_x^2\right)\partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) \\ &= u_{xxt} + b(uu_x)_{xx} + \Lambda^{-2}\partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) - \partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right). \end{aligned} \tag{3.7}$$

Combining (1.1) and (3.7), we obtain

$$y_t = -buu_x + b(uu_x)_{xx} - \partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right). \tag{3.8}$$

Substituting the identity (3.8) into $dE(t)/dt$ and using (3.5), we obtain

$$\begin{aligned} \frac{dE(t)}{dt} &= -b\int_{\mathbb{R}} e^{x}uu_x dx + b\int_{\mathbb{R}} e^{x}(uu_x)_{xx}dx - \int_{\mathbb{R}} e^{x}\partial_x\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)dx \\ &= \int_{\mathbb{R}} e^{x}\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)dx. \end{aligned} \tag{3.9}$$

Therefore, in the lifespan of the solution, we have

$$E(t) = \int_0^t\int_{\mathbb{R}} e^{x}\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)dx > 0. \tag{3.10}$$

By the same argument, one can check that the following identity for $F(t)$ is true:

$$F(t) = -\int_0^t\int_{\mathbb{R}} e^{-x}\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)dx < 0. \tag{3.11}$$

In order to complete the proof, it is sufficient to let $L(t) = (1/2)E(t)$ and $l(t) = (1/2)F(t)$, respectively. □

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