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Research Article

A Fixed Point Approach to Superstability of Generalized Derivations on Non-Archimedean Banach Algebras

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We investigate the superstability of generalized derivations in non-Archimedean algebras by using a version of fixed point theorem via Cauchy functional equation.

1. Introduction

A functional equation (ξ) is *superstable* if every approximately solution of (ξ) is an exact solution of it.

The stability of functional equations was first introduced by Ulam [1] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940.

Given a metric group $G(\cdot,\rho)$, a number $\varepsilon > 0$, and a mapping $f : G \to G$ which satisfies the inequality $\rho(f(x \cdot y), f(x) \cdot f(y)) \le \varepsilon$ for all x, y in G, does there exist an automorphism a of G and a constant $k > 0$, depending only on G such that $\rho(a(x), f(x)) \le k\varepsilon$ for all $x \in G$?

If the answer is affirmative, we would call the equation $a(x \cdot y) = a(x) \cdot a(y)$ of automorphism is stable. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1978, Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$, $(\epsilon > 0,\ p \in [0, 1))$. In 1991, Gajda [4] answered the question for the case

$p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias or generalized Hyers-Ulam stability of functional equations [5, 6].

In 1992, Găvruţa [7] generalized the Th. M. Rassias Theorem as follows.

Suppose that $(G, +)$ is an ablian group, X is a Banach space $\varphi : G \times G \to [0, \infty)$ which satisfies

$$\widetilde{\varphi}(x, y) = \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty, \tag{1.1}$$

for all $x, y \in G$. If $f : G \to X$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y), \tag{1.2}$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to X$ such that $T(x+y) = T(x)+T(y)$ and $\|f(x) - T(x)\| \le \widetilde{\varphi}(x, x)$ for all $x, y \in G$.

In 1949, Bourgin [8] proved the following result, which is sometimes called the superstability of ring homomorphisms: suppose that A and B are Banach algebras with unit. If $f : A \to B$ is a surjective mapping such that

$$\begin{gathered} \|f(x+y) - f(x) - f(y)\| \le \epsilon, \\ \|f(xy) - f(x)f(y)\| \le \delta, \end{gathered} \tag{1.3}$$

for some $\epsilon \ge 0$, $\delta \ge 0$ and for all $x, y \in A$, then f is a ring homomorphism.

Badora [9] and Miura et al. [10] proved the Ulam-Hyers stability and the Isac and Rassias-type stability of derivations [11] (see also [12, 13]); Savadkouhi et al. [14] have contributed works regarding the stability of ternary Jordan derivations. Jung and Chang [15] investigated the stability and superstability of higher derivations on rings. Recently, Ansari-Piri and Anjidani [16] discussed the superstability of generalized derivations on Banach algebras. In this paper, we investigate the superstability of generalized derivations on non-Archimedean Banach algebras by using the fixed point methods.

2. Preliminaries

In 1897, Hensel [17] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [18, 19].

A non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0, |rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$ (see [20, 21]).

Definition 2.1. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA_1) $\|x\| = 0$ if and only if $x = 0$,

(NA_2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,

(NA_3) $\|x + y\| \le \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (the strong triangle inequality).

A sequence $\{x_m\}$ in a non-Archimedean space is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent. A non-Archimedean normed algebra is a non-Archimedean normed space A with a linear associative multiplication, satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A non-Archimedean complete normed algebra is called a non-Archimedean Banach algebra (see [22]).

Example 2.2. Let p be a prime number. For any nonzero rational number $x = (a/b)p^{n_x}$ such that a and b are integers not divisible by p, define the p–adic absolute value $|x|_p := p^{-n_x}$. Then, $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field.

Definition 2.3. Let X be a nonempty set and $d : X \times X \to [0, \infty]$ satisfy the following properties:

(D$_1$) $d(x, y) = 0$ if and only if $x = y$,

(D$_2$) $d(x, y) = d(y, x)$ (symmetry),

(D$_3$) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (strong triangle in equality),

for all $x, y, z \in X$. Then, (X, d) is called a non-Archimedean generalized metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.

Definition 2.4. Let A be a non-Archimedean algebra. An additive mapping $D : A \to A$ is said to be a ring derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. An additive mapping $H : A \to A$ is said to be a generalized ring derivation if there exists a ring derivation $D : A \to A$ such that

$$H(xy) = xH(y) + D(x)y, \tag{2.1}$$

for all $x, y \in A$.

We need the following fixed point theorem (see [23, 24]).

Theorem 2.5 (non-Archimedean alternative Contraction Principle). *Suppose that (X, d) is a non-Archimedean generalized complete metric space and $\Lambda : X \to X$ is a strictly contractive mapping; that is,*

$$d(\Lambda x, \Lambda y) \leq Ld(x, y), \quad (x, y \in X), \tag{2.2}$$

for some $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:

(a) *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ,*

(b) *x^* is a unique fixed point of Λ in*

$$X^* = \left\{y \in X \mid d\left(\Lambda^k x, y\right) < \infty\right\}, \tag{2.3}$$

(c) *if $y \in X^*$, then*

$$d(y, x^*) \leq d(\Lambda y, y). \tag{2.4}$$

3. Non-Archimedean Superstability of Generalized Derivations

Hereafter, we will assume that A is a non-Archimedean Banach algebra with unit over a non-Archimedean field $\mathbb{K}$.

Theorem 3.1. *Let $\varphi : A \times A \to [0, \infty)$ be a function. Suppose that $f, g : A \to A$ are mappings such that g is additive and*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|f(xy) - xf(y) - g(x)y\| \leq \varphi(x, y), \tag{3.2}$$

for all $x, y \in A$. If there exists a natural number $k \in \mathbb{K}$ and $0 < L < 1$,

$$|k|^{-1}\varphi(kx, ky), |k|^{-1}\varphi(kx, y), |k|^{-1}\varphi(x, ky) \leq L\varphi(x, y), \tag{3.3}$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

Proof. By induction on i, we prove that

$$\|f(ix) - if(x)\| \leq \max\{\varphi(0,0), \varphi(x,x), \varphi(2x,x), \ldots, \varphi((i-1)x, x)\}, \tag{3.4}$$

for all $x \in A$ and $i \geq 2$. Let $x = y$ in (3.1). Then,

$$\|f(2x) - 2f(x)\| \leq \max\{\varphi(0,0), \varphi(x,x)\}, \quad n \in \mathbb{N}_0,\ x \in A. \tag{3.5}$$

This proves (3.4) for $i = 2$. Let (3.4) holds for $i = 1, 2, \ldots, j$. Replacing x by jx and y by x in (3.1) for each $n \in \mathbb{N}_0$, and for all $x \in A$, we get

$$\|f((j+1)x) - f(jx) - f(x)\| \leq \max\{\varphi(0,0), \varphi(jx, x)\}. \tag{3.6}$$

Since

$$\begin{aligned} &f((j+1)x) - f(jx) - f(x) \\ &\quad = f((j+1)x) - (j+1)f(x) + (j+1)f(x) - f(jx) - f(x) \\ &\quad = f((j+1)x) - (j+1)f(x) + jf(x) - f(jx), \end{aligned} \tag{3.7}$$

for all $x \in A$, it follows from induction hypothesis and (3.6) that

$$\begin{aligned} &\|f((j+1)x) - (j+1)f(x)\| \\ &\quad \leq \max\{\|f((j+1)x) - f(jx) - f(x)\|, \|jf(x) - f(jx)\|\} \\ &\quad \leq \max\{\varphi(0,0), \varphi(x,x), \varphi(2x,x), \ldots, \varphi((j)x, x)\}, \end{aligned} \tag{3.8}$$

for all $x \in A$. This proves (3.4) for all $i \geq 2$. In particular,

$$\left\| f(kx) - kf(x) \right\| \leq \psi(x), \tag{3.9}$$

for all $x \in A$ where

$$\psi(x) = \max\left\{ \varphi(0,0), \varphi(x,x), \varphi(2x,x), \ldots, \varphi((k-1)x,x) \right\} \quad (x \in A). \tag{3.10}$$

Let X be the set of all functions $r : A \to A$. We define $d : X \times X \to [0,\infty]$ as follows:

$$d(r,s) = \inf\left\{ \alpha > 0 : \|r(x) - s(x)\| \leq \alpha\psi(x)\ \forall x \in A \right\}. \tag{3.11}$$

It is easy to see that d defines a generalized complete metric on X. Define $J : X \to X$ by $J(r)(x) = k^{-1}r(kx)$. Then, J is strictly contractive on X, in fact, if

$$\|r(x) - s(x)\| \leq \alpha\psi(x), \quad (x \in A), \tag{3.12}$$

then by (3.3),

$$\|J(r)(x) - J(s)(x)\| = |k|^{-1}\|r(kx) - s(kx)\| \leq \alpha|k|^{-1}\psi(kx) \leq L\alpha\psi(x), \quad (x \in A). \tag{3.13}$$

It follows that

$$d(J(r), J(s)) \leq Ld(r,s) \quad (r, s \in X). \tag{3.14}$$

Hence, J is a strictly contractive mapping with Lipschitz constant L. By (3.9),

$$\begin{gathered} \left\| (Jf)(x) - f(x) \right\| = \left\| k^{-1}f(kx) - f(x) \right\|, \\ |k|^{-1}\left\| f(kx) - kf(x) \right\| \leq |k|^{-1}\psi(x) \quad (x \in A). \end{gathered} \tag{3.15}$$

This means that $d(J(f), f) \leq 1/|k|$. By Theorem 2.5, J has a unique fixed point $h : A \to A$ in the set

$$U = \left\{ r \in X : d\left(r, J(f)\right) < \infty \right\}, \tag{3.16}$$

and for each $x \in A$,

$$h(x) = \lim_{m\to\infty} J^m\left(f(x)\right) = \lim k^{-m}f(k^m x). \tag{3.17}$$

Therefore,

$$\begin{aligned}
&\|h(x+y)-h(x)-h(y)\| \\
&\quad = \lim_{m\to\infty}|k|^{-m}\|f(k^m(x+y))-f(k^m x)-f(k^m y)\| \\
&\quad \le \lim_{m\to\infty}|k|^{-m}\max\{\varphi(0,0),\varphi(k^n x,k^n y)\} \\
&\quad \le \lim_{m\to\infty}L^m\varphi(x,y)=0,
\end{aligned} \tag{3.18}$$

for all $x, y \in A$. This shows that h is additive.

Replacing x by $k^n x$ in (3.2) to get

$$\|f(k^n xy)-k^n x f(y)-g(k^n x)y\| \le \varphi(k^n x,y), \tag{3.19}$$

and so

$$\left\|\frac{f(k^n xy)}{k^n}-xf(y)-\frac{g(k^n x)}{k^n}y\right\| \le \frac{1}{|k|^n}\varphi(k^n x,y) \le L^n\varphi(x,y), \tag{3.20}$$

for all $x, y \in A$ and all $n \in \mathbb{N}$. By taking $n \to \infty$, we have

$$h(xy)=xf(y)+\lim_{n\to\infty}\frac{g(k^n x)}{k^n}y, \tag{3.21}$$

for all $x, y \in A$.

Fix $m \in \mathbb{N}$. By (3.21), we have

$$\begin{aligned}
xf(k^m y) &= h(k^m xy)-\lim_{n\to\infty}\left(\frac{g(k^n x)}{k^n}(k^m y)\right) \\
&= k^m xf(y)+\lim_{n\to\infty}\left(\frac{g(k^n k^m x)}{k^n}y\right)-k^m\lim_{n\to\infty}\left(\frac{g(k^n x)}{k^n}y\right) \\
&= k^m xf(y)+k^m\lim_{n\to\infty}\left(\frac{g(k^{n+m}x)}{k^{n+m}}y\right)-k^m\lim_{n\to\infty}\left(\frac{g(k^n x)}{k^n}y\right) \\
&= k^m xf(y),
\end{aligned} \tag{3.22}$$

for all $x, y \in A$. Then, $xf(y) = x(f(k^m y)/k^m)$ for all $x, y \in A$ and each $m \in \mathbb{N}$, and so by taking $m \to \infty$, we have $xf(y) = xh(y)$. Now, we obtain $h = f$, since A is with unit. Replacing y by $k^n y$ in (3.2), we obtain

$$\|f(k^n(xy))-xf(k^n y)-k^n g(x)y\| \le \varphi(x,k^n y), \tag{3.23}$$

and hence,

$$\left\| \frac{f(k^n xy)}{k^n} - x\frac{f(k^n y)}{k^n} - g(x)y \right\| \leq \frac{1}{|k|^n}\varphi(x, k^n y) \leq L^n \varphi(x, y), \tag{3.24}$$

for all $x, y \in A$ and each $n \in \mathbb{N}$. Letting n tends to infinite, we have

$$f(xy) = xf(y) + g(x)y. \tag{3.25}$$

Now, we show that g is a ring derivation. By (3.25), we get

$$\begin{aligned} g(xy)z &= f(xyz) - xyf(z) \\ &= xf(yz) + g(x)yz - xyf(z) \\ &= (xg(y) + g(x)y)z, \end{aligned} \tag{3.26}$$

for all $x, y, z \in A$. Therefore, we have $g(xy) = xg(y) + g(x)y$. □

The proof of following theorem is similar to that in Theorem 3.1, hence it is omitted.

Theorem 3.2. *Let $\varphi : A \times A \to [0, \infty)$ be a function. Suppose that $f, g : A \to A$ are mappings such that g is additive and*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \varphi(x, y), \\ \|f(xy) - xf(y) - g(x)y\| &\leq \varphi(x, y), \end{aligned} \tag{3.27}$$

for all $x, y \in A$. If there exists a natural number $k \in \mathbb{K}$ and $0 < L < 1$,

$$|k|\varphi\left(k^{-1}x, k^{-1}y\right), |k|\varphi\left(k^{-1}x, y\right), |k|\varphi\left(x, k^{-1}y\right) \leq L\varphi(x, y), \tag{3.28}$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

The following results are immediate corollaries of Theorems 3.1 and 3.2 and Example 2.3.

Corollary 3.3. *Let A be a non-Archimedean Banach algebra over $\mathbb{Q}_p$, $\varepsilon > 0$, and $p_1, p_2 \in (1, \infty)$. Suppose that $f, g : A \to A$ are mappings such that g is additive and*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \varepsilon\left(\|x\|^{p_1}\|y\|^{p_2}\right), \\ \|f(xy) - xf(y) - g(x)y\| &\leq \varepsilon\left(\|x\|^{p_1}\|y\|^{p_2}\right), \end{aligned} \tag{3.29}$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

Corollary 3.4. *Let A be a non-Archimedean Banach algebra over $\mathbb{Q}_p$, $\varepsilon > 0$ and $p_1, p_2, p_1 + p_2 \in (-\infty, 1)$. Suppose that $f, g : A \to A$ are mappings such that g is additive and*

$$\begin{aligned} \| f(x+y) - f(x) - f(y) \| &\le \varepsilon \left(\|x\|^{p_1} \|y\|^{p_2} \right), \\ \| f(xy) - x f(y) - g(x) y \| &\le \varepsilon \left(\|x\|^{p_1} \|y\|^{p_2} \right), \end{aligned} \tag{3.30}$$

for all $x, y \in A$. Then, f is a generalized ring derivation and g is a ring derivation.

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