Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 369571, 15 pages
doi:10.1155/2011/369571

Research Article

Asymptotic Behavior of the Navier-Stokes Equations with Nonzero Far-Field Velocity

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Received 18 June 2011; Revised 7 October 2011; Accepted 17 October 2011

Academic Editor: Weiqing Ren

Concerning the nonstationary Navier-Stokes flow with a nonzero constant velocity at infinity, the temporal stability has been studied by Heywood (1970, 1972) and Masuda (1975) in L^2 space and by Shibata (1999) and Enomoto-Shibata (2005) in L^p spaces for $p \geq 3$. However, their results did not include enough information to find the spatial decay. So, Bae-Roh (2010) improved Enomoto-Shibata's results in some sense and estimated the spatial decay even though their results are limited. In this paper, we will prove temporal decay with a weighted function by using $L^r - L^p$ decay estimates obtained by Roh (2011). Bae-Roh (2010) proved the temporal rate becomes slower by $(1+\sigma)/2$ if a weighted function is $|x|^\sigma$ for $0 < \sigma < 1/2$. In this paper, we prove that the temporal decay becomes slower by σ, where $0 < \sigma < 3/2$ if a weighted function is $|x|^\sigma$. For the proof, we deduce an integral representation of the solution and then establish the temporal decay estimates of weighted L^p-norm of solutions. This method was first initiated by He and Xin (2000) and developed by Bae and Jin (2006, 2007, 2008).

1. Introduction

When a boat is sailing with a constant velocity $\mathbf{u}_\infty$, we may think that the water is flowing around the fixed boat with opposite velocity $-\mathbf{u}_\infty$ like the water flow around an island. As we have seen, behind the boat the motion of the water is significantly different from other areas, which is called the wake. The motion of nonstationary flow of an incompressible viscous fluid past an isolated rigid body is formulated by the following initial boundary value problem of the Navier-Stokes equations:

$$\begin{gathered}\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \lim_{|x|\to\infty}\mathbf{u}(x,t) = \mathbf{u}_\infty,\end{gathered} \tag{1.1}$$

where Ω is an exterior domain in R^3 with a smooth boundary $\partial\Omega$ and $\mathbf{u}_\infty$ denotes a given constant vector describing the velocity of the fluid at infinity. For $\mathbf{u}_\infty = 0$, the temporal decay and weighted estimates for solutions of (1.1) have been studied in [1–13].

In this paper, we consider a nonzero constant $\mathbf{u}_\infty$. We set $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$ in (1.1) and have

$$\begin{gathered}
\frac{\partial}{\partial t}\mathbf{v} - \Delta\mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p_1 = \mathbf{f}, \quad \text{in } \Omega \times (0,\infty), \\
\nabla \cdot \mathbf{v} = 0, \\
\mathbf{v}|_{t=0} = \mathbf{u}_0 - \mathbf{u}_\infty, \qquad \mathbf{v}|_{\partial\Omega} = -\mathbf{u}_\infty, \qquad \lim_{|x|\to\infty} \mathbf{v}(x,t) = 0.
\end{gathered} \tag{1.2}$$

Consider the following linear equations of (1.2):

$$\begin{gathered}
\frac{\partial}{\partial t}\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0,\infty), \\
\mathbf{u}|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0,
\end{gathered} \tag{1.3}$$

which is referred to as the Oseen equations; see [14].

In order to formulate the problem (1.3), Enomoto and Shibata [15] used the Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega), \tag{1.4}$$

where $1 < p < \infty$,

$$\begin{gathered}
L_p(\Omega)^n = \left\{u = (u_1,\ldots,u_n) : \ u_j \in L_p(\Omega), \ j = 1,\ldots,n\right\}, \\
C_{0,\sigma}^\infty = \left\{u = (u_1,\ldots,u_n) \in C_0^\infty(\Omega)^n \ : \nabla \cdot u = 0 \text{ in } \Omega\right\}, \\
J_p(\Omega) = \text{the completion of } C_{0,\sigma}^\infty(\Omega) \quad \text{in } L_p(\Omega)^n, \\
G_p(\Omega) = \left\{\nabla\pi \in L_p(\Omega)^n : \pi \in L_{p,\mathrm{loc}}\left(\overline{\Omega}\right)\right\}.
\end{gathered} \tag{1.5}$$

The Helmholtz decomposition of $L_p(\Omega)^n$ was proved by Fujiwara-Morimoto [16], Miyakawa [17], and Simader-Sohr [18]. Let P be a continuous projection from $L_p(\Omega)^n$ onto $J_p(\Omega)^n$.

By applying P into (1.3) and setting $\mathcal{O}_{\mathbf{u}_\infty} = P(-\Delta + \mathbf{u}_\infty \cdot \nabla)$, one has

$$\mathbf{u}_t + \mathcal{O}_{\mathbf{u}_\infty}\mathbf{u} = 0, \quad \text{for } t > 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{1.6}$$

where the domain of $\mathcal{O}_{\mathbf{u}_\infty}$ is given by

$$\mathfrak{D}_p(\mathcal{O}_{\mathbf{u}_\infty}) = \left\{u \in J_p(\Omega) \cap W_p^2(\Omega)^n : \ u|_{\partial\Omega} = 0\right\}. \tag{1.7}$$

Then, Enomoto and Shibata [15] proved that $\mathcal{O}_{\mathbf{u}_\infty}$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ which is called the Oseen semigroup (one can also refer to [17, 19]) and obtained the following properties.

Proposition 1.1. *Let* $\sigma_0 > 0$ *and assume that* $|\mathbf{u}_\infty| \leq \sigma_0$. *Let* $1 \leq r \leq q \leq \infty$. *Then,*

$$\|T(t)a\|_{L^q(\Omega)} \leq C_{r,q,\sigma_0} t^{-3/2(1/r-1/q)} \|a\|_{L^r(\Omega)}, \quad t > 0, \tag{1.8}$$

where $(r,q) \neq (1,1)$ *and* (∞,∞),

$$\|\nabla T(t)a\|_{L^q(\Omega)} \leq C_{r,q,\sigma_0} t^{-3/2(1/r-1/q)-1/2} \|a\|_{L^r(\Omega)}, \quad t > 0, \tag{1.9}$$

where $1 \leq r \leq q \leq 3$ *and* $(r,q) \neq (1,1)$.

By using Proposition 1.1, Bae-Jin [20] considered the spatial stability of stationary solution $\mathbf{w}$ of (1.3) and obtained the following: if $|x|\mathbf{u}_0, \mathbf{u}_0 \in L^r(\Omega)$ with $\nabla \cdot \mathbf{u}_0 = 0$, then for any $t > 0$,

$$\||x|\mathbf{u}(t)\|_p \leq Ct^{-3/2(1/r-1/p)} \||x|\mathbf{u}_0\|_{L^r(\Omega)} + C|\mathbf{u}_\infty| t^{-3/2(1/r-1/p)+1} \|\mathbf{u}_0\|_{L^r(\Omega)}, \tag{1.10}$$

where $p \geq 3$ and $1 < r < 3$.

And, for the nonstationary Navier-Stokes equations, we discuss the stability of stationary solution $\mathbf{w}$ of the nonlinear Navier-Stokes equation (1.2), and $\mathbf{w}$ satisfies the following equations:

$$\begin{gathered} -\Delta \mathbf{w} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p_2 = \mathbf{f}, \quad \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}|_{\partial\Omega} = -\mathbf{u}_\infty, \qquad \lim_{|x|\to\infty} \mathbf{w}(x) = 0. \end{gathered} \tag{1.11}$$

For suitable $\mathbf{f}$, Shibata [21] proved that for any given $0 < \delta < 1/4$ there exists ϵ such that if $0 < |\mathbf{u}_\infty| \leq \epsilon$, then one has

$$\|\mathbf{w}\|_{L^{3/(1+\delta_1)}(\Omega)} + \|\mathbf{w}\|_{L^{3/(1-\delta_2)}(\Omega)} + \|\nabla \mathbf{w}\|_{L^{3/(2+\delta_1)}(\Omega)} + \|\nabla \mathbf{w}\|_{L^{3/(2-\delta_2)}(\Omega)} \leq C|\mathbf{u}_\infty|^{1/2}, \tag{1.12}$$

for small δ_1, δ_2, where C is independent of $\mathbf{u}_\infty$.

By setting $\mathbf{u} = \mathbf{v} - \mathbf{w}$ and $p = p_1 - p_2$ for $\mathbf{v}, p_1, \mathbf{w}, p_2$ in (1.2) and (1.11), we have the following equations in Ω:

$$\begin{gathered} \frac{\partial}{\partial t}\mathbf{u} - \Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \\ \nabla \cdot \mathbf{u}(t,x) = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{for } x \in \Omega, \\ \mathbf{u}(x,t) = 0 \quad \text{for } x \in \partial\Omega, \qquad \lim_{|x|\to\infty} \mathbf{u}(x,t) = 0. \end{gathered} \tag{1.13}$$

Here, in fact, the initial data should be $\mathbf{u}_0-\mathbf{u}_\infty-\mathbf{w}$, but for our convenience we denote by $\mathbf{u}_0$ for $\mathbf{u}_0-\mathbf{u}_\infty-\mathbf{w}$ if there is no confusion. Heywood [22, 23], Masuda [24], Shibata [21], Enomoto-Shibata [15], Bae-Roh [25], and Roh [26] have studied the temporal decay for solutions of (1.13), and we have the followings in [26].

Proposition 1.2. *There exists small $\epsilon(p,q,r)$ such that if $0<|\mathbf{u}_\infty|\le\epsilon$, and $\|\mathbf{u}_0\|_{L^3(\Omega)}<\epsilon$, then a unique solution $\mathbf{u}(x,t)$ of (1.13) has*

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^p(\Omega)} &\le C_\epsilon t^{-3/2(1/r-1/p)}\|\mathbf{u}_0\|_r \quad \text{for } 1<r<p\le\infty,\ t>0,\\ \|\nabla\mathbf{u}(t)\|_{L^q(\Omega)} &\le C_\epsilon t^{-3/2(1/r-1/q)-(1/2)}\|\mathbf{u}_0\|_r \quad \text{for } 1<r<q\le 3,\ t>0,\end{aligned} \tag{1.14}$$

where $\mathbf{u}_0\in L^3(\Omega)\cap L^r(\Omega)$.

Now, we are in the position to introduce our main theorems which are the weighted stability of stationary solution $\mathbf{w}$.

Theorem 1.3. *Let $1<r<p<\infty$ and $(1/r-1/p)>2/3$. Then there exists small $\epsilon(p,r)$ such that if $0<|\mathbf{u}_\infty|\le\epsilon$, $\|\mathbf{u}_0\|_{L^3(\Omega)}<\epsilon$, $|x|\mathbf{u}_0\in L^{3r/(3-2r)}(\Omega)$, and $\nabla\cdot\mathbf{u}_0=0$, then the solution $\mathbf{u}(x,t)$ of (1.13) satisfies*

$$\||x|\mathbf{u}(t)\|_{L^p(\Omega)} \le C_\epsilon t^{-3/2(1/r-1/p)+1}\|\mathbf{u}_0\|_r, \quad \forall t>0, \tag{1.15}$$

where $\mathbf{u}_0\in L^3(\Omega)\cap L^r(\Omega)$.

Remark 1.4. In Theorem 1.3, the assumption $|x|\mathbf{u}_0\in L^{3r/(3-2r)}(\Omega)$ is for simple calculations. We also can obtain a similar result where $|x|\mathbf{u}_0\in L^r(\Omega)$. For the proof we have to consider delay solution $\mathbf{u}(t)=\mathbf{u}(t+t_0)$. Then we can follow the method in Bae and Roh [4].

Theorem 1.5. *Let $1/r-1/p>2\sigma/3$ for $1<\sigma<3/2$ and $1<r<p<\infty$. Then there exists small $\epsilon(p,r)$ such that if $0<|\mathbf{u}_\infty|\le\epsilon$, $\|\mathbf{u}_0\|_{L^3(\Omega)}<\epsilon$, $|x|^\sigma\mathbf{u}_0\in L^{3r/(3-2r)}(\Omega)$, and $\nabla\cdot\mathbf{u}_0=0$, then the solution $\mathbf{u}(x,t)$ of (1.13) satisfies*

$$\||x|^\sigma\mathbf{u}(t)\|_{L^p(\Omega)} \le C_\epsilon t^{-3/2(1/r-1/p)+\sigma}\|\mathbf{u}_0\|_r, \quad \forall t\ge 1, \tag{1.16}$$

where $\mathbf{u}_0\in L^3(\Omega)\cap L^r(\Omega)$.

Remark 1.6. For the exterior Navier-Stokes flows with $\mathbf{u}_\infty=0$, temporal decay rate with weight function $|x|^\sigma$ becomes slower by $\sigma/2$; refer to [1–4, 8, 13]. However, for $\mathbf{u}_\infty\neq 0$, we found out from Theorems 1.3 and 1.5 that temporal decay rate with weight function $|\mathrm{x}|^\sigma$ becomes slower by σ for $0\le\sigma<3/2$. In fact, Bae and Roh [25] concluded that it becomes slower by $(1+\sigma)/2$ for $0<\sigma<1/2$. Hence, our decay rate is little faster than the one in Bae and Roh [25] for $0<\sigma<1/2$.

One of the difficulties for the exterior Navier-Stokes equations is dealing with the boundary of Ω because a pressure representation in terms of velocity is not a simple problem. So to remove the pressure term, we adapt an indirect method by taking a weight function ϕ

vanishing near the boundary. This astonied method for exterior problem was initiated by He and Xin [27] and then developed by Bae and Jin [1, 2, 4, 20].

2. Proof of Main Theorems

In this section, we will prove the weighted stability of stationary solutions of the Navier-Stokes equations with nonzero far-field velocity. We first consider $|x|$ for a weight function and then $|x|^\sigma$ for $\sigma < 3/2$. Our method can be applied to the Oseen equations. As a result, we note that we can improve the result of Bae-Jin [1] by the same method.

2.1. Proof of Theorem 1.3

We define $\phi_R(x) = |x|\chi(|x|)(1-\chi(|x|/R))$ for large $R > 0$, where χ is a nonnegative cutoff function with $\chi \in C^\infty[0,\infty)$, $\chi(s) = 0$ for $s \le 1$, and $\chi(s) = 1$ for $s \ge 2$. When there is no confusion, we use the same notation ϕ instead of ϕ_R for convenience.

As in [1], we set

$$\mathbf{v}(x) \equiv \int_{\mathbb{R}^3} N(x-y)\left[\phi(y)(\nabla\times\mathbf{u})(y)\right]dy, \tag{2.1}$$

where N is the fundamental function of $-\Delta$, that is, $N = N(x-y) = 1/(4\pi|x-y|)$. By the definition of $\mathbf{v}$, we have $-\Delta\mathbf{v} = \phi\nabla\times\mathbf{u}$. Moreover,

$$\nabla\times\mathbf{v} = \int_\Omega N(x-y)\nabla\times\left[\phi(\nabla\times\mathbf{u})\right](y)dy = \phi\mathbf{u} + \mathbf{R}_0, \tag{2.2}$$

where

$$\mathbf{R}_0 := \nabla N * \left[(\mathbf{u}\cdot\nabla)\phi\right] - \nabla\times N * \left[(\nabla\phi)\times\mathbf{u}\right]. \tag{2.3}$$

We first estimate $\|\nabla\times\mathbf{v}(t)\|_p$ and then obtain the estimate of $\|\phi\mathbf{u}(t)\|_p = \||x|\mathbf{u}(t)\|_p$.

Now, we consider the fundamental solutions for the nonstationary Oseen equations, written as

$$V_t^i(x) = V^i(x,t) = \Gamma_t(x)\mathbf{e}^i + \nabla\frac{\partial}{\partial x_i}(N * \Gamma_t)(x), \tag{2.4}$$

where $\Gamma_t(x) = \Gamma(x,t) = (4\pi t)^{-3/2}e^{-|x-t\mathbf{u}_\infty|^2/4t}$ (refer to [15, 28]). In fact, Γ is a translation in the direction of x by $t\mathbf{u}_\infty$ of the heat kernel $K(x,t) = (4\pi t)^{-3/2}e^{-|x|^2/4t}$, that is, $\Gamma(x,t) = K(x - t\mathbf{u}_\infty, t)$. Set $\omega_t^i(x) = \omega^i(x,t) = (N * \Gamma_t)(x)\mathbf{e}^i$, $i = 1,2,3$, where $\mathbf{e}^i$ is the standard unit vector of which the ith term is 1. Then, we have

$$\nabla\times\nabla\times\omega^i = -\Delta\omega^i + \nabla\operatorname{div}\omega^i = V^i. \tag{2.5}$$

Hence, we have the identity

$$\nabla_y \times \left[\phi(y)\nabla_y \times \omega^i(x-y,t-\tau)\right] = \phi(y)V^i(x-y,t-\tau) + R_1^i(x,y,t-\tau), \tag{2.6}$$

where

$$R_1^i(x,y,t-\tau) = \nabla\phi(y) \times \nabla_y \times \omega^i(x-y,t-\tau). \tag{2.7}$$

From straightforward calculations we have that for $1 \le s \le \infty$,

$$\left\|\partial^\beta \Gamma_{t-\tau}\right\|_s \le c(t-\tau)^{-3/2(1-1/s)-(|\beta|/2)}. \tag{2.8}$$

One might note that we may sometimes use $\|V^i\|_s \le \|\Gamma_t\|_{3s/(3+s)} < ct^{-1+3/2s}$ instead of $\|V^i\|_s \le \|\Gamma_t\|_s < ct^{-3/2(1-1/s)}$ because of technical reason. By the definition of V^i, both inequalities hold for any $s \ge 3/2$. We multiply (1.13) by $\nabla_y \times [\phi(y)\nabla_y \times \omega^i(x-y,t-\tau)]$ and integrate over $\Omega \times (0,t-\epsilon)$, and then we have

$$\begin{aligned}
&\int_0^{t-\epsilon}\int_\Omega \left(\frac{\partial \mathbf{u}}{\partial\tau} - \Delta_y\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla_y)\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \\
&\qquad \cdot \nabla_y \times \left[\phi(y)\nabla_y \times \omega^i(x-y,t-\tau)\right] dy\, d\tau \\
&\quad = -\int_0^{t-\epsilon}\int_\Omega \nabla p(y)\cdot\nabla_y \times \left[\phi(y)\nabla_y\times\omega^i(x-y,t-\tau)\right] dy\, d\tau = 0.
\end{aligned} \tag{2.9}$$

We finally get the following integral representation for $\nabla \times \mathbf{v}$ (refer to [2, 3] for the detail):

$$\begin{aligned}
(\nabla_x \times \mathbf{v})_i &= (\nabla_x \times \mathbf{v}_0) * \Gamma_t \\
&\quad - \int_0^t\int_\Omega \mathbf{u}\cdot\left[\partial_\tau + \Delta_y + (\mathbf{u}_\infty\cdot\nabla_y)\right] R_1(x,y,t-\tau)\, dy\, d\tau \\
&\quad - \int_0^t\int_\Omega \mathbf{u}\cdot\left[R_2^i(x,y,t-\tau)\right] dy\, d\tau \\
&\quad - \int_0^t\int_\Omega \mathbf{u}\cdot V^i(x-y,t-\tau)(\mathbf{u}_\infty\cdot\nabla_y)\phi(y)\, dy\, d\tau \\
&\quad - \int_0^t\int_\Omega (w_k\mathbf{u} + u_k\mathbf{w})\cdot\left[\partial_{y_k}\left(\phi(y)V^i(x-y,t-\tau)\right) + \partial_{y_k}R_1^i(x,y,t-\tau)\right] dy\, d\tau \\
&\quad - \int_0^{t-\epsilon}\int_\Omega u_k\mathbf{u}\cdot\left[\partial_{y_k}\left(\phi(y)V^i(x-y,t-\tau)\right) + \partial_{y_k}R_1^i(x,y,t-\tau)\right] dy\, d\tau \\
&= I + II + III + IV + V + VI,
\end{aligned} \tag{2.10}$$

where

$$R_2^i(x,y,t-\tau) = 2(\nabla_y\phi(y)\cdot\nabla_y)V^i(x-y,t-\tau) + \Delta_y\phi(y)V^i(x-y,t-\tau). \tag{2.11}$$

Applying Young's convolution and the Calderon-Zygmund inequalities, we obtain

$$\begin{aligned}\|I\|_p &= \left\|(\nabla\times\mathbf{v}_0)*\Gamma_t\right\|_p \le \|\mathbf{u}_0\phi*\Gamma_t\|_p + \left\|\nabla N*[\mathbf{u}_0\nabla\phi]*\Gamma_t\right\|_p \\ &\le \|\mathbf{u}_0\phi\|_{3r/(3-2r)}\|\Gamma_t\|_{3pr/(5pr+3r-3p)} + \|\nabla N*\mathbf{u}_0\nabla\phi\|_{3r/(3-2r)}\|\Gamma_t\|_{3pr/(5pr+3r-3p)} \\ &\le Ct^{-3/2(1/r-1/p)+1}\|\phi\mathbf{u}_0\|_{3r/3-2r} + Ct^{-3/2(1/r-1/p)+1}\|\mathbf{u}_0\|_r, \quad \forall t>0,\end{aligned} \tag{2.12}$$

if $\phi\mathbf{u}_0 \in L^{3r/(3-2r)}$ and $\mathbf{u}_0 \in L^r$.

And II is bounded by as follows:

$$\begin{aligned}\|II\|_p &\le c\int_0^t \|\mathbf{u}\|_{s_1}\left\|\nabla^2\phi\right\|_\infty\left\|\partial_k\nabla\times\omega_{t-\tau}^i\right\|_{s_2} + \|\mathbf{u}\|_{s_3}\left\|\nabla\Delta\phi\right\|_3\left\|\nabla\times\omega_{t-\tau}^i\right\|_{s_4} \\ &\quad + |\mathbf{u}_\infty|\left\||x|^{-1}\mathbf{u}\right\|_{s_5}\left\|\nabla\times\omega_{t-\tau}^i\right\|_{s_6}d\tau \\ &= II_1 + II_2 + II_3,\end{aligned} \tag{2.13}$$

where $1/s_1 + 1/s_2 = 1 + 1/p$, $1/s_3 + 1/s_4 = 1 + 1/p - 1/3$ and $1/s_5 + 1/s_6 = 1 + 1/p$.

We have

$$II_1 \le C\|\mathbf{u}_0\|_r\int_0^t \tau^{-3/2(1/r-1/s_1)}(t-\tau)^{-3/2(1-1/s_2)}d\tau \le C\|\mathbf{u}_0\|_{\mathrm{r}}t^{-3/2(1/r-1/p)+1}, \quad \forall t>0, \tag{2.14}$$

where $1/r - 1/s_1 < 2/3$ and $s_2 < 3$. Also, we obtain

$$II_2 \le C\|\mathbf{u}_0\|_{\mathrm{r}}\int_0^t \tau^{-3/2(1/r-1/s_3)}(t-\tau)^{-1+3/2s_4}d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t>0, \tag{2.15}$$

where $1/r - 1/s_3 < 2/3$. Finally, we get

$$\begin{aligned}II_3 &\le C\int_0^t \|\nabla\mathbf{u}\|_{s_5}\left\|\nabla\times\omega_{t-\tau}^i\right\|_{s_6}d\tau \le C\|\mathbf{u}_0\|_r\int_0^t \tau^{-3/2(1/r-1/s_5)-1/2}(t-\tau)^{-1+3/2s_6}d\tau \\ &\le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t>0,\end{aligned} \tag{2.16}$$

where $1/r - 1/s_5 < 1/3$. Hence, for any $t > 0$, we have

$$\|I\|_p + \|II\|_p \le Ct^{-3/2(1/r-1/p)+1}\left\|\phi\mathbf{u}_0\right\|_{3r/(3-2r)} + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}. \tag{2.17}$$

Also, we obtain

$$\begin{aligned}\|III\|_p &\le \int_0^t \|(\mathbf{u}\partial_j\phi) * \partial_j V^i\|_p + \|(\mathbf{u}\Delta\phi) * V^i\|_p d\tau \\ &\le \int_0^t \|\mathbf{u}\|_s \|\partial_j V^i\|_{ps/(ps+s-p)} + \|\mathbf{u}\|_{s_1}\|\nabla^2\phi\|_\infty \|\partial_k \nabla \times \omega^i_{t-\tau}\|_{s_2} d\tau \\ &\le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0,\end{aligned} \tag{2.18}$$

where $1/s_1 + 1/s_2 = 1 + 1/p$, $1/r - 1/s_1 < 2/3$ and $s_2 < 3$. In the above calculation, we used $\|\partial V^i(t)\|_q \le t^{-(3/2)(1-1/q)}$ instead of $\|\partial V^i(t)\|_q \le t^{-3/2(1-1/q)-1/2}$ because of simplicity of calculations.

And we have

$$\|IV\|_p \le c|\mathbf{u}_\infty| \int_0^t \|\mathbf{u}\|_{s_7} \|\nabla\phi\|_\infty \left\|V^i\right\|_{s_8} d\tau \le C\|\mathbf{u}_0\|_r t^{-(3/2)(1/r-1/p)+1}, \quad \forall t > 0, \tag{2.19}$$

where $1/s_7 + 1/s_8 = 1 + 1/p, 1/r - 1/s_7 < 2/3$ and $s_8 < 3$.

Next, for V, we have

$$\begin{aligned}V = &-\int_0^t \int_\Omega (w_k\mathbf{u} + u_k\mathbf{w}) \cdot \Big[(\partial_{y_k}\phi(y)) V^i(x-y, t-\tau) \\ &+\phi(y)\partial_{y_k} V^i(x-y,t-\tau) + \partial_{y_k} R^i_1(x-y,t-\tau)\Big] dy\, d\tau \\ &\le V_1 + V_2 + V_3.\end{aligned} \tag{2.20}$$

We get

$$\|V_1\|_p \le c\int_0^t \|\mathbf{u}\|_{r_1} \|\mathbf{w}\|_3 \|\nabla\phi\|_\infty \left\|V^i\right\|_{r_2} d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0, \tag{2.21}$$

where $1/r_1 + 1/r_2 = 2/3 + 1/p, 1/r - 1/r_1 < 2/3$ and $r_2 < 3$. In the above calculation, we used $\|V^i(t)\|_q \le t^{-3/2(1-1/q)+1/2}$ instead of $\|V^i(t)\|_q \le t^{-3/2(1-1/q)}$ because of simplicity of calculations.

Since $\||x|\mathbf{w}\|_\infty < C$ (see [21]), we have

$$\|V_2\|_p \le c\int_0^t \|\mathbf{u}\|_{r_3} \|\phi\mathbf{w}\|_\infty \left\|\nabla V^i\right\|_{r_4} d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0, \tag{2.22}$$

where $1/r_3 + 1/r_4 = 1 + 1/p$, $1/r - 1/r_3 < 2/3$ and $r_4 < 3$. In the above calculation, we used $\|\partial V^i(t)\|_q \le t^{-3/2(1-1/q)}$instead of $\|\partial V^i(t)\|_q \le t^{-3/2(1-1/q)-1/2}$ because of simplicity of calculations.

Next, for any $t > 0$, we have

$$\|V_3\|_p \le \int_0^t \|\mathbf{u}\|_{r_5}\|\mathbf{w}\|_3 \left\|\nabla^2\phi\right\|_\infty \left\|\nabla\times\omega^i\right\|_{r_6} + \|\mathbf{u}\|_{r_7}\|\mathbf{w}\|_\infty \|\nabla\phi\|_\infty \left\|\partial_k\nabla\times\omega^i\right\|_{r_8} d\tau$$
$$\le \|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \tag{2.23}$$

where $1/r_5 + 1/r_6 = 1/r_7 + 1/r_8 = 2/3 + 1/p, 1/r - 1/r_5 < 2/3$, $1/r - 1/r_7 < 2/3$ and $r_8 < 3$.

Hence, we have

$$\|V\|_p \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0. \tag{2.24}$$

Consider VIas follows:

$$VI = -\int_0^t \int_\Omega u_k\mathbf{u}\cdot\Big[\left(\partial_{y_k}\phi(y)\right)V^i(x-y,t-\tau)$$
$$+\phi(y)\partial_{y_k}V^i(x-y,t-\tau) + \partial_{y_k}R_1^i(x-y,t-\tau)\Big]dy\,d\tau$$
$$\le VI_1 + VI_2 + VI_3. \tag{2.25}$$

We have, for any $t > 0$,

$$\|VI_1\|_p \le \int_0^t \|\mathbf{u}\|_{s_1}\|\mathbf{u}\|_{s_2}\|\nabla\phi\|_\infty \left\|V^i\right\|_{s_3} d\tau \le C_2\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \tag{2.26}$$

where $1/s_1 + 1/s_2 + 1/s_3 = 1 + 1/p$, $1/r - 1/s_1 + 1/s_2 < 1/3$, and $\|\mathbf{u}(t)\|_{s_2} \le ct^{-1/2+3/2s_2}$. In the above calculation, we used $\|V^i(t)\|_q \le t^{-3/2(1-1/q)+1/2}$ instead of $\|V^i(t)\|_q \le t^{-3/2(1-1/q)}$ because of technical reason.

Similar to VI_1, we get

$$\|VI_3\|_p \le \int_0^t \|\mathbf{u}\|_{r_1}\|\mathbf{u}\|_{r_2} \left\|\nabla^2\phi\right\|_\infty \left\|\nabla\times\omega^i\right\|_{r_3} + \|\mathbf{u}\|_{s_1}\|\mathbf{u}\|_{s_2}\|\nabla\phi\|_\infty \left\|\partial_k\nabla\times\omega^i\right\|_{s_3} d\tau$$
$$\le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0, \tag{2.27}$$

where $1/r_1 + 1/r_2 + 1/r_3 = 1 + 1/p = 1/s_1 + 1/s_2 + 1/s_3$, $1/r - 1/r_1 + 1/r_2 < 1/3$, $\|\mathbf{u}(t)\|_{r_2} \le ct^{-1/2+3/2r_2}$, and $1/r - (1/s_1 + 1/s_2) < 1/3$, $\|\mathbf{u}(t)\|_{s_2} \le ct^{-1/2+3/2s_2}$.

Note that

$$\left\|\phi\mathbf{u}(t)\right\|_p \le \|\nabla\times\mathbf{v}(t)\|_p + \left\|\nabla N * \mathbf{u}(t)\nabla\phi\right\|_p \le \|\nabla\times\mathbf{v}(t)\|_p + \|\mathbf{u}(t)\|_{3p/(3+2p)}$$
$$\le \|\nabla\times\mathbf{v}(t)\|_p + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0. \tag{2.28}$$

Since for $t > 0$, $\|\mathbf{u}(t)\|_9 \le \sqrt{\epsilon} t^{-1/3}$, from Shibata [21], we have

$$\begin{aligned}\|VI_2\|_p &\le \int_0^t \left\|\phi\mathbf{u}(\tau)\right\|_p \|\mathbf{u}(\tau)\|_9 \left\|\nabla V^i(t-\tau)\right\|_{9/8} d\tau \\ &\le \epsilon \int_0^t \|\nabla\times\mathbf{v}(\tau)\|_p \tau^{-1/3}(t-\tau)^{-2/3} d\tau + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}.\end{aligned} \tag{2.29}$$

Hence, we have

$$\|VI\|_p \le \epsilon \int_0^t \|\nabla\times\mathbf{v}(\tau)\|_p \tau^{-1/3}(t-\tau)^{-2/3} d\tau + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}. \tag{2.30}$$

Thus, by (2.17)–(2.19), (2.24), (2.28), and (2.30), for all $t > 0$, we obtain

$$\|\nabla\times\mathbf{v}(t)\|_p \le \epsilon \int_0^t \|\nabla\times\mathbf{v}(\tau)\|_p \tau^{-1/3}(t-\tau)^{-2/3} d\tau + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}. \tag{2.31}$$

Now, we use the following lemma (refer to [25]).

Lemma 2.1. *Let a function $S(t)$ satisfy the inequality, for some $\alpha < 2/3$,*

$$S(t) \le ct^{-\alpha} + \varepsilon \int_0^t S(\tau)\tau^{-1/3}(t-\tau)^{-2/3} d\tau \quad \forall t > 0. \tag{2.32}$$

One also assumes that

$$\lim_{t\to 0+} t^{-\varepsilon} \int_0^t \tau^{-1/3} S(\tau) d\tau = 0. \tag{2.33}$$

Then, there is ε_0 so that if $\varepsilon \le \varepsilon_0$, then one has

$$S(t) \le ct^{-\alpha} \tag{2.34}$$

for some c independent of t.

Since

$$\begin{aligned}\|\nabla\times\mathbf{v}(t)\|_p &\le \left\|\phi\mathbf{u}(t)\right\|_p + \left\|\nabla N * \mathbf{u}(t)\nabla\phi\right\|_p \le R\|\mathbf{u}(t)\|_p + \|\mathbf{u}(t)\|_{3p/(3+p)} \\ &\le CR\|\mathbf{u}_0\|_3 t^{-1/2+3/2p} + C_2\|\mathbf{u}_0\|_3 t^{-3/2(1/3-1/p)+1/2}, \quad \forall t > 0,\end{aligned} \tag{2.35}$$

condition (2.33) satisfies

$$\lim_{t\to 0+} t^{-\varepsilon}\int_0^t \tau^{-1/3}\|\nabla\times \mathbf{v}(\tau)\|_p d\tau = \lim_{t\to 0+} t^{-\varepsilon}\int_0^t \tau^{-1/3}\left(CR\|\mathbf{u}_0\|_3\tau^{-1/2+3/2p} + C_2\|\mathbf{u}_0\|_3\tau^{3/2p}\right)d\tau$$
$$= \lim_{t\to 0+}\left(CR\|\mathbf{u}_0\|_3\tau^{-\epsilon+1/3+3/2p} + C_2\|\mathbf{u}_0\|_3\tau^{-\epsilon+1+3/2p}\right) = 0, \tag{2.36}$$

for $\epsilon < (1/3 + 3/2p)$.

So, by Lemma 2.1, we have

$$\|\nabla\times \mathbf{v}(t)\|_p \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}. \tag{2.37}$$

Hence, by (2.28), for any $t > 0$, we have

$$\left\|\phi\mathbf{u}(t)\right\|_p \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \tag{2.38}$$

and by taking $R \to \infty$, we complete the proof of Theorem 1.3.

2.2. Proof of Theorem 1.5

By using the results in previous section, for any $0 < \alpha < 1$, we have small $\beta > 0$ such that

$$\left\||x|^\alpha \mathbf{u}(t)\right\|_s \le \||x|^\alpha \mathbf{u}^\alpha\|_{3/(\alpha-3\beta)}\left\|\mathbf{u}^{1-\alpha}\right\|_{1/(1-\alpha-\beta)}$$
$$\le \left[Ct^{-3/2(1/r-(\alpha-3\beta)/3\alpha)+1}\right]^\alpha\left[Ct^{-3/2(1/r-(1-\alpha-\beta)/(1-\alpha))}\right]^{1-\alpha} \le Ct^{-3/2(1/r-1/s)+\alpha}, \tag{2.39}$$

where $1 - 2\alpha/3 - 2\beta = 1/s$.

Now, in this section, we consider $\phi(x) = |x|^\sigma\chi(|x|)$, where $1 < \sigma < 3/2$.

Similar to previous section, for $\|I\|_p$, II_1, and II_2, we obtain the same decay rate with previous section. And for any $t > 0$, we have

$$II_3 \le C\int_0^t \left\||x|^{\sigma-2}\right\|_{3/(2-\sigma)^2}\|\mathbf{u}\|_{s_1}\left\|\nabla\times\omega^i_{t-\tau}\right\|_{s_2} d\tau$$
$$\le C\|\mathbf{u}_0\|_r\int_0^t \tau^{-3/2(1/r-1/s_1)}(t-\tau)^{-1+3/2s_2}d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+3/2-(2-\sigma)^2/2}, \tag{2.40}$$

where $1/s_1 + 1/s_2 = 1 + 1/p$ and $1/r - 1/s_1 < 2/3$. Also, for $\|III\|_p$, we obtain

$$\begin{aligned}\|III\|_p &\le \int_0^t \left\|(\mathbf{u}\partial_j\phi) * \partial_j V^i\right\|_p + \left\|(\mathbf{u}\Delta\phi) * V^i\right\|_p d\tau \\ &\le \int_0^t \left\||x|^{\sigma-1}\mathbf{u}\right\|_s \left\|\partial_j V^i\right\|_{ps/(ps+s-p)} + \|\mathbf{u}\|_{s_1} \left\|\nabla^2\phi\right\|_\infty \left\|\partial_k \nabla \times \omega^i_{t-\tau}\right\|_{s_2} d\tau \\ &\le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2} + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0,\end{aligned} \tag{2.41}$$

where $1/s_1 + 1/s_2 = 1 + 1/p$, $1/r - 1/s_1 < 2/3$ and $s_2 < 3$.

Next, we have

$$\|IV\|_p \le c|\mathbf{u}_\infty| \int_0^t \left\||x|^{\sigma-1}\mathbf{u}\right\|_{s_1} \left\|V^i\right\|_{s_2} d\tau \le C|\mathbf{u}_\infty| \|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma}, \quad \forall t > 0, \tag{2.42}$$

where $1/s_1 + 1/s_2 = 1 + 1/p$, $s_2 < 3$, and $1/r - 1/s_1 < 2\sigma/3$. Also, since $\||x|\mathbf{w}\|_\infty < C$, we get

$$\|V_1\|_p \le c\int_0^t \|\mathbf{u}\|_{r_1} \left\||x|^{\sigma-1}\mathbf{w}\right\|_\infty \left\|V^i\right\|_{r_2} d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1}, \quad \forall t > 0, \tag{2.43}$$

where $1/r_1 + 1/r_2 = 1 + 1/p$, $1/r - 1/r_1 < 2/3$, and $r_2 < 3$.

And we obtain

$$\|V_2\|_p \le c\int_0^t \left\||x|^{\sigma-1}\mathbf{u}\right\|_{r_3} \||x|\mathbf{w}\|_\infty \left\|\nabla V^i\right\|_{r_4} d\tau \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2}, \quad \forall t > 0, \tag{2.44}$$

where $1/r_3 + 1/r_4 = 1 + 1/p$, $1/r - 1/r_3 < 2\sigma/3$, and $r_4 < 3/2$.

Next, for any $t > 0$, we have

$$\begin{aligned}\|V_3\|_p &\le \int_0^t \|\mathbf{u}\|_{r_5} \|\mathbf{w}\|_3 \left\|\nabla^2\phi\right\|_\infty \left\|\nabla \times \omega^i\right\|_{r_6} + \left\||x|^{\sigma-1}\mathbf{u}\right\|_{r_7} \|\mathbf{w}\|_3 \left\|\partial_k \nabla \times \omega^i\right\|_{r_8} d\tau \\ &\le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1} + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2},\end{aligned} \tag{2.45}$$

where $1/r_5 + 1/r_6 = 1/r_7 + 1/r_8 = 2/3 + 1/p$, $1/r - 1/r_5 < 2/3$, $1/r - 1/r_7 < 2\sigma/3$, and $r_8 < 3$.

Hence, we have

$$\|V\|_p \le C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1} + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2}, \quad \forall t > 0. \tag{2.46}$$

Consider VI as follows:

$$VI = -\int_0^t \int_\Omega u_k \mathbf{u} \cdot \Big[(\partial_{y_k} \phi(y)) V^i(x-y, t-\tau) + \phi(y) \partial_{y_k} V^i(x-y, t-\tau) + \partial_{y_k} R_1^i(x-y, t-\tau) \Big] dy\, d\tau \leq VI_1 + VI_2 + VI_3. \tag{2.47}$$

We have, for any $t > 0$,

$$\|VI_1\|_p \leq \int_0^t \left\| |x|^{\sigma-1} \mathbf{u} \right\|_{s_1} \|\mathbf{u}\|_{s_2} \left\| V^i \right\|_{s_3} d\tau \leq C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2}, \tag{2.48}$$

where $1/s_1 + 1/s_2 + 1/s_3 = 1 + 1/p$, $s_3 < 3$, $1/r - 1/s_1 < 2\sigma/3$, and $\|\mathbf{u}(t)\|_{s_2} \leq ct^{-1/2+3/2s_2}$.

Similar to VI_1, we get

$$\begin{aligned} \|VI_3\|_p &\leq \int_0^t \|\mathbf{u}\|_{r_1} \|\mathbf{u}\|_{r_2} \left\| \nabla^2 \phi \right\|_\infty \left\| \nabla \times \omega^i \right\|_{r_3} + \left\| |x|^{\sigma-1} \mathbf{u} \right\|_{s_1} \|\mathbf{u}\|_{s_2} \left\| \partial_k \nabla \times \omega^i \right\|_{s_3} d\tau \\ &\leq C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+1/2} + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma}, \quad \forall t > 0, \end{aligned} \tag{2.49}$$

where $1/r_1 + 1/r_2 + 1/r_3 = 1 + 1/p = 1/s_1 + 1/s_2 + 1/s_3$, $1/r - 1/r_1 < 2/3$, $1/r - (1/r_1 + 1/r_2) < 1/3$, $1/r - 1/s_1 < 2\sigma/3$, $1/r - 1/s_2 < 2/3$, $1/r - (1/s_1 + 1/s_2) < (2\sigma - 1)/3$, $\|\mathbf{u}(t)\|_{s_2} \leq ct^{-1/2+3/2s_2}$, and $\|\mathbf{u}(t)\|_{r_2} \leq ct^{-1/2+3/2r_2}$. In the above calculation, we used $\|V^i(t)\|_q \leq t^{-3/2(1-1/q)+1/2}$ instead of $\|V^i(t)\|_q \leq t^{-3/2(1-1/q)}$ because of technical reason. Now, we have

$$\begin{aligned} \|VI_2\|_p &\leq \int_0^t \||x| \mathbf{u}(\tau)\|_{s_1} \left\| |x|^{\sigma-1} \mathbf{u}(\tau) \right\|_{s_2} \left\| \nabla V^i(t-\tau) \right\|_{s_3} d\tau \\ &\leq C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma+1/2-3/2r_1}, \end{aligned} \tag{2.50}$$

where $1/s_1 + 1/s_2 + 1/s_3 = 1 + 1/p$, $s_2 < 3/2$, $\||x|^{\sigma-1} \mathbf{u}(\tau)\|_{s_2} < Ct^{-3/2(1/r_1-1/s_2)+\sigma-1}$, and $r_1 < 3$ (≈ 3).

So, we obtain

$$\begin{aligned} \left\| \phi \mathbf{u}(t) \right\|_p &\leq \|\nabla \times \mathbf{v}(t)\|_p + \left\| \nabla N * \mathbf{u}(t) \nabla \phi \right\|_p \leq \|\nabla \times \mathbf{v}(t)\|_p + \left\| |x|^{\sigma-1} \mathbf{u}(t) \right\|_{3p/(3+p)} \\ &\leq \|\nabla \times \mathbf{v}(t)\|_p + C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma-1/2}, \quad \forall t > 0 \\ &\leq C\|\mathbf{u}_0\|_r t^{-3/2(1/r-1/p)+\sigma}, \quad \forall t \geq 1, \end{aligned} \tag{2.51}$$

which completes the proof.

Acknowledgments

The author would like to express her appreciation to Professors Hyeong-Ohk Bae and Bum Ja Jin for valuable comments. The author would like to thank the referee for helpful comments. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0023386).

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