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Research Article

Necessary and Sufficient Conditions for Schur Geometrical Convexity of the Four-Parameter Homogeneous Means

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The necessary and sufficient conditions for Schur geometrical convexity of the four-parameter means are given. This gives a unified treatment for Schur geometrical convexity of Stolarsky and Gini means.

1. Introduction and Main Result

Let $p, q \in \mathbb{R}$ and $a, b > 0$. For $a \neq b$ the Stolarsky means are defined as

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q}{p}\dfrac{a^p - b^p}{a^q - b^q}\right)^{1/(p-q)}, & pq(p-q) \neq 0, \\ L^{1/p}(a^p, b^p), & p \neq 0,\ q = 0, \\ L^{1/q}(a^q, b^q), & q \neq 0,\ p = 0, \\ I^{1/p}(a^p, b^p), & p = q \neq 0, \\ \sqrt{ab}, & p = q = 0, \end{cases} \tag{1.1}$$

and $S_{p,q}(a,a) = a$ (see [1]), where

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & x \neq y, \\ x & x = y, \end{cases} \tag{1.2}$$

$$I(x,y) = \begin{cases} \left(\dfrac{x^x}{y^y}\right)^{1/(x-y)}, & x \neq y, \\ x, & x = y \end{cases} \tag{1.3}$$

are the logarithmic mean and identric (exponential) mean of positive numbers x and y, respectively.

Another two-parameter family of means was introduced by Gini in [2]. That are defined as

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{1/(p-q)}, & p \neq q, \\ \exp\left(\dfrac{a^p \ln a + b^p \ln b}{a^p + b^p}\right), & p = q. \end{cases} \tag{1.4}$$

Stolarsky and Gini means both are contained in the so-called four-parameter means [3], which are defined as follows.

Definition 1.1. Let $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$ and $(p,q), (r,s) \in \mathbb{R} \times \mathbb{R}$. Then the four-parameter homogeneous means denoted by $\mathbf{F}(p,q;r,s;a,b)$ are defined as follows:

$$\mathbf{F}(p,q;r,s;a,b) = \left(\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\frac{L(a^{qs},b^{qs})}{L(a^{qr},b^{qr})}\right)^{1/(p-q)(r-s)} \quad \text{if } pqrs(p-q)(r-s) \neq 0, \tag{1.5}$$

or

$$\mathbf{F}(p,q;r,s;a,b) = \left(\frac{a^{pr} - b^{pr}}{a^{ps} - b^{ps}}\frac{a^{qs} - b^{qs}}{a^{qr} - b^{qr}}\right)^{1/(p-q)(r-s)} \quad \text{if } pqrs(p-q)(r-s) \neq 0. \tag{1.6}$$

If $pqrs(p-q)(r-s) = 0$, then $\mathbf{F}(p,q;r,s;a,b)$ are defined as their corresponding limits, for example:

$$\begin{gathered} \mathbf{F}(p,p;r,s;a,b) = \lim_{q \to p} \mathbf{F}(p,q;r,s;a,b) = \left(\frac{I(a^{pr},b^{pr})}{I(a^{ps},b^{ps})}\right)^{1/p(r-s)}, \quad \text{if } prs(r-s) \neq 0,\ p = q, \\ \mathbf{F}(p,0;r,s;a,b) = \lim_{q \to 0} \mathbf{F}(p,q;r,s;a,b) = \left(\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\right)^{1/p(r-s)}, \quad \text{if } prs(r-s) \neq 0,\ q = 0, \\ \mathbf{F}(0,0;r,s;a,b) = \lim_{p \to 0} \mathbf{F}(p,0;r,s;a,b) = G(a,b), \quad \text{if } rs(r-s) \neq 0,\ p = q = 0, \end{gathered} \tag{1.7}$$

where $L(x,y), I(x,y)$ denote logarithmic mean and identric (exponential) mean, respectively, $G(a,b)=\sqrt{ab}$.

The Schur convexity of $S_{p,q}(a,b)$ and $G_{p,q}(a,b)$ on $(0,\infty)\times(0,\infty)$ with respect to (a,b) was investigated by Qi et al. [4], Shi et al. [5], Li and Shi [6], and Chu and Zhang [7]. Until now, they have been perfectly solved by Chu and Zhang [7], Wang and Zhang [8], respectively. Recently, Chu and Xia also proved the same result as Wang and Zhang [9].

The Schur convexity of $S_{p,q}(a,b)$ and $G_{p,q}(a,b)$ on $[0,\infty)\times[0,\infty)$ and $(-\infty,0]\times(-\infty,0]$ with respect to (p,q) was investigated by Qi [10] and Sándor [11], respectively. Now Schur convexity of a four-parameter homogeneous means family containing Stolarsky and Gini means on $(-\infty,\infty)\times(-\infty,\infty)$ with respect to (p,q) has been perfectly solved by Yang [12].

The Schur geometrical convexity was introduced by Zhang [13]. In [8, 14], Wand and Zhang proved that $G_{p,q}(a,b)$ is Schur geometrically convex (Schur geometrically concave) on $(0,\infty)\times(0,\infty)$ with respect to (a,b) if $p+q\geq(\leq)0$. Chu et al. [15] pointed out that this conclusion is also true for $S_{p,q}(a,b)$. Shi et al. [5, 16], Li and Shi [6], and Gu and Shi [17] also obtained similar results.

The purpose of this paper is to present the necessary and sufficient conditions for Schur geometrical convexity of the four-parameter homogeneous means. This gives a unified treatment for Schur geometrical convexity of Stolarsky and Gini means with respect to (a,b).

Our main result is as follows.

Theorem 1.2. *For fixed* $(p,q),(r,s)\in\mathbb{R}\times\mathbb{R}$ *the four-parameter homogeneous means* $\mathbf{F}(p,q;r,s;a,b)$ *are Schur geometrically convex (Schur geometrically concave) on* $(0,\infty)\times(0,\infty)$ *with respect to* (a,b) *if and only if* $(p+q)(r+s)>(<)0$.

2. Definitions and Lemmas

Definition 2.1 (see [18, 19]). Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ $(n\geq 2)$.

(i) x is said to by majorized by y (in symbol $\mathbf{x}\prec\mathbf{y}$) if

$$\sum_{i=1}^{k}x_{[i]}\leq\sum_{i=1}^{k}y_{[i]}\quad\text{for } 1\leq k\leq n-1,\qquad\sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]},\tag{2.1}$$

where $x_{[1]}\geq x_{[2]}\cdots\geq x_{[n]}$ and $y_{[1]}\geq y_{[2]}\cdots\geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a decreasing order.

(ii) $\mathbf{x}\geq\mathbf{y}$ means $x_i\geq y_i$ for all $i=1,2,\ldots,n$. Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$. The function $\phi:\Omega\to\mathbb{R}$ is said to be increasing if $\mathbf{x}\geq\mathbf{y}$ implies $\phi(\mathbf{x})\geq\phi(\mathbf{y})$. ϕ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for all $\mathbf{x}$ and $\mathbf{y}$, where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

(iv) Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a set with nonempty interior. Then $\phi:\Omega\to\mathbb{R}$ is said to be Schur convex if $\mathbf{x}\prec\mathbf{y}$ on Ω implies $\phi(\mathbf{x})\leq\phi(\mathbf{y})$. ϕ is said to be Schur concave if $-\phi$ is Schur convex.

Definition 2.2 (see [13, 20]). Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$ $(n \geq 2)$. Denote

$$\ln \mathbf{x} = (\ln x_1, \ln x_2, \ldots, \ln x_n), \qquad \ln \mathbf{y} = (\ln y_1, \ln y_2, \ldots, \ln y_n). \tag{2.2}$$

(i) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\mathbf{x}$ and $\mathbf{y}$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let $\Omega \subset \mathbb{R}_+^n$ $(n \geq 2)$ be a set with nonempty interior. Then function $\phi : \Omega \to \mathbb{R}_+$ is said to be Schur geometrically convex on Ω if $\ln \mathbf{x} \prec \ln \mathbf{y}$ on Ω implies $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. ϕ is said to be Schur geometrically concave if $-\phi$ is Schur geometrically convex.

Definition 2.3 (see [18]). (i) $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is called symmetric set if $\mathbf{x} \in \Omega$ implies $P\mathbf{x} \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) The function $\phi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\phi(P\mathbf{x}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

Lemma 2.4 (see [18, 19]). *Let $\Omega \subset \mathbb{R}^n$ be a symmetric set with nonempty interior Ω^0 and $\phi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur convex (Schur concave) on Ω if and only if ϕ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \tag{2.3}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

Lemma 2.5 (see [13, Theorem 1.4, page 108]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric set with a nonempty interior geometrically convex set Ω^0. Let $\phi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. Then ϕ is Schur geometrically convex (Schur geometrically concave) on Ω if and only if ϕ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \phi}{\partial x_1} - x_2 \frac{\partial \phi}{\partial x_2}\right) \geq (\leq) 0 \tag{2.4}$$

holds for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega^0$.

3. Schur Geometrical Convexity of Two-Parameter Homogeneous Functions

The more general form of two-parameter homogeneous means is the so-called two-parameter homogenous functions first introduced by Yang [21]. For conveniences, we record it as follows.

Definition 3.1. Assume that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ is n-order homogeneous, continuous and exists first partial derivatives and $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$, $(p,q) \in \mathbb{R} \times \mathbb{R}$.

If $f(x,y) > 0$ for $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x,x) : x \in \mathbb{R}_+\}$ and $f(x,x) = 0$ for all $x \in \mathbb{R}_+$, then define

$$\begin{aligned}\mathscr{H}_f(p,q;a,b) &= \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{1(p-q)} \quad \text{if } p \neq q,\ pq \neq 0,\\ \mathscr{H}_f(p,p;a,b) &= \lim_{q\to p} \mathscr{H}_f(p,q;a,b) = G_{f,p}(a,b) \quad \text{if } p = q \neq 0,\end{aligned} \tag{3.1}$$

where

$$G_{f,p}(a,b) = G_f^{1/p}(a^p,b^p), \qquad G_f(x,y) = \exp\left(\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right), \tag{3.2}$$

$f_x(x,y)$ and $f_y(x,y)$ denote first-order partial derivatives with respect to first and second component of $f(x,y)$, respectively.

If $f(x,y) > 0$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, then define further

$$\begin{aligned}\mathscr{H}_f(p,0;a,b) &= \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{1/p} \quad \text{if } p \neq 0,\ q = 0;\\ \mathscr{H}_f(0,q;a,b) &= \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{1/q} \quad \text{if } p = 0,\ q \neq 0;\\ \mathscr{H}_f(0,0;a,b) &= a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)} \quad \text{if } p = q = 0.\end{aligned} \tag{3.3}$$

Since $f(x,y)$ is a homogeneous function, $\mathscr{H}_f(p,q;a,b)$ is also one and called a homogeneous function with parameters p and q and simply denoted by $\mathscr{H}_f(p,q)$ or $\mathscr{H}_f$ sometimes.

Concerning the monotonicity and log-convexity of two-parameter homogeneous functions, there have been some literatures such as [3, 21, 22], which yield some new and interesting inequalities for means.

The two-parameter homogeneous functions $\mathscr{H}_f(p,q;a,b)$ have some well properties (see [21–23]) such as the following lemma.

Lemma 3.2 (see [23]). *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a homogenous and differentiable function and*

$$T(t) = T(t;a,b) := \ln f(a^t,b^t), \quad (t;a,b) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{3.4}$$

Then we have

$$\frac{\partial T(t;a,b)}{\partial t} = \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)}, \tag{3.5}$$

$$\ln \mathscr{H}_f(p,q;a,b) = \int_0^1 \frac{\partial T(tp+(1-t)q;a,b)}{\partial t}dt. \tag{3.6}$$

Next we give another property.

Lemma 3.3. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a homogenous and m-time differentiable function. Then $\mathscr{H}_f(p,q;a,b) \in C^{m-1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+)$.*

Proof. Since $f(x,y)$ has continuous partial derivatives of m order with respect to x, y on $\mathbb{R}_+ \times \mathbb{R}_+$, the integrand in (3.6) has continuous partial derivatives of $m-1$ order with respect to p,q,a,b on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$, that is $\mathscr{H}_f(p,q;a,b) \in C^{m-1}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+)$. □

For the Schur geometrical convexity, we have the following result.

Theorem 3.4. *Assume that $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, n-order homogeneous, continuous, and three-time differentiable function. If for any $(x,y) \in \mathbb{R}_+\times\mathbb{R}_+$ with $x \neq y$*

$$\mathcal{N}(x,y) = (x-y)\left(x(\ln f)_x - y(\ln f)_y - 2xy\mathcal{I}\ln\left(\frac{x}{y}\right)\right) > (<)0, \quad \textit{where } \mathcal{I} = (\ln f)_{xy}, \tag{3.7}$$

then $\mathscr{H}_f(p,q;a,b)$ is Schur geometrically convex on $(0,\infty)\times(0,\infty)$ with respect to (a,b) if and only if $p+q > (<)0$ and Schur geometrically concave if and only if $p+q < (>)0$.

Proof. (1) In the case of $p \neq q$. We have

$$\ln \mathscr{H}_f(p,q;a,b) = \frac{\ln f(a^p,b^p) - \ln f(a^q,b^q)}{p-q}. \tag{3.8}$$

Some simple partial derivative computations yield

$$\begin{aligned}\frac{\partial \ln \mathscr{H}_f}{\partial a} &= \frac{1}{\mathscr{H}_f}\frac{\partial \mathscr{H}_f}{\partial a} = \frac{1}{p-q}\left(\frac{pa^{p-1}f_x(a^p,b^p)}{f(a^p,b^p)} - \frac{qa^{q-1}f_x(a^q,b^q)}{f(a^q,b^q)}\right),\\ \frac{\partial \ln \mathscr{H}_f}{\partial b} &= \frac{1}{\mathscr{H}_f}\frac{\partial \mathscr{H}_f}{\partial b} = \frac{1}{p-q}\left(\frac{pb^{p-1}f_y(a^p,b^p)}{f(a^p,b^p)} - \frac{qb^{q-1}f_y(a^q,b^q)}{f(a^q,b^q)}\right),\end{aligned} \tag{3.9}$$

hence,

$$\frac{1}{\mathscr{H}_f}\left(a\frac{\partial \mathscr{H}_f}{\partial a} - b\frac{\partial \mathscr{H}_f}{\partial b}\right) = \frac{g(p)-g(q)}{p-q}, \tag{3.10}$$

where

$$g(t) = \frac{ta^t f_x(a^t,b^t)}{f(a^t,b^t)} - \frac{tb^t f_y(a^t,b^t)}{f(a^t,b^t)}. \tag{3.11}$$

It is easy to verify that $g(t)$ is even on $(-\infty,\infty)$. In fact, since $f(x,y)$ is n-order homogeneous and symmetric, for arbitrary $\lambda > 0$, we have

$$\begin{gathered} f(\lambda x,\lambda y) = \lambda^n f(x,y), \quad f_x(\lambda x,\lambda y) = \lambda^{n-1}f_x(x,y), \quad f_y(\lambda x,\lambda y) = \lambda^{n-1}f_y(x,y),\\ f(x,y) = f(y,x), \quad f_x(x,y) = f_y(y,x), \quad f_y(x,y) = f_x(y,x). \end{gathered} \tag{3.12}$$

Thus,

$$\begin{aligned} g(-t) &= \frac{-ta^{-t} f_x\left(a^{-t}, b^{-t}\right)}{f\left(a^{-t}, b^{-t}\right)} - \frac{-tb^{-t} f_y\left(a^{-t}, b^{-t}\right)}{f\left(a^{-t}, b^{-t}\right)} \\ &= \frac{-ta^{-t}\left(a^t b^t\right)^{-(n-1)} f_x\left(b^t, a^t\right)}{\left(a^t b^t\right)^{-n} f\left(b^t, a^t\right)} - \frac{-tb^{t}\left(a^t b^t\right)^{-(n-1)} f_y\left(b^t, a^t\right)}{\left(a^t b^t\right)^{-n} f\left(b^t, a^t\right)} \\ &= -\frac{tb^t f_y\left(a^t, b^t\right)}{f\left(a^t, b^t\right)} + \frac{ta^t f_x\left(a^t, b^t\right)}{f\left(a^t, b^t\right)} = g(t). \end{aligned} \tag{3.13}$$

Let $a^t = x$, $b^t = y$. Then

$$\begin{aligned} g'(t) = {}& x(\ln f)_x + t\left(\left(\frac{xf_x(x,y)}{f(x,y)}\right)_x \frac{dx}{dt} + \left(\frac{xf_x(x,y)}{f(x,y)}\right)_y \frac{dy}{dt}\right) \\ & - y(\ln f)_y - t\left(\left(\frac{yf_y(x,y)}{f(x,y)}\right)_x \frac{dx}{dt} + \left(\frac{yf_y(x,y)}{f(x,y)}\right)_y \frac{dy}{dt}\right) \\ = {}& x(\ln f)_x + t\left(x\left(\frac{xf_x(x,y)}{f(x,y)}\right)_x \ln a + y\left(\frac{xf_x(x,y)}{f(x,y)}\right)_y \ln b\right) \\ & - y(\ln f)_y - t\left(x\left(\frac{yf_y(x,y)}{f(x,y)}\right)_x \ln a + y\left(\frac{yf_y(x,y)}{f(x,y)}\right)_y \ln b\right). \end{aligned} \tag{3.14}$$

Note $xf_x(x,y)/f(x,y)$ and $yf_y(x,y)/f(x,y)$ both are 0-order homogeneous with respect to x and y, then

$$\begin{aligned} x\left(\frac{xf_x(x,y)}{f(x,y)}\right)_x + y\left(\frac{xf_x(x,y)}{f(x,y)}\right)_y &= 0, \\ x\left(\frac{yf_y(x,y)}{f(x,y)}\right)_x + y\left(\frac{yf_y(x,y)}{f(x,y)}\right)_y &= 0, \end{aligned} \tag{3.15}$$

and then

$$\begin{aligned} x\left(\frac{xf_x(x,y)}{f(x,y)}\right)_x &= -y\left(\frac{xf_x(x,y)}{f(x,y)}\right)_y = -xy\mathcal{J}, \\ y\left(\frac{yf_y(x,y)}{f(x,y)}\right)_y &= -x\left(\frac{yf_y(x,y)}{f(x,y)}\right)_x = -xy\mathcal{J}. \end{aligned} \tag{3.16}$$

Therefore,

$$g'(t) = x(\ln f)_x + txy\mathcal{I}(\ln b - \ln a) - y(\ln f)_y - txy\mathcal{I}(\ln a - \ln b)$$
$$= x(\ln f)_x - y(\ln f)_y - 2txy\mathcal{I}(\ln a - \ln b) \tag{3.17}$$
$$= x(\ln f)_x - y(\ln f)_y - 2xy\mathcal{I}\ln\left(\frac{x}{y}\right) = \frac{\mathcal{N}(x,y)}{x-y} \quad \text{for } x \neq y.$$

By the mean values theorem, there is a ξ between $|p|$ and $|q|$ such that

$$\frac{g(p)-g(q)}{p-q} = \frac{g(|p|)-g(|q|)}{p-q} = \frac{|p|-|q|}{p-q}g'(\xi) = \frac{p+q}{|p|+|q|}g'(\xi) = \frac{p+q}{|p|+|q|}\frac{\mathcal{N}(x,y)}{x-y}, \quad \text{for } x \neq y, \tag{3.18}$$

where $x = a^{\xi}$, $y = b^{\xi}$. Thus we have

$$(\ln a - \ln b)\left(a\frac{\partial \mathcal{H}_f}{\partial a} - b\frac{\partial \mathcal{H}_f}{\partial b}\right) = \mathcal{H}_f \frac{p+q}{|p|+|q|}\ln\left(\frac{a}{b}\right)\frac{\mathcal{N}(x,y)}{x-y}$$
$$= \mathcal{H}_f \frac{p+q}{|p|+|q|}\frac{\mathcal{N}(x,y)}{\xi}\frac{\ln x - \ln y}{x-y} \tag{3.19}$$
$$= \begin{cases} > 0 & \text{if } p+q > (<)0, \\ < 0 & \text{if } p+q < (>)0. \end{cases}$$

By Lemma 2.5, our required result is derived immediately.

(2) In the case of $p = q \neq 0$. By Lemma 3.3 together with (3.10) and (3.17), we have

$$\frac{1}{\mathcal{H}_f(p,p)}\left(a\frac{\partial \mathcal{H}_f(p,p)}{\partial a} - b\frac{\partial \mathcal{H}_f(p,p)}{\partial b}\right) = \lim_{q\to p}\frac{1}{\mathcal{H}_f(p,q)}\left(a\frac{\partial \mathcal{H}_f(p,q)}{\partial a} - b\frac{\partial \mathcal{H}_f(p,q)}{\partial b}\right)$$
$$= \lim_{q\to p}\frac{g(p)-g(q)}{p-q} = g'(p) = \frac{\mathcal{N}(x,y)}{x-y}, \tag{3.20}$$

where $x = a^p$, $y = b^p$. Hence we have

$$
\begin{aligned}
(\ln a - \ln b)\left(a\frac{\partial \mathcal{H}_f(p,p)}{\partial a} - b\frac{\partial \mathcal{H}_f(p,p)}{\partial b} \right) &= \mathcal{H}_f(p,p)(\ln a - \ln b)\frac{\mathcal{N}(x,y)}{x-y} \\
&= p^{-1}\mathcal{H}_f(p,p)\mathcal{N}(x,y)\frac{\ln x - \ln y}{x-y} \\
&= \begin{cases} > 0 & \text{if } p > (<)0, \\ < 0 & \text{if } p < (>)0. \end{cases}
\end{aligned}
\tag{3.21}
$$

By Lemma 2.5, the required result holds.

(3) In the case of $p = q = 0$. By Lemma 3.3 and (3.20), we have

$$
\frac{1}{\mathcal{H}_f(0,0)}\left(a\frac{\partial \mathcal{H}_f(0,0)}{\partial a} - b\frac{\partial \mathcal{H}_f(0,0)}{\partial b} \right) = \lim_{p\to 0}\left(a\frac{\partial \mathcal{H}_f(p,p)}{\partial a} - b\frac{\partial \mathcal{H}_f(p,p)}{\partial b} \right) = \lim_{p\to 0} g'(p).
\tag{3.22}
$$

However,

$$
\begin{aligned}
g'(0) &= \left(x(\ln f)_x - y(\ln f)_y - 2xy\mathcal{J}\ln\left(\frac{x}{y}\right)\right)\Big|_{x=1,y=1} \\
&= 1\cdot\frac{f_x(1,1)}{f(1,1)} - 1\cdot\frac{f_y(1,1)}{f(1,1)} - 2\cdot 1\cdot 1\cdot \mathcal{J}(1,1)\cdot\ln\left(\frac{1}{1}\right) = 0,
\end{aligned}
\tag{3.23}
$$

where $f_x(1,1) = f_y(1,1)$ due to the symmetry of $f(x,y)$. Thus

$$
(\ln a - \ln b)\left(a\frac{\partial \mathcal{H}_f(p,p)}{\partial a} - b\frac{\partial \mathcal{H}_f(p,p)}{\partial b} \right) = 0.
\tag{3.24}
$$

Summarizing the above three cases, this proof of Theorem 3.4 is complete. □

4. Proof of Main Result

Establishing the Theorem 3.4, we are in a position to prove main result.

Proof of Theorem 1.2. It follows from [3, Section 1], that $\mathbf{F}(p,q;r,s;a,b) = \mathcal{H}_{\mathcal{H}_L}(p,q;a,b)$, where $\mathcal{H}_L = \mathcal{H}_L(r,s) = \mathcal{H}_L(r,s;x,y) = S_{r,s}(x,y)$ is symmetric with respect to x and y. From Lemma 3.3, it follows that $\mathcal{H}_L = \mathcal{H}_L(r,s;x,y) \in C^{\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+)$. Thus we have

$$(\ln\mathcal{H}_L(r,r))_x = \lim_{s\to r}(\ln\mathcal{H}_L(r,s))_x, \tag{4.1}$$

$$(\ln\mathcal{H}_L(r,r))_y = \lim_{s\to r}(\ln\mathcal{H}_L(r,s))_y, \tag{4.2}$$

$$(\ln\mathcal{H}_L(r,r))_{xy} = \lim_{s\to r}(\ln\mathcal{H}_L(r,s))_{xy}, \tag{4.3}$$

$$(\ln\mathcal{H}_L(r,0))_x = \lim_{s\to 0}(\ln\mathcal{H}_L(r,s))_x, \tag{4.4}$$

$$(\ln\mathcal{H}_L(r,0))_y = \lim_{s\to 0}(\ln\mathcal{H}_L(r,s))_y, \tag{4.5}$$

$$(\ln\mathcal{H}_L(r,0))_{xy} = \lim_{s\to r}(\ln\mathcal{H}_L(r,s))_{xy}, \tag{4.6}$$

$$(\ln\mathcal{H}_L(0,0))_x = \lim_{r\to 0}(\ln\mathcal{H}_L(r,r))_x, \tag{4.7}$$

$$(\ln\mathcal{H}_L(0,0))_y = \lim_{r\to 0}(\ln\mathcal{H}_L(r,r))_y, \tag{4.8}$$

$$(\ln\mathcal{H}_L(0,0))_{xy} = \lim_{r\to 0}(\ln\mathcal{H}_L(r,r))_{xy}. \tag{4.9}$$

(1) *In the case of* $rs(r-s)\neq 0$.

Simple partial derivative calculations yield

$$\begin{aligned}
\ln\mathcal{H}_L &= \frac{1}{r-s}\left(\ln|s| + \ln\left|x^r - y^r\right| - \ln|r| - \ln\left|x^s - y^s\right|\right),\\
(\ln\mathcal{H}_L)_x &= \frac{1}{r-s}\left(\frac{rx^{r-1}}{x^r-y^r} - \frac{sx^{s-1}}{x^s-y^s}\right),\\
(\ln\mathcal{H}_L)_y &= \frac{1}{r-s}\left(\frac{-ry^{r-1}}{x^r-y^r} + \frac{sy^{s-1}}{x^s-y^s}\right),\\
\mathcal{I} = (\ln\mathcal{H}_L)_{xy} &= \frac{1}{xy(r-s)}\left(\frac{r^2x^ry^r}{(x^r-y^r)^2} - \frac{s^2x^sy^s}{(x^s-y^s)^2}\right).
\end{aligned} \tag{4.10}$$

Hence,

$$\begin{aligned}
\mathcal{N}(x,y) &= (x-y)\left(x(\ln \mathscr{H}_L)_x - y(\ln \mathscr{H}_L)_y - 2xy\mathcal{I}\ln\left(\frac{x}{y}\right)\right) \\
&= \frac{x-y}{r-s}\left(\frac{r(x^r+y^r)}{x^r-y^r} - \frac{2r^2x^ry^r\ln(x/y)}{(x^r-y^r)^2}\right) \\
&\quad - \frac{x-y}{r-s}\left(\frac{s(x^s+y^s)}{x^s-y^s} - \frac{2s^2x^sy^s\ln(x/y)}{(x^s-y^s)^2}\right) \\
&= (x-y)\frac{P(r)-P(s)}{r-s},
\end{aligned} \tag{4.11}$$

where

$$P(t) = t\left(\frac{x^t+y^t}{x^t-y^t} - \frac{2x^ty^t\ln(x^t/y^t)}{(x^t-y^t)^2}\right). \tag{4.12}$$

It is easy to check that $P(t)$ is even and increasing (decreasing) on $(0,\infty)$ if $x > (<) y$. Indeed,

$$P(-t) = -t\left(\frac{x^{-t}+y^{-t}}{x^{-t}-y^{-t}} - \frac{2x^{-t}y^{-t}\ln(x^{-t}/y^{-t})}{(x^{-t}-y^{-t})^2}\right) = P(t). \tag{4.13}$$

With $(x/y)^t = u$, then $t = \ln u/\ln(x/y)$, and then $P(t)$ can be written as

$$P(t) = \frac{1}{\ln(x/y)}\left(\frac{u+1}{u-1}\ln u - \frac{2u\ln^2 u}{(u-1)^2}\right). \tag{4.14}$$

Direct computation yields

$$\begin{aligned}
P'(t) &= \frac{1}{\ln(x/y)}\left(\frac{u+1}{u-1}\ln u - \frac{2u\ln^2 u}{(u-1)^2}\right)'\frac{du}{dt} \\
&= u\left((u+1)\frac{(u-1)/u-\ln u}{(u-1)^2} + \frac{\ln u}{u-1} - \frac{2\ln^2 u}{(u-1)^2} - 4u\frac{\ln u}{u-1}\frac{(u-1)/u-\ln u}{(u-1)^2}\right) \\
&\underline{\underline{(u-1)/\ln u = L}}\, \frac{(u+1)L^2 - 6uL + 2u(u+1)}{(u-1)L^2} \\
&= \frac{2L(((u+1)/2)L-u) + 4u((u+1)/2-L)}{(u-1)L^2}.
\end{aligned} \tag{4.15}$$

From

$$\frac{u+1}{2}L - u = \frac{u^2-1}{\ln u^2} - \sqrt{u^2} > 0,$$
$$L - \frac{u+1}{2} < 0, \tag{4.16}$$

it follows that $P'(t) > 0$ if $u - 1 > 0$, that is, $x > y$ and $P'(t) < 0$ if $x < y$. Namely,

$$(x-y)P'(t) > 0 \quad \text{for } t > 0 \text{ with } x \neq y. \tag{4.17}$$

By the mean values theorem, there is a η between $|r|$ and $|s|$ such that

$$P(|r|) - P(|s|) = (|r| - |s|)P'(\eta), \tag{4.18}$$

and then

$$\begin{aligned}\mathcal{N}(x,y) &= (x-y)\frac{P(r)-P(s)}{r-s} = (x-y)\frac{r+s}{|r|+|s|}\frac{P(|r|)-P(|s|)}{|r|-|s|} \\ &= \frac{r+s}{|r|+|s|}\cdot(x-y)P'(\eta) \\ &= \begin{cases} > 0 & \text{if } r+s > 0, \\ < 0 & \text{if } r+s < 0. \end{cases}\end{aligned} \tag{4.19}$$

Using Theorem 3.4, for fixed $(p,q),(r,s) \in \mathbb{R}\times\mathbb{R}$ with $rs(r-s)\neq 0$, the four-parameter homogeneous means $\mathbf{F}(p,q;r,s;a,b)$ are Schur geometrically convex on $(0,\infty)\times(0,\infty)$ with respect to (a,b) if and only if $(p+q)(r+s) > 0$ and Schur geometrically concave if and only if $(p+q)(r+s) < 0$.

(2) *In the case of* $s = 0$, $r \neq 0$.

From (4.11) together with (4.4)–(4.6) and (4.19), there is a η_1 between 0 and $|r|$ such that

$$\begin{aligned}\mathcal{N}(x,y) &= (x-y)\left(x(\ln\mathcal{H}_L(r,0))_x - y(\ln\mathcal{H}_L(r,0))_y - 2xy(\ln\mathcal{H}_L(r,0))_{xy}\ln\left(\frac{x}{y}\right)\right) \\ &= \lim_{s\to 0}\left((x-y)\left(x(\ln\mathcal{H}_L(r,s))_x - y(\ln\mathcal{H}_L(r,s))_y - 2xy(\ln\mathcal{H}_L(r,s))_{xy}\ln\left(\frac{x}{y}\right)\right)\right) \\ &= \lim_{s\to 0}(x-y)\frac{P(r)-P(s)}{r-s} = \lim_{s\to 0}\frac{r+s}{|r|+|s|}\cdot\lim_{s\to 0}(x-y)P'(\eta_1) \\ &= \begin{cases} > 0 & \text{if } r > 0, \\ < 0 & \text{if } r < 0, \end{cases} \quad (\text{by } (4.17)).\end{aligned} \tag{4.20}$$

(3) *In the case of* $r = 0$, $s \neq 0$.

Since $\mathcal{H}_L(r,s;x,y)$ is symmetric with respect to r and s, it follows from case 2 that

$$\begin{aligned} \mathcal{N}(x,y) &= (x-y)\left(x(\ln \mathcal{H}_L(0,s))_x - y(\ln \mathcal{H}_L(0,s))_y - 2xy(\ln \mathcal{H}_L(r,s))_{xy}\ln\left(\frac{x}{y}\right)\right) \\ &= \begin{cases} > 0 & \text{if } s > 0, \\ < 0 & \text{if } s < 0. \end{cases} \end{aligned} \tag{4.21}$$

(4) *In the case of* $r = s \neq 0$.

From (4.11) together with (4.1)–(4.3), we have

$$\begin{aligned} \mathcal{N}(x,y) &= (x-y)\left(x(\ln \mathcal{H}_L(r,r))_x - y(\ln \mathcal{H}_L(r,r))_y - 2xy(\ln \mathcal{H}_L(r,r))_{xy}\ln\left(\frac{x}{y}\right)\right) \\ &= \lim_{s\to r}\left((x-y)\left(x(\ln \mathcal{H}_L(r,s))_x - y(\ln \mathcal{H}_L(r,s))_y - 2xy(\ln \mathcal{H}_L(r,s))_{xy}\ln\left(\frac{x}{y}\right)\right)\right) \\ &= (x-y)\lim_{s\to r}\frac{P(r)-P(s)}{r-s} = (x-y)P'(r) \\ &= \begin{cases} > 0 & \text{if } r > 0, \\ < 0 & \text{if } r < 0. \end{cases} \quad (\text{by } (4.17)) \end{aligned} \tag{4.22}$$

(5) *In the case of* $r = s = 0$.

From (4.22) together with (4.7)–(4.9), we have

$$\begin{aligned} \mathcal{N}(x,y) &= (x-y)\left(x(\ln \mathcal{H}_L(0,0))_x - y(\ln \mathcal{H}_L(0,0))_y - 2xy(\ln \mathcal{H}_L(0,0))_{xy}\ln\left(\frac{x}{y}\right)\right) \\ &= \lim_{r\to 0}\left((x-y)\left(x(\ln \mathcal{H}_L(r,r))_x - y(\ln \mathcal{H}_L(r,r))_y - 2xy(\ln \mathcal{H}_L(r,r))_{xy}\ln\left(\frac{x}{y}\right)\right)\right) \\ &= (x-y)\lim_{r\to 0}P'(r). \end{aligned} \tag{4.23}$$

But by (4.15) and some limit computations, we obtain

$$\begin{aligned} \lim_{t\to 0}P'(t)\underline{\underline{(x/y)^t = u}}\lim_{u\to 1}u\Bigg((u+1)\frac{(u-1)/u-\ln u}{(u-1)^2} + \frac{\ln u}{u-1} - \frac{2\ln^2 u}{(u-1)^2} \\ - 4u\frac{\ln u}{u-1}\frac{(u-1)/u-\ln u}{(u-1)^2}\Bigg) = 0, \end{aligned} \tag{4.24}$$

which implies $\mathcal{N}(x,y) = 0$.

Summarizing the above five cases, our required results are derived.

This proof ends. □

5. Other Corollaries

The four-parameter homogeneous means $\mathbf{F}(p,q;r,s;a,b)$ also contain many other two-parameter means, for instance, for the identric (exponential) mean defined by (1.3), its two-parameter means are defined as follows [21, Example 2.3]:

$$\mathscr{H}_I(p,q;a,b) = \begin{cases} \left(\dfrac{I(a^p,b^p)}{I(a^q,b^q)}\right)^{1/(p-q)}, & p\neq q,\ pq\neq 0, \\ G_{I,p}(a,b), & p=q\neq 0, \\ I^{1/p}(a^p,b^p), & p\neq 0,\ q=0, \\ I^{1/q}(a^q,b^q), & p=0,\ q\neq 0, \\ G(a,b), & p=q=0, \end{cases} \tag{5.1}$$

where $G_{I,p}(a,b)=Y^{1/p}(a^p,b^p):=Y_p(a,b),\ Y(a,b)=Ie^{1-G^2/L^2}$.

By [3], we see that

$$\mathscr{H}_I(p,q;a,b)=\mathbf{F}(p,q;1,1;a,b). \tag{5.2}$$

And then according to Theorem 1.2, we have the following corollary.

Corollary 5.1. *For fixed $(p,q)\in\mathbb{R}\times\mathbb{R}$, the two-parameter identric (exponential) means $\mathscr{H}_I(p,q;a,b)$ are Schur geometrically convex on $(0,\infty)\times(0,\infty)$ with respect to (a,b) if and only if $p+q>0$ and Schur geometrically concave if and only if $p+q<0$.*

As another example, for Heronian mean defined by

$$\mathrm{He}=\frac{a+\sqrt{ab}+b}{3}, \tag{5.3}$$

its two-parameter means are defined as follows:

$$\mathscr{H}_{\mathrm{He}}(p,q;a,b) = \begin{cases} \left(\dfrac{a^p+\left(\sqrt{ab}\right)^p+b^p}{a^q+\left(\sqrt{ab}\right)^q+b^q}\right)^{1/(p-q)}, & p\neq q,\ pq\neq 0, \\ a^{(a^p+(\sqrt{ab})^p/2)/(a^p+(\sqrt{ab})^p+b^p)}\, b^{((\sqrt{ab})^p/2+b^p)/(a^p+(\sqrt{ab})^p+b^p)}, & p=q\neq 0, \\ \mathrm{He}^{1/p}(a^p,b^p), & p\neq 0,\ q=0, \\ \mathrm{He}^{1/q}(a^q,b^q), & p=0,\ q\neq 0, \\ G(a,b), & p=q=0. \end{cases} \tag{5.4}$$

By [3], we see that

$$\mathscr{H}_{He}(p,q;\mathrm{a},b) = \mathbf{F}(p,q;3/2,1/2;a,b). \tag{5.5}$$

And then according to Theorem 1.2, we have the following corollary.

Corollary 5.2. *For fixed $(p,q) \in \mathbb{R} \times \mathbb{R}$, the two-parameter Heronian means $\mathscr{H}_{He}(p,q;a,b)$ are Schur geometrically convex on $(0,\infty) \times (0,\infty)$ with respect to (a,b) if and only if $p+q > 0$ and Schur geometrically concave if and only if $p+q < 0$.*

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