Research Article

A Second-Order Boundary Value Problem with Nonlinear and Mixed Boundary Conditions: Existence, Uniqueness, and Approximation

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A second-order boundary value problem with nonlinear and mixed two-point boundary conditions is considered, $Lx = f(t, x, x')$, $t \in (a, b)$, $g(x(a), x(b), x'(a), x'(b)) = 0$, $x(b) = x(a)$ in which L is a formally self-adjoint second-order differential operator. Under appropriate assumptions on L, f, and g, existence and uniqueness of solutions is established by the method of upper and lower solutions and Leray-Schauder degree theory. The general quasilinearization method is then applied to this problem. Two monotone sequences converging quadratically to the unique solution are constructed.

1. Introduction

The investigation of boundary value problems (denoted as BVPs for short) of ordinary differential equations is of great significance. On one hand, it makes a great impact on the studies of partial differential equations [1]. On the other hand, BVPs of ordinary differential equations can be used to describe a large number of mechanical, physical, biological, and chemical phenomena; see [2–5] for example. So far a lot of work has been carried out, including second-order, third-order, and higher-order BVPs with various boundary conditions.

As far as we know, for a long term most of works focused on existence and uniqueness of solutions. The works relating to approximation of solutions are relatively rare. In recent years, some approximate methods, such as the shooting method [6], monotone iterative technique [7], homotopy analysis method [8], and general quasilinearization method have been applied to BVPs for obtaining approximations of solutions. Among these methods, the general quasilinearization method becomes more and more popular.

The quasilinearization method was originally proposed by Bellman and Kalaba [9]. It is a very powerful approximation technique and unlike perturbation methods, is not

dependent on the existence of a small or large parameter. The method, whose sequence of solutions of linear problems convergences to the solution of the original nonlinear problem, is quadratic and monotone, which is one of the reasons for the popularity of this technique. This method was generalized by Lakshmikantham and Vatsala [10] in which the convexity or concavity assumption on the nonlinear functions involved in the problems is relaxed.

So far, the general quasilinearization method, coupled with the method of upper and lower solutions, has been applied to obtain approximation of solutions for a large number of nonlinear problems, for example, BVPs of ordinary differential equations, such as first-order BVP with nonlinear boundary condition [11] and second-order BVPs with Dirichlet boundary condition [12], periodic boundary condition [13], three-point boundary condition [14], four-point boundary condition [15], and m-point boundary condition [16]; BVPs of partial differential equations, such as parabolic initial-boundary value problem [17], elliptic problems with nonlinear boundary condition [18] and p-Laplacian equations with nonlinear boundary condition [19], and so forth; BVPs of impulsive differential equations [20] and impulse functional differential equations with anti-periodic boundary condition [21]; BVPs of some practically nonlinear problems, such as Duffing equation involving both integral and nonintegral forcing terms with Robin boundary condition [22] and forced Duffing equation with discontinuous-type integral boundary condition [23]; as well as some abstract problems such as fixed point theorems in ordered Banach space [24].

Recently, El-Gebeily and O'Regan [25, 26] consider the singular and nonsingular second-order ordinary differential equations

$$l(u(t)) = f(t, u(t)), \quad t \in (a, b) \tag{1.1}$$

with nonlinear boundary conditions as follows:

$$g(u(a), u(b), pu'(a)) = 0, \tag{1.2}$$

$$h(u(a), u(b), pu'(b)) = 0, \tag{1.3}$$

in which $lu = -(pu')' + qu$ is a formally self-adjoint second-order differential operator. This type of BVPs arises in a variety of problems in applied mathematics and physics [27]. By defining the upper and lower solutions of BVP (1.1)–(1.3) suitably, Gebeily and Regan established the existence and uniqueness of solutions and constructed two monotonic iterative sequences converging to the unique solution quadratically. However, it can be noted that the $u'(b)$ and $u'(a)$ terms are not involved in (1.2) and (1.3), respectively. For second-order two-point BVPs, if all the terms $u(a), u(b), u'(a)$ and $u'(b)$ are involved in the boundary conditions, it seems quite difficult to deal with.

In this paper, we consider a second-order BVP with nonlinear and mixed two-point boundary conditions as follows:

$$Lx = f(t, x, x'), \quad t \in (a, b), \tag{1.4}$$

$$g(x(a), x(b), x'(a), x'(b)) = 0, \tag{1.5}$$

$$x(b) = x(a), \tag{1.6}$$

where $Lx = -(px')' + qx$, in which $p \in C^1[a, b]$, $p > 0$, and $q \in C[a, b]$, $q \geq 0$.

In BVP (1.4)–(1.6), it can be found that the boundary condition (1.5) is dependent on all the $x(a), x(b), x'(a)$, and $x'(b)$ terms. First, existence and uniqueness of solutions of BVP (1.4)–(1.6) is established by combining the method of upper and lower solutions with Leray-Schauder degree theory. Then, the general quasilinearization method is applied to construct the approximations of the unique solution. Two monotone sequences of iterations converging to the unique solution quadratically are obtained.

2. Preliminaries

In this section, several definitions and lemmas needed to the main results are given first.

Definition 2.1. $\beta(t), \alpha(t) \in C^2[a, b]$ are called the upper and lower solutions of BVP (1.4)–(1.6), respectively, if

$$
\begin{aligned}
L\beta &\geq f(t, \beta, \beta'), \quad t \in (a, b), \\
g(\beta(a), \beta(b), \beta'(a), \beta'(b)) &\leq 0, \\
\beta(b) &= \beta(a), \\
L\alpha &\leq f(t, \alpha, \alpha'), \quad t \in (a, b), \\
g(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b)) &\geq 0, \\
\alpha(b) &= \alpha(a).
\end{aligned}
\tag{2.1}
$$

Definition 2.2. Let E be a subset of $[a, b] \times R^2$; it is said that the right-hand side function of (1.4) satisfies *Nagumo* condition on E if

$$
|f(t, x, x')| \leq h(|x'|) = O\left(|x'|^2\right)
\tag{2.2}
$$

holds for $(t, x, x') \in E$ and $|x'| \to +\infty$.

Lemma 2.3 (see [28]). *Let $f : [a, b] \times R^2 \to R$ be a continuous function satisfying Nagumo condition on*

$$
E = \left\{ (t, x, x') \in [a, b] \times R^2 : \alpha(t) \leq x(t) \leq \beta(t) \right\},
\tag{2.3}
$$

where $\alpha, \beta : [a, b] \to R$ are continuous functions such that $\alpha(t) \leq \beta(t)$ for all $t \in [a, b]$. Then there exists a constant $N > 0$ such that every solution $x(t)$ of second-order equations $x'' = f(t, x, x')$ with

$$
\alpha(t) \leq x(t) \leq \beta(t), \quad t \in [a, b]
\tag{2.4}
$$

satisfies $\|x'\|_\infty \leq N$, in which N is called the Nagumo constant.

Lemma 2.4. *Boundary value problem as follows:*

$$Lx = -x, \quad t \in (a, b), \tag{2.5}$$

$$x(a) = 0, \quad x(b) = 0 \tag{2.6}$$

has only the trivial solution.

Proof. Assume that $x_0(t)$ is an arbitrarily nontrivial solution of BVP (2.5)-(2.6). From the boundary conditions (2.6), it can be concluded that $x_0(t)$ can achieve its positive maximum or negative minimum in the interior of $[a, b]$, suppose at t_0, $t_0 \in (a, b)$.

If $x_0(t)$ achieves its positive maximum, then

$$x_0(t_0) > 0, \quad x_0'(t_0) = 0, \quad x_0''(t_0) \leq 0 \tag{2.7}$$

which means that

$$Lx_0(t_0) = -p(t_0)x_0''(t_0) - p'(t_0)x_0'(t_0) + q(t_0)x(t_0) \geq 0. \tag{2.8}$$

On the other hand, it can be derived from (2.5) that

$$Lx_0(t_0) = -x_0(t_0) < 0. \tag{2.9}$$

It is a contradiction.

If $x_0(t)$ achieves its negative minimum, similar arguments lead to a contradiction too. Hence, BVP (2.5)-(2.6) has only the trivial solution. $\qquad\square$

Lemma 2.5 (see [26]). *Define a linear operator*

$$l : C^1[a, b] \longrightarrow C[a, b] \times R \times R \tag{2.10}$$

by

$$lx(t) = \left(px'(a) - px'(t) + \int_a^t q(s)x(s)\,ds, x(a), x(b) \right). \tag{2.11}$$

Then l^{-1} exists and is continuous.

Lemma 2.6. *Assume that*

(1) $\alpha(t), \beta(t)$ *are the lower and upper solutions of BVP* (1.4)–(1.6)*, respectively;*

(2) $f(t, x, x')$ *is continuous on* $[a, b] \times R^2$ *and is strictly decreasing in x on*

$$D = \left\{ (t, x, x') \in [a, b] \times R^2 : \min\{\alpha(t), \beta(t)\} \leq x \leq \max\{\alpha(t), \beta(t)\} \right\}; \tag{2.12}$$

(3) g is continuously differentiable on R^4, strictly decreasing in the first and second variables, and nondecreasing and nonincreasing, respectively, in the third and forth variables.

Then,

$$\alpha(t) \leq \beta(t), \quad t \in [a,b]. \tag{2.13}$$

Proof. Suppose that $\alpha(t) > \beta(t)$ for some $t \in [a,b]$. Then there exist some $t_0 \in [a,b]$ such that

$$\max_{t \in [a,b]} [\alpha(t) - \beta(t)] = \alpha(t_0) - \beta(t_0) > 0. \tag{2.14}$$

If $t_0 \in (a,b)$, then $\alpha(t_0) > \beta(t_0)$, $\alpha'(t_0) = \beta'(t_0)$, and $\alpha''(t_0) \leq \beta''(t_0)$ and consequently,

$$L[\alpha(t_0) - \beta(t_0)] = L\alpha(t_0) - L\beta(t_0) \geq 0. \tag{2.15}$$

However, it follows from Definition 2.1 and mean value theorem that

$$
\begin{aligned}
L\alpha(t_0) - L\beta(t_0) &\leq f\left(t_0, \alpha(t_0), \alpha'(t_0)\right) - f\left(t_0, \beta(t_0), \beta'(t_0)\right) \\
&= f_x'\left(t_0, \xi, \beta'(t_0)\right)\left(\alpha(t_0) - \beta(t_0)\right) \\
&< 0,
\end{aligned}
\tag{2.16}
$$

in which $\xi \in (\beta(t_0), \alpha(t_0))$, and the last inequality follows from the strictly decreasing property of f in x. It is a contradiction.

If $t_0 = a$, then $\alpha(a) > \beta(a)$ and

$$\alpha'(a) - \beta'(a) = \lim_{t \to a^+} \frac{[\alpha(t) - \beta(t)] - [\alpha(a) - \beta(a)]}{t - a} \leq 0. \tag{2.17}$$

By the definitions of the lower and upper solutions, we have

$$\alpha(b) = \alpha(a) > \beta(a) = \beta(b). \tag{2.18}$$

Moreover,

$$\alpha'(b) - \beta'(b) = \lim_{t \to b^-} \frac{(\alpha(t) - \beta(t)) - (\alpha(b) - \beta(b))}{t - b} = \lim_{t \to b^-} \frac{(\alpha(t) - \beta(t)) - (\alpha(a) - \beta(a))}{t - b} \geq 0. \tag{2.19}$$

Consequently, in view of the monotonicity of g in its variables, it follows that

$$0 \geq g\left(\beta(a), \beta(b), \beta'(a), \beta'(b)\right) > g\left(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b)\right) \geq 0, \tag{2.20}$$

which is a contradiction.

If $t_0 = b$, similar deductions lead to a contradiction too. Hence, Lemma 2.6 is proved. □

3. Existence and Uniqueness of Solutions

Theorem 3.1. *Assume that*

(1) $\alpha(t), \beta(t)$ *are the lower and upper solutions of BVP* (1.4)–(1.6), *respectively;*

(2) $f(t, x, x')$ *is continuous on* $[a, b] \times R^2$ *and is strictly decreasing in* x *on*

$$D = \left\{ (t, x, x') \in [a, b] \times R^2 : \alpha(t) \le x(t) \le \beta(t) \right\} \tag{3.1}$$

and satisfies Nagumo condition on D;

(3) g *is continuously differentiable, strictly decreasing in the first and second variables, nondecreasing and nonincreasing, respectively, in the third and forth variables on* $[\alpha(a), \beta(a)] \times [\alpha(b), \beta(b)] \times R^2$.

Then there exists a unique solution $x(t) \in C^2[a.b]$ *of BVP* (1.4)–(1.6) *such that*

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [a, b]. \tag{3.2}$$

Proof. Define

$$w(t, x) = \begin{cases} \alpha(t), & x(t) < \alpha(t), \\ x(t), & \alpha(t) \le x(t) \le \beta(t), \\ \beta(t), & x(t) > \beta(t). \end{cases} \tag{3.3}$$

Introduce the following auxiliary BVPs with homotopy character:

$$\begin{aligned} Lx &= \lambda f(t, w(t, x), x') - x + \lambda w(t, x), \\ x(a) &= \lambda \big[g(w(a, x(a)), w(b, x(b)), x'(a), x'(b)) + w(a, x(a)) \big], \\ x(b) &= x(a) \end{aligned} \tag{3.4}$$

in which $\lambda \in [0, 1]$ is called the embedded parameter.

By the continuity of f and g, and by the boundedness of $\alpha(t)$ and $\beta(t)$, we can select a sufficiently large constant $M_0 > 0$ such that

$$-M_0 < \alpha(t) \le \beta(t) < M_0, \tag{3.5}$$

$$f(t, \beta(t), 0) + \beta(t) < M_0, \tag{3.6}$$

$$g(\beta(a), \beta(b), 0, 0) + \beta(a) < M_0, \tag{3.7}$$

$$-M_0 < f(t, \alpha(t), 0) + \alpha(t), \tag{3.8}$$

$$-M_0 < g(\alpha(a), \alpha(b), 0, 0) + \alpha(a). \tag{3.9}$$

In what follows, the proof of the existence of solutions is divided into four steps.

Step 1. Show that, for $\lambda \in [0,1]$, every solution $x(t)$ of BVPs (3.4) satisfies

$$|x(t)| < M_0, \quad t \in [a,b]. \tag{3.10}$$

Suppose that the estimate $|x(t)| < M_0$ is not true. Then, there must be some points in $[a,b]$ such that either $x(t) \geq M_0$ or $x(t) \leq -M_0$. For the former case, $x(t)$ has a positive maximum, suppose at t_1, that is,

$$\max_{t \in [a,b]} x(t) = x(t_1)(\geq M_0 > 0) \tag{3.11}$$

can be assumed. We have three cases to consider.

Case 1 $(t_1 \in (a,b))$. In this case, $x(t_1) \geq M_0 > \beta(t_1)$, $x'(t_1) = 0$ and $x''(t_1) \leq 0$ which leads to $Lx(t_1) \geq 0$. On the other hand, since the definition of $w(t,x)$ yields $w(t_1, x(t_1)) = \beta(t_1)$; hence, for $\lambda \in (0,1]$, we have

$$
\begin{aligned}
Lx(t_1) &= \lambda f(t_1, \beta(t_1), 0) - x(t_1) + \lambda \beta(t_1) \\
&= \lambda \big[f(t_1, \beta(t_1), 0) + \beta(t_1) \big] - x(t_1) \\
&\leq \lambda \big[f(t_1, \beta(t_1), 0) + \beta(t_1) - M_0 \big] < 0,
\end{aligned}
\tag{3.12}
$$

in which the last inequality is obtained by the inequality (3.6). It is a contradiction.

For $\lambda = 0$, it can be derived that

$$0 \leq Lx(t_1) = -x(t_1) \leq -M_0 < 0, \tag{3.13}$$

which is also a contradiction.

Case 2 $(t_1 = a)$. In this case, $x(a) \geq M_0 > \beta(a)$, $x'(a) \leq 0$. Furthermore, we have

$$
\begin{aligned}
x(b) &= x(a) \geq M_0 > \beta(b), \\
x'(b) &= \lim_{t \to b^-} \frac{x(t) - x(b)}{t - b} = \lim_{t \to b^-} \frac{x(t) - x(a)}{t - b} \geq 0.
\end{aligned}
\tag{3.14}
$$

The definition of $w(t,x)$ means that $w(a, x(a)) = \beta(a)$ and $w(b, x(b)) = \beta(b)$. Hence, for $\lambda \in (0,1]$, in view of $x'(a) \leq 0$ and $x'(b) \geq 0$, it follows from the monotonicity of g in its third and forth variables and the inequality (3.7) that

$$
\begin{aligned}
M_0 \leq x(a) &= \lambda \big[g(w(a, x(a)), w(b, x(b)), x'(a), x'(b)) + w(a, x(a)) \big] \\
&= \lambda \big[\beta(a) + g(\beta(a), \beta(b), x'(a), x'(b)) \big] \\
&\leq \lambda \big[\beta(a) + g(\beta(a), \beta(b), 0, 0) \big] \\
&< \lambda M_0 \leq M_0
\end{aligned}
\tag{3.15}
$$

which is a contradiction.

For $\lambda = 0$, we can obtain from the second equation in (3.4) that

$$0 < M_0 \leq x(a) = 0, \tag{3.16}$$

which is a contradiction too.

Case 3 ($t_1 = b$). In this case, $x(b) \geq M_0 > \beta(b)$, $x'(b) \geq 0$, and furthermore,

$$x(a) = x(b) \geq M_0 > \beta(a),$$

$$x'(a) = \lim_{t \to a^+} \frac{x(t) - x(a)}{t - a} = \lim_{t \to a^+} \frac{x(t) - x(b)}{t - a} \leq 0. \tag{3.17}$$

The same deductions with those in Case 2 yield, for $\lambda \in (0, 1]$, that

$$M_0 \leq x(b) = x(a) \leq \lambda [\beta(a) + g(\beta(a), \beta(b), 0, 0)]$$

$$< \lambda M_0 \leq M_0. \tag{3.18}$$

It is also a contradiction. For $\lambda = 0$, it can be deduced from the boundary conditions of (3.4) that

$$0 < M_0 \leq x(b) = x(a) = 0, \tag{3.19}$$

which is a contradiction too.

Step 2. Show that there exists a positive constant M_1 such that, for $\lambda \in [0, 1]$, every solution $x(t)$ of BVPs (3.4) satisfies

$$|x'(t)| < M_1, \quad t \in [a, b]. \tag{3.20}$$

Let

$$E = \left\{ (t, x, x') \in [a, b] \times R^2 : |x(t)| < M_0 \right\}. \tag{3.21}$$

Define

$$F_\lambda(t, x, x') = \frac{1}{-p} [p'x' - qx + \lambda f(t, w(t, x), x') - x + \lambda w(t, x)], \quad (t, x, x') \in E. \tag{3.22}$$

Consequently, (1.4) can be rewritten as

$$x'' = F_\lambda(t, x, x'). \tag{3.23}$$

Since the function f satisfies *Nagumo* condition on E,

$$
\begin{aligned}
\left| F_\lambda(t, x, x') \right| &= \frac{1}{p} \left| p'x' - qx + \lambda f(t, w(t, x), x') - x + \lambda w(t, x) \right| \\
&= O\left(|x'|^2 \right), \quad \text{for } (t, x, x') \in E, \ |x'| \longrightarrow +\infty
\end{aligned}
\tag{3.24}
$$

is obvious if the boundedness of $x(t)$ and $w(t, x)$, $p \in C^1[a, b]$, $p > 0$, and $q \in C[a, b]$ is kept in mind. That is, $F_\lambda(t, x, x')$ satisfies *Nagumo* condition on E. Hence, by **Lemma 2.3**, the estimate $|x'(t)| < M_1$, $t \in [a, b]$, can be obtained.

Step 3. Show that for $\lambda = 1$, BVP (3.4) has at least one solution $x_1(t)$.

Define a nonlinear operator

$$
N_\lambda : C^1[a, b] \longrightarrow C[a, b] \times R \times R
\tag{3.25}
$$

by

$$
N_\lambda x(t) = \left(\int_a^t \left[\lambda f(s, w(s, x), x') - x + \lambda w(s, x) \right] ds, \, A_\lambda, \, B_\lambda \right)
\tag{3.26}
$$

with

$$
\begin{aligned}
A_\lambda &= \lambda \left[g(w(a, x(a)), w(b, x(b)), x'(a), x'(b)) + w(a, x(a)) \right], \\
B_\lambda &= x(a).
\end{aligned}
\tag{3.27}
$$

Consequently, BVPs (3.4) are equivalent to the following operator equations:

$$
\left(I - l^{-1} N_\lambda \right) x = 0,
\tag{3.28}
$$

in which l is the linear operator defined in **Lemma 2.5**, and I is the unit operator.

Define the norm in $C[a, b] \times R \times R$ as

$$
\|x(t)\|_{C[a,b] \times R \times R} = \|x(t)\|_{C[a,b]} + |R_1| + |R_2|
\tag{3.29}
$$

for $x(t) \in C[a, b] \times R \times R$, in which R_1, R_2 are two real numbers.

Let $\{x_n(t)\}$ be the bounded sequence on $C^1[a, b]$. It then follows from Steps 1 and 2 that $\{N_\lambda x_n\}$ and $\{N'_\lambda x_n\}$ are both uniformly bounded in the sense of the norm defined above. Thus, $\{N_\lambda x_n\}$ is equicontinuous on $C[a, b] \times R \times R$. Consequently, Arzela-Ascoli theorem yields that $\{N_\lambda x_n\}$ is compact on $C[a, b] \times R \times R$. Therefore, $N_\lambda x$ is a completely continuous operator.

Furthermore, the operator

$$
T_\lambda : C^1[a, b] \longrightarrow C^1[a, b]
\tag{3.30}
$$

defined by

$$T_\lambda = l^{-1} N_\lambda \tag{3.31}$$

is also a completely continuous operator.

Define a bounded and open domain as follows:

$$\Omega = \left\{ x(t) \in C^1[a,b] : |x(t)| < M_0, \ |x'(t)| < M_1 \right\}. \tag{3.32}$$

It follows from Steps 1 and 2 that

$$\left(I - l^{-1} N_\lambda \right) \partial \Omega \neq 0. \tag{3.33}$$

Therefore, the degree $\mathrm{Deg}(I - l^{-1} N_\lambda, \Omega, 0)$ is well defined. Then the invariance of degree under homotopy yields

$$\mathrm{Deg}\left(I - l^{-1} N_1, \Omega, 0 \right) = \mathrm{Deg}\left(I - l^{-1} N_0, \Omega, 0 \right). \tag{3.34}$$

Since the operator equation $(I - l^{-1} N_0)x = 0$ is equivalent to BVP (2.5)-(2.6) which has only the trivial solution, therefore

$$\mathrm{Deg}\left(I - l^{-1} N_1, \Omega, 0 \right) = \mathrm{Deg}\left(I - l^{-1} N_0, \Omega, 0 \right) = \pm 1. \tag{3.35}$$

Consequently, the operator equation $(I - l^{-1} N_1)x = 0$ gives

$$Lx = f\left(s, w(s,x), x' \right) - x + w(s,x),$$
$$x(a) = g\left(w(a, x(a)), w(b, x(b)), x'(a), x'(b) \right) + w(a, x(a)), \tag{3.36}$$
$$x(b) = x(a),$$

which has at least one solution $x_1(t) \in C^2[a,b]$.

Step 4. Show that every solution $x(t) \in C^2[a,b]$ of BVP (3.36) satisfies

$$\alpha(t) \leq x(t) \leq \beta(t), \quad t \in [a,b]. \tag{3.37}$$

The right-hand side of this inequality is first proved. For the sake of contradiction, suppose that $x(t) > \beta(t)$ for some $t \in [a,b]$. Define $h(t) = x(t) - \beta(t)$, then $h(t)$ has a positive maximum at some $t_1 \in [a,b]$.

Case 1 ($t_1 \in (a,b)$). In this case, $h(t_1) = x(t_1) - \beta(t_1) > 0$, $h'(t_1) = x'(t_1) - \beta'(t_1) = 0$, and $h''(t_1) \leq 0$ and consequently, $Lh(t_1) \geq 0$.

On the other hand,

$$
\begin{aligned}
Lh(t_1) &= Lx(t_1) - L\beta(t_1) \\
&\leq f(t_1, w(t_1, x(t_1)), x'(t_1)) - x(t_1) + w(t, x(t_1)) - f(t_1, \beta(t_1), \beta'(t_1)) \\
&= f(t_1, \beta(t_1), \beta'(t_1)) - x(t_1) + \beta(t_1) - f(t_1, \beta(t_1), \beta'(t_1)) \\
&= -x(t_1) + \beta(t_1) < 0,
\end{aligned}
\tag{3.38}
$$

which is a contradiction.

Case 2 ($t_1 = a$). In this case,

$$
h(a) = x(a) - \beta(a) > 0, \qquad h'(a) = x'(a) - \beta'(a) \leq 0.
\tag{3.39}
$$

Furthermore, we have

$$
\beta(b) = \beta(a) < x(a) = x(b),
$$

$$
x'(b) - \beta'(b) = h'(b) = \lim_{t \to b^-} \frac{h(t) - h(b)}{t - b} = \lim_{t \to b^-} \frac{h(t) - h(a)}{t - b} \geq 0.
\tag{3.40}
$$

The definition of $w(t, x)$ yields $w(a, x(a)) = \beta(a)$ and $w(b, x(b)) = \beta(b)$. In view of $x'(a) \leq \beta'(a)$ and $x'(b) \geq \beta'(b)$, it then follows from the monotonicity of g in its third and forth variables that

$$
\begin{aligned}
\beta(a) < x(a) &= g(w(a, x(a)), w(b, x(b)), x'(a), x'(b)) + w(a, x(a)) \\
&\leq g(\beta(a), \beta(b), \beta'(a), \beta'(b)) + \beta(a) \\
&\leq \beta(a).
\end{aligned}
\tag{3.41}
$$

It is also a contradiction.

Case 3 ($t_1 = b$). In this case,

$$
h(b) = x(b) - \beta(b) > 0, \qquad h'(b) = x'(b) - \beta'(b) \geq 0.
\tag{3.42}
$$

Furthermore,

$$
\beta(a) = \beta(b) < x(b) = x(a),
$$

$$
x'(a) - \beta'(a) = h'(a) = \lim_{t \to a^+} \frac{h(t) - h(a)}{t - a} = \lim_{t \to a^+} \frac{h(t) - h(b)}{t - a} \leq 0.
\tag{3.43}
$$

Similar deductions lead to a contradiction too.

Therefore, the inequality $x(t) \leq \beta(t)$ holds. In a similar way, $\alpha(t) \leq x(t)$ can be proved.

Till now, by Steps 1–4, the proof of the existence of solutions is completed. In what follows, we turn to the proof of the uniqueness of solutions.

We may assume that $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of BVP (1.4)–(1.6). Define

$$y(t) = x_1(t) - x_2(t), \quad t \in [a,b]. \tag{3.44}$$

If $y(t) \equiv 0$ for $t \in [a,b]$, the uniqueness of solutions is obvious. Otherwise, there must be some points in $[a,b]$ such that either $y(t) > 0$ or $y(t) < 0$. For the former case, we define

$$\max_{t \in [a,b]} y(t) := y(t_1). \tag{3.45}$$

Similarly, we only need to consider the following three cases.

Case 1 $(t_1 \in (a,b))$. In this case,

$$y(t_1) = x_1(t_1) - x_2(t_1) > 0, \qquad y'(t_1) = x_1'(t_1) - x_2'(t_1) = 0, \qquad y''(t_1) \leq 0. \tag{3.46}$$

Thus, $0 \leq Ly(t_1)$. On the other hand, the mean value theorem and the strictly decreasing property of $f(t,x,x')$ in x results that

$$\begin{aligned}
Ly(t_1) &= Lx_1(t_1) - Lx_2(t_1) \\
&= f\big(t_1, x_1(t_1), x_1'(t_1)\big) - f\big(t_1, x_2(t_1), x_2'(t_1)\big) \\
&= f_x'\big(t_1, \theta, x_1'(t_1)\big)\big(x_1(t_1) - x_2(t_1)\big) < 0,
\end{aligned} \tag{3.47}$$

which is a contradiction, where $\theta \in (x_2(t_1), x_1(t_1))$.

Case 2 $(t_1 = a)$. In this case,

$$y(a) = x_1(a) - x_2(a) > 0, \qquad y'(a) = x_1'(a) - x_2'(a) \leq 0. \tag{3.48}$$

Moreover,

$$y(b) = x_1(b) - x_2(b) = x_1(a) - x_2(a) = y(a) > 0,$$

$$x_1'(b) - x_2'(b) = y'(b) = \lim_{t \to b^-} \frac{y(t) - y(b)}{t - b} = \lim_{t \to b^-} \frac{y(t) - y(a)}{t - b} \geq 0. \tag{3.49}$$

Nevertheless, in view of the monotonicity of g in its variables, it follows from the mean value theorem that

$$\begin{aligned}
0 &= g\big(x_1(a), x_1(b), x_1'(a), x_1'(b)\big) - g\big(x_2(a), x_2(b), x_2'(a), x_2'(b)\big) \\
&= g_1'[x_1(a) - x_2(a)] + g_2'[x_1(b) - x_2(b)] + g_3'[x_1'(a) - x_2'(a)] + g_4'[x_1'(b) - x_2'(b)] < 0,
\end{aligned} \tag{3.50}$$

which is a contradiction, in which $g_i' = g_i'(\xi_1, \xi_2, \xi_3, \xi_4)$, $i = 1, 2, 3, 4$, $\xi_1 \in (x_2(a), x_1(a))$ and ξ_2, ξ_3, ξ_4 can be located by analogy.

Case 3 ($t_1 = b$). For this case, in the same way, it can be proved that this case is also impossible.

Consequently, the conclusion in Theorem 3.1 is proved. □

4. Approximations of the Unique Solution

In this section, a series of boundary value problems whose sequence of solutions converging to the unique solution of BVP (1.4)–(1.6) is constructed.

Theorem 4.1. *Assume that*

(1) *there exist the upper and lower solutions $\beta_0(t), \alpha_0(t)$ of BVP (1.4)–(1.6);*

(2) *$f(t, x, x')$ is continuous on $[a, b] \times R^2$ and satisfies Nagumo condition on*

$$D = \left\{ (t, x, x') \in [a, b] \times R^2 : \alpha_0(t) \le x(t) \le \beta_0(t) \right\}; \tag{4.1}$$

(3) *$f_{xx}(t, x, x')$ exists and is continuous on $[a, b] \times R^2$ and $f_x(t, x, x') < 0$, $f_{xx}(t, x, x') \le 0$ on $D \times [-C, C]$, where*

$$C \ge \max\left\{ N, \max_{t \in [a,b]} \alpha_0'(t), \max_{t \in [a,b]} \beta_0'(t) \right\} \tag{4.2}$$

is chosen sufficiently large, and N is the Nagumo constant;

(4) *$g(u, v, z, w) \in C^1([\alpha_0(a), \beta_0(a)] \times [\alpha_0(b), \beta_0(b)] \times R^2, R)$, and it is strictly decreasing in u and v, no-decreasing and nonincreasing, respectively, in z and w.*

Then there exist a monotone nonincreasing sequence $\{\beta_n(t)\}$ and a monotone nondecreasing sequence $\{\alpha_n(t)\}$ converging uniformly to the unique solution of BVP (1.4)–(1.6). Moreover, the rate of the convergence is quadratic.

Proof. Define a piecewise function $q(x') = \max\{-C, \min\{x', C\}\}$ for $x' \in R$, and introduce $f(t, x, q(x'))$ on $D \times [-C, C]$ and

$$F(t, x, x'; z) = f(t, z, q(x')) + f_x(t, z, q(x'))(x - z) \tag{4.3}$$

for $(t, x, x'; z) \in D \times [-C, C] \times [\alpha_0(t), \beta_0(t)]$.

Obviously, $F(t, x, x'; z)$ is linear in x if z is viewed as a known quantity. Moreover, $f(t, x, q(x'))$ and $F(t, x, x'; z)$ both satisfy the *Nagumo* condition.

We begin by considering the following BVP:

$$Lx = F(t, x, x'; \beta_0),$$

$$g(x(a), x(b), x'(a), x'(b)) = 0, \tag{4.4}$$

$$x(b) = x(a).$$

In view of the definition of the upper solution $\beta_0(t)$ of BVP (1.4)–(1.6), it can be verified directly that

$$
\begin{aligned}
L\beta_0 &\geq f(t, \beta_0, \beta_0') = F(t, \beta_0, \beta_0'; \beta_0), \\
g(\beta_0(a), \beta_0(b), \beta_0'(a), \beta_0'(b)) &\leq 0, \\
\beta_0(b) &= \beta_0(a).
\end{aligned}
\tag{4.5}
$$

Hence, $\beta_0(t)$ is the upper solution of BVP (4.4).

According to condition (3), it can be deduced that

$$
\begin{aligned}
F(t, &\alpha_0, \alpha_0'; \beta_0) - f(t, \alpha_0, \alpha_0') \\
&= f(t, \beta_0, \alpha_0') + f_x(t, \beta_0, \alpha_0')(\alpha_0 - \beta_0) - f(t, \alpha_0, \alpha_0') \\
&= f(t, \beta_0, \alpha_0') + f_x(t, \beta_0, \alpha_0')(\alpha_0 - \beta_0) \\
&\quad - \left[f(t, \beta_0, \alpha_0') + f_x(t, \beta_0, \alpha_0')(\alpha_0 - \beta_0) + \frac{1}{2} f_{xx}(t, \xi, \alpha_0')(\alpha_0 - \beta_0)^2 \right] \\
&= -\frac{1}{2} f_{xx}(t, \xi_1, \alpha_0')(\alpha_0 - \beta_0)^2 \geq 0,
\end{aligned}
\tag{4.6}
$$

in which $|\alpha_0'| < C$ has been taken into account, and $\xi_1 \in (\alpha_0, \beta_0)$. Consequently, by taking the definition of the lower solution $\alpha_0(t)$ of BVP (1.4)–(1.6) into account, we have

$$
\begin{aligned}
L\alpha_0 &\leq f(t, \alpha_0, \alpha_0') \leq F(t, \alpha_0, \alpha_0'; \beta_0), \\
g(\alpha_0(a), \alpha_0(b), \alpha_0'(a), \alpha_0'(b)) &\geq 0, \\
\alpha_0(b) &= \alpha_0(a),
\end{aligned}
\tag{4.7}
$$

which indicates that $\alpha_0(t)$ is the lower solution of BVP (4.4). Since BVP (4.4) satisfies all the requirements in Theorem 3.1, hence BVP (4.4) has a unique solution $\beta_1(t) \in C^2[a, b]$ with $\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t)$ and $|\beta_1'(t)| < C_1, t \in [a, b]$, where $C_1 \leq C$ is a constant.

Next, we consider

$$
\begin{aligned}
Lx &= F(t, x, x'; \beta_1), \\
g(x(a), x(b), x'(a), x'(b)) &= 0, \\
x(b) &= x(a).
\end{aligned}
\tag{4.8}
$$

It can be shown that

$$
\begin{aligned}
L\beta_1 &= F(t, \beta_1, \beta_1'; \beta_0) \\
&= f(t, \beta_0, \beta_1') + f_x(t, \beta_0, \beta_1')(\beta_1 - \beta_0) \\
&\geq f(t, \beta_0, \beta_1') + f_x(t, \beta_0, \beta_1')(\beta_1 - \beta_0) + \frac{1}{2} f_{xx}(t, \xi_2, \beta_1')(\beta_1 - \beta_0)^2 \\
&= f(t, \beta_1, \beta_1') = F(t, \beta_1, \beta_1'; \beta_1)
\end{aligned}
\tag{4.9}
$$

in which $\xi_2 \in (\beta_1, \beta_0)$. Thus, $\beta_1(t)$ is the upper solution of BVP (4.8) since

$$
\begin{aligned}
g(\beta_1(a), \beta_1(b), \beta_1'(a), \beta_1'(b)) &\leq 0, \\
\beta_1(b) &= \beta_1(a)
\end{aligned}
\tag{4.10}
$$

are obvious if we keep that $\beta_1(t)$ is the solution of BVP (4.4) in mind.

Similarly, we can show that $\alpha_0(t)$ is the lower solution of BVP (4.8). Hence, according to Theorem 3.1, there exists a unique solution $\beta_2(t) \in C^2[a,b]$ of BVP (4.8) such that $\alpha_0(t) \leq \beta_2(t) \leq \beta_1(t)$ and $|\beta_2'(t)| < C_1$, $t \in [a,b]$, where C_1 is a constant.

By repeating the above procedure iteratively, we obtain a monotone nonincreasing sequence $\{\beta_n(t)\}$ with

$$
\alpha_0(t) \leq \cdots \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t), \quad |\beta_i'(t)| \leq C_1, \ t \in [a,b],
\tag{4.11}
$$

where $\{\beta_n(t)\}$ are, respectively, the unique solution of the BVPs as follows:

$$
\begin{aligned}
Lx &= F(t, x, x'; \beta_{n-1}), \\
g(x(a), x(b), x'(a), x'(b)) &= 0, \\
x(b) &= x(a).
\end{aligned}
\tag{4.12}
$$

Formula (4.11), the properties of the operator L, and the continuity of F yield the uniform boundedness of $\{\beta_n''(t)\}$ for $t \in [a,b]$. Therefore, the sequences $\{\beta_n^i(t)\}$ $(i = 0, 1)$ are uniformly bounded and equicontinuous on $[a,b]$. Hence, Ascoli-Arzela theorem guarantees the existence of subsequences $\{\beta_{n_k}^i(t)\} \subset \{\beta_n^i(t)\}$ and a function $x(t) \in C^1[a,b]$ with $\{\beta_{n_k}^i(t)\}$ converging uniformly to $x^i(t)$ $(i = 0, 1)$ on $[a,b]$ as $k \to +\infty$.

Now let

$$
\begin{aligned}
f_k &= F(t, \beta_{n_k}, \beta_{n_k}'; \beta_{n_k-1}), \\
g_k &= g(\beta_{n_k}(a), \beta_{n_k}(b), \beta_{n_k}'(a), \beta_{n_k}'(b)).
\end{aligned}
\tag{4.13}
$$

The continuity of the functions F and g imply that, as $k \to +\infty$,

$$f_k = F(t, \beta_{n_k}, \beta'_{n_k}; \beta_{n_k-1}) \longrightarrow f(t, x, x'),$$

$$g_k = g\big(\beta_{n_k}(a), \beta_{n_k}(b), \beta'_{n_k}(a), \beta'_{n_k}(b)\big) \longrightarrow g\big(x(a), x(b), x'(a), x'(b)\big). \tag{4.14}$$

The Lebeague dominated convergence theorem then yields

$$\int_a^t f_k(s)\,ds \longrightarrow \int_a^t f(s, x(s), x'(s))\,ds \tag{4.15}$$

for each $t \in [a, b]$ as $k \to +\infty$.

Integrating the equation

$$L\beta_{n_k} = F(t, \beta_{n_k}, \beta'_{n_k}; \beta_{n_k-1}) \tag{4.16}$$

from a to t results in

$$p\beta'_{n_k}(a) - p\beta'_{n_k}(t) + \int_a^t q(s)\beta_{n_k}(s)\,ds = \int_a^t f_k(s)\,ds. \tag{4.17}$$

Passing to the limit in (4.17), we obtain

$$px'(a) - px'(t) + \int_a^t q(s)x(s)\,ds = \int_a^t f(s, x(s), x'(s))\,ds. \tag{4.18}$$

Hence, $x(t)$ is the unique solution of BVP (1.4)–(1.6).

To show the quadratic rate of convergence, we define the error function as follows:

$$e_n(t) = \beta_n(t) - x(t). \tag{4.19}$$

Hence, $e_n(t) \geq 0$ for $t \in [a, b]$.

$$
\begin{aligned}
Le_n &= L(\beta_n - x) = L\beta_n - Lx \\
&= F(t, \beta_n, \beta'_n; \beta_{n-1}) - f(t, x, x') \\
&= f(t, \beta_{n-1}, \beta'_n) + f_x(t, \beta_{n-1}, \beta'_n)(\beta_n - \beta_{n-1}) - f(t, x, x') \\
&= f(t, x, \beta'_n) - f(t, x, x') + f(t, \beta_{n-1}, \beta'_n) - f(t, x, \beta'_n) \\
&\quad + f_x(t, \beta_{n-1}, \beta'_n)(\beta_n - x) + f_x(t, \beta_{n-1}, \beta'_n)(x - \beta_{n-1}) \\
&= f_{x'}(t, \beta_{n-1}, \eta_1)e'_n + f_x(t, \beta_{n-1}, \beta'_n)e_n - \frac{1}{2}f_{xx}(t, \eta_2, \beta'_n)(x - \beta_{n-1})^2 \\
&\leq f_{x'}(t, \beta_{n-1}, \eta_1)e'_n + K_1 e_n + K_2\|e_{n-1}\|^2,
\end{aligned}
\tag{4.20}
$$

in which η_1 lies between β'_n and x', $\eta_2 \in (x, \beta_{n-1})$, $K_1 = \max_{t \in [a,b]} f_x(t, \beta_{n-1}, \beta'_n)$ is a negative constant, K_2 is a constant with $0 \le -(1/2) f_{xx}(t, \eta_2, \beta'_n) \le K_2$ for $(t, \eta_2, \beta'_n) \in D \times [-C, C]$, and $\|e_{n-1}\| = \max_{t \in [a,b]} |e_{n-1}(t)|$.

Moreover, since $\beta_n(t), x(t)$ are the solutions of BVPs (4.12) and BVP (1.4)–(1.6), respectively, therefore, mean value theorem yields

$$g'_1 e_n(a) + g'_2 e_n(b) + g'_3 e'_n(a) + g'_2 e'_n(b) = 0, \tag{4.21}$$

$$e_n(a) = e_n(b), \tag{4.22}$$

in which $g'_i = g'_i(\eta_3, \eta_4, \eta_5, \eta_6)$, $i = 1, 2, 3, 4$, where $\eta_3 \in (x(a), \beta_n(a))$, and so forth.

Define a constant function by

$$W(t) = -\frac{K_2}{K_1} \|e_{n-1}\|^2 \ge 0, \quad t \in [a, b]. \tag{4.23}$$

Then, $W(t)$ satisfies

$$LW \ge f_{x'}(t, \beta_{n-1}, \eta_1) W' + K_1 W + K_2 \|e_{n-1}\|^2 \tag{4.24}$$

and moreover, since $W(t)$ is a nonnegative constant function, therefore, we have $W(a) \ge 0$, $W(b) \ge 0$ and $W'(a) = 0$, $W'(b) = 0$. Consequently, the condition (4) guarantees that

$$g'_1 W(a) + g'_2 W(b) + g'_3 W'(a) + g'_2 W'(b) \le 0, \tag{4.25}$$

$$W(a) = W(b) \tag{4.26}$$

in which g'_i, $i = 1, 2, 3, 4$, are defined in (4.21).

If $e_n(t) \le W(t)$ holds uniformly for $t \in [a, b]$, the quadratic convergence of the iteration has been proved. Otherwise, there must be some points $t \in [a, b]$ such that $e_n(t) > W(t)$. We then introduce a function

$$q_n(t) = e_n(t) - W(t), \quad t \in [a, b] \tag{4.27}$$

and define

$$\max_{t \in [a,b]} q_n(t) := q_n(t_1). \tag{4.28}$$

Case 1 ($t_1 \in (a, b)$). In this case, $q_n(t_1) = e_n(t_1) - W(t_1) > 0$, $q'_n(t_1) = e'_n(t_1) - W'(t_1) = 0$, and $q''_n(t_1) = e''_n(t_1) - W''(t_1) \le 0$. Hence, $Lq_n(t_1) \ge 0$. On the other hand, it can be deduced that

$$\begin{aligned} Lq_n(t_1) &= Le_n(t_1) - LW(t_1) \\ &\le f_{x'}(t_1, \beta_{n-1}, \eta_1)(e'_n(t_1) - W'(t_1)) + K_1(e_n(t_1) - W(t_1)) \\ &= K_1 q_n(t_1) < 0 \end{aligned} \tag{4.29}$$

which is a contradiction.

Case 2 ($t_1 = a$). In this case, $q_n(a) > 0$, $q_n'(a) \le 0$. Furthermore, we can obtain

$$q_n(b) = e_n(b) - W(b) = e_n(a) - W(a) = q_n(a) > 0,$$

$$q_n'(b) = \lim_{t \to b^-} \frac{q_n(t) - q_n(b)}{t - b} = \lim_{t \to b^-} \frac{q_n(t) - q_n(a)}{t - b} \ge 0. \tag{4.30}$$

Consequently, the monotonicity of g given in the condition (4) yields

$$g_1' q_n(a) + g_2' q_n(b) + g_3' q_n'(a) + g_2' q_n'(b) < 0. \tag{4.31}$$

However, equation (4.21) minus inequality (4.25) gives

$$g_1' q_n(a) + g_2' q_n(b) + g_3' q_n'(a) + g_2' q_n'(b) \ge 0 \tag{4.32}$$

which contradicts to (4.31).

Similarly, we can prove that Case 3, that is, $t_1 = b$, is also impossible. Hence, the inequality

$$e_n(t) \le W(t) = -\frac{K_2}{K_1} \|e_{n-1}\|^2 \tag{4.33}$$

holds uniformly for $t \in [a, b]$, that is,

$$\|e_n(t)\| \le K \|e_{n-1}\|^2, \quad t \in [a, b] \tag{4.34}$$

holds uniformly, in which $K = -K_2 / K_1 \ge 0$ and $\|e_n(t)\| = \max_{t \in [a,b]} |e_n(t)|$. $\qquad \square$

5. An Example

Consider a nonlinear BVP as follows:

$$-\left(\left(t^2 + 1\right) x'\right)' + kx = -\left(t^2 + 3\right) e^x - 2(x')^2 + t^2 + 1, \quad t \in (-1, 1),$$

$$-x(-1) - x(1) + \frac{1}{3}(x'(-1))^3 - \frac{1}{3}(x'(1))^3 = 0, \tag{5.1}$$

$$x(-1) = x(1)$$

in which k is a constant to be determined, that is,

$$Lx = -\left(\left(t^2 + 1\right) x'\right)' + kx, \qquad f(t, x, x') = -\left(t^2 + 3\right) e^x - 2(x')^2 + t^2 + 1,$$

$$g(x, y, z, w) = -x - y + \frac{1}{3} z^3 - \frac{1}{3} w^3 \tag{5.2}$$

in this BVP.

It can be verified that $\alpha_0(t) = -2$ and $\beta_0(t) = t^2$ are, respectively, the lower and upper solutions of BVP (5.1) if $k \geq (3 - e^2)/2e^2$ is chosen. Moreover, it is easy to show that all the assumptions in Theorem 4.1 are satisfied. Therefore, by Theorem 4.1, BVP (5.1) has a unique solution and it can be approximated quadratically by two monotone sequences.

Acknowledgment

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