Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 286040, 12 pages
doi:10.1155/2010/286040

Research Article

The Lyapunov Stability for the Linear and Nonlinear Damped Oscillator with Time-Periodic Parameters

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Received 27 May 2010; Accepted 11 August 2010

Academic Editor: Stefan Siegmund

Let $a(t), b(t)$ be continuous T-periodic functions with $\int_0^T b(t)dt = 0$. We establish one stability criterion for the linear damped oscillator $x''+b(t)x'+a(t)x = 0$. Moreover, based on the computation of the corresponding Birkhoff normal forms, we present a sufficient condition for the stability of the equilibrium of the nonlinear damped oscillator $x''+b(t)x'+a(t)x+c(t)x^{2n-1}+e(t,x) = 0$, where $n \geq 2$, $c(t)$ is a continuous T-periodic function, $e(t,x)$ is continuous T-periodic in t and dominated by the power x^{2n} in a neighborhood of $x = 0$.

1. Introduction

This paper is motivated by the study of the Lyapunov stability of the equilibrium of the linear and nonlinear time-periodic differential equations, which is an interesting problem in the theory of ordinary differential equations and dynamical systems. Let $a(t), b(t)$ be continuous T-periodic functions, that is, $a, b \in L^1(\mathbb{S}_T)$, $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$. Recall that the linear oscillator

$$x'' + b(t)x' + a(t)x = 0 \tag{1.1}$$

is stable (in the sense of Lyapunov) if any solution $x(t)$ of (1.1) satisfies

$$\sup_{t\in\mathbb{R}}\left(|x(t)| + \left|x'(t)\right|\right) < \infty. \tag{1.2}$$

The research on the stability of second-order linear differential equations goes back to the time of Lyapunov. Many theoretical results concerning this problem can be found in textbooks such as [1]. One famous stability criterion was given by Magnus and Winkler [2] for the Hill equation

$$x'' + a(t)x = 0. \tag{1.3}$$

That is, (1.3) is stable if

$$a(t) > 0, \quad \int_0^T a(t)dt \leq \frac{4}{T}, \tag{1.4}$$

which can be shown using a Poincaré inequality. Such a stability criterion has been generalized and improved by Zhang and Li in [3], which now is the so-called L^p-criterion. Recently, Zhang in [4] has extended such a criterion to the linear planar Hamiltonian system

$$\begin{aligned} x' &= m(t)y, \\ y' &= -n(t)x, \end{aligned} \tag{1.5}$$

where m, n are continuous and T-periodic functions. See Lemma 2.1 in next section.

Based on Lemma 2.1, in Section 2, we establish one new stability criterion for (1.1) under the following condition:

(A) $a, b \in L^1(\mathbb{S}_T)$ and $\int_0^T b(t)dt = 0$.

It is interesting to compare our new stability criterion with those in the literature. In fact, there have been many stability results for (1.1) in the literature [5–11]. We refer to [7, 8] for the discussions for (1.1) when $b(t)$ is positive and $a(t) = k^2$ is constant. On the other hand, if $b(t)$ is a continuously differential function, (1.1) can be transformed into

$$y'' + q(t)y = 0, \tag{1.6}$$

where

$$y(t) = x(t)\exp\left(\frac{1}{2}\int_0^t b(s)ds\right), \qquad q(t) = a(t) - \frac{1}{2}b'(t) - \frac{1}{4}b^2(t). \tag{1.7}$$

Therefore, every stability criterion for the Hill equation (1.6) can produce a stability criterion for (1.1). As far as the authors know, in the literature there are few stability results for the case that $b(t)$ is continuous (may not be differential) and $b(t)$ can change sign, which is just our case. Here we refer the reader to [12, 13] for some recent stability criterion of planar Hamiltonian systems.

In Section 3, we present a sufficient condition for the stability of the equilibrium of the nonlinear damped oscillator

$$x'' + b(t)x' + a(t)x + c(t)x^{2n-1} + e(t,x) = 0, \tag{1.8}$$

where $a(t), b(t)$ satisfy (A) and $c(t)$ is a continuous, T-periodic functions, $n \geq 2, e : \mathbb{R} \times B_\epsilon(0) \longrightarrow \mathbb{R}$ is a continuous function with continuous derivatives of all orders with respect to the second variable, T-periodic in t, and

$$e(t,x) = O\left(|x|^{2n}\right), \quad x \longrightarrow 0, \quad \text{uniformly with respect to } t \in \mathbb{R}. \tag{1.9}$$

It is well-known that (1.8) can be unstable when linear oscillator (1.1) is stable. For the case $b(t) = a(t) \equiv 0$, Liu proved in [14] that the equilibrium of (1.8) is stable if and only if $\int_0^T c(t)dt > 0$. Recently, Liu et al. [15] has extended such a result to the case $a(t) \equiv 0$ and $b \in L^1(\mathbb{S}_T)$ with $\int_0^T b(t)dt = 0$. In this paper, we will deal with the case that $a, b \in L^1(\mathbb{S}_T)$ with $\int_0^T b(t)dt = 0$.

During the last decade, an analytical method for studying time-periodic Lagrangian equations has been developed recently by Ortega in a series of papers [16–19]. A nice illustrating example for Ortega's approach is the so-called swing [19, 20]

$$x'' + a(t)\sin x = 0, \quad a(t) > 0,\ a(t) \in L^1(\mathbb{S}_T). \tag{1.10}$$

That is, the equilibrium $x(t) = 0$ of swing (1.10) is stable if and only if its cubic approximation

$$x'' + a(t)x - \frac{a(t)}{6}x^3 = 0 \tag{1.11}$$

is stable. After that, some researchers have extended the applications of such an analytical method, and some important stability results for several types of Lagrangian equations have been established, for example, see [21–23] and the references therein. We also refer the reader to [24] for its extension to the planar nonlinear system.

We will extend such an analytical method to the nonlinear damped oscillator (1.8). The proof is based on a careful computation of certain Birkhoff normal forms together with some stability results of fixed points of area-preserving maps in the plane. As an application, when condition (A) is satisfied, we present a quite complete Lyapunov's stability result for the equilibrium position of the pendulum of variable length with damping term

$$x'' + b(t)x' + a(t)\sin x = 0, \quad a(t) > 0. \tag{1.12}$$

That is, the stability of its linearized equation implies the stability of the equilibrium of (1.12).

2. Stability Criteria for Linear Damped Oscillator

Let

$$M(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \phi'_1(t) & \phi'_2(t) \end{pmatrix} \tag{2.1}$$

be a fundamental matrix solution for the linear damped oscillator (1.1), where $\phi_1(t)$ and $\phi_2(t)$ are real-valued solutions of (1.1) satisfying

$$\phi_1(0) = 1, \qquad \phi_1'(0) = 0, \qquad \phi_2(0) = 0, \qquad \phi_2'(0) = 1. \tag{2.2}$$

It is easy to verify that $F(t)$=det$M(t)$ satisfies the following equation:

$$F'(t) = -b(t)F(t), \tag{2.3}$$

which implies that

$$F(t) = F(0)\exp\left(-\int_0^t b(s)ds\right) = \exp\left(-\int_0^t b(s)ds\right), \tag{2.4}$$

in which we have used the fact

$$F(0) = \phi_1(0)\phi_2'(0) - \phi_2(0)\phi_1'(0) = 1. \tag{2.5}$$

The Poincaré matrix of (1.1) is

$$M(T) = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \phi_1'(T) & \phi_2'(T) \end{pmatrix}. \tag{2.6}$$

Using the condition (A) and (2.4), we have

$$\det M(T) = 1. \tag{2.7}$$

The eigenvalues $\lambda_{1,2}$ of $M(T)$ are called the Floquet multipliers of (1.1). Obviously $\lambda_1 \cdot \lambda_2 = 1$. Therefore we can distinguish (1.1) in the following three cases:

(i) *elliptic*: $\lambda_1 = \overline{\lambda}_2,\ |\lambda_1| = 1,\ \lambda_1 \neq \pm 1$;

(ii) *hyperbolic*: $0 < |\lambda_1| < 1 < |\lambda_2|$;

(iii) *parabolic*: $\lambda_1 = \lambda_2 = \pm 1$.

It is well-known that (1.1) is stable in the sense of Lyapunov if and only if (1.1) is elliptic or is parabolic with further property that all solutions of (1.1) are T-periodic solutions in case $\lambda_1 = \lambda_2 = 1$ or $2T$-periodic solutions in case $\lambda_1 = \lambda_2 = -1$. See, for example, [1, Theorem 7.2].

In order to state our new stability criteria for (1.1), we recall the following stability results for the planar Hamiltonian system (1.5).

Lemma 2.1 (see [4]). *Let $m \in L^1_+(\mathbb{S}_T) = \{m \in L^1(\mathbb{S}_T) : \text{ess inf}_t m(t) > 0\}$ and $n \in L^p(\mathbb{S}_T)$, $p \in [1, \infty]$. Then system* (1.5) *is stable if one of the following two conditions is satisfied:*

(a) L^1*–criterion*

$$\|m\|_{1,T} \cdot \|n_+\|_{1,T} \le \mathbf{M}^2(\infty) = 4; \tag{2.8}$$

(b) L^p*–criterion,* $p \in (1, \infty]$

$$\|m\|_{1,T}^{1+1/p^*} \cdot \left\| n_+ / m^{1/p^*} \right\|_{p,T} < \mathbf{M}^2(2p^*), \quad p \in (1, \infty], \tag{2.9}$$

where $n_+(t) = \max\{n(t), 0\}$, $\|\cdot\|_{p,T}$ is the usual L^p norm for $p \in [1, \infty]$, p^* is the conjugate exponent of $p : (1/p) + (1/p^*) = 1$, and $\mathbf{M}(q)$ is given by

$$\mathbf{M}(q) = \begin{cases} \left(\dfrac{2\pi}{q}\right)^{1/2} \left(\dfrac{2}{q+2}\right)^{1/2-1/q} \dfrac{\mathbf{\Gamma}(1/q)}{\mathbf{\Gamma}((1/2) + (1/q))}, & 1 \le q < \infty, \\ 2, & q = \infty, \end{cases} \tag{2.10}$$

where $\mathbf{\Gamma}(\cdot)$ is the Gamma function of Euler. See [4].

Now we are in a position to state the new stability criteria for the linear damped oscillator.

Theorem 2.2. *Assume that (A) holds. Then* (1.1) *is stable if one of the following two conditions is satisfied:*

(a) L^1*–criterion*

$$\left(\int_0^T \exp\left(-\int_0^t b(s)ds\right)dt\right) \cdot \left(\int_0^T a_+(t) \exp\left(\int_0^t b(s)ds\right)dt\right) \le \mathbf{M}^2(\infty) = 4; \tag{2.11}$$

(b) L^p*–criterion,* $p \in (1, \infty]$

$$\begin{aligned} &\left(\int_0^T \exp\left(-\int_0^t b(s)ds\right)dt\right)^{1+(1/p^*)} \cdot \left(\int_0^T a_+^p(t) \exp\left((2p-1)\int_0^t b(s)ds\right)dt\right)^{1/p} \\ &\quad < \mathbf{M}^2(2p^*), \quad p \in (1, \infty]. \end{aligned} \tag{2.12}$$

Proof. The linear damped oscillator (1.1) is equivalent to the following planar system:

$$\begin{aligned} x' &= y, \\ y' &= -a(t)x - b(t)y. \end{aligned} \tag{2.13}$$

Using the change of variable

$$z(t) = y(t)\exp\left(\int_0^t b(s)ds\right), \tag{2.14}$$

system (2.13) can be written as

$$\begin{aligned} x' &= \exp\left(-\int_0^t b(s)ds\right)z(t),\\ z' &= -a(t)\exp\left(\int_0^t b(s)ds\right)x(t). \end{aligned} \tag{2.15}$$

System (2.15) is stable if and only if

$$\sup_{t\in\mathbb{R}}(|x(t)| + |z(t)|) = \sup_{t\in\mathbb{R}}\left(|x(t)| + \exp\left(\int_0^t b(s)ds\right)\left|x'(t)\right|\right) < \infty. \tag{2.16}$$

Under the condition (A), if (2.16) is satisfied, one may easily see that

$$\sup_{t\in\mathbb{R}}\left(|x(t)| + \left|x'(t)\right|\right) < \infty, \tag{2.17}$$

which implies that (1.1) is stable. Using Lemma 2.1, system (2.15) is stable when the inequality (2.11) or the inequality

$$\left(\int_0^T \exp\left(-\int_0^t b(s)ds\right)dt\right)^{1+(1/p^*)} \cdot \left(\int_0^T B^p(a,b)dt\right)^{1/p} < \mathbf{M}^2(2p^*), \quad p \in (1,\infty] \tag{2.18}$$

is satisfied, where

$$B(a,b) = \frac{a_+(t)\exp\left(\int_0^t b(s)ds\right)}{\exp\left(-(1/p^*)\int_0^t b(s)ds\right)}. \tag{2.19}$$

One may easily verify that (2.18) is just inequality (2.12). □

The following corollary is a direct result of Theorem 2.2.

Corollary 2.3. *Assume that (A) holds and $a(t) = k^2$. Then there exists a positive constant $L(b)$ such that* (1.1) *is stable when $k \in (0, L(b))$.*

Theorem 2.2 can be applied to the damped Mathieu's equation

$$x'' + b(t)x' + \lambda(1 + \epsilon \cos t)x = 0, \tag{2.20}$$

where $\epsilon \in [-1, 1]$ and $\lambda > 0$.

Corollary 2.4. *Assume that $b(t)$ is a continuous T-periodic function with $\int_0^T b(t)dt = 0$. Then there exists a constant $H(\epsilon, b) > 0$ such that (2.20) is stable when $\lambda < H(\epsilon, b)$.*

Remark 2.5. In this paper, the condition $\int_0^T b(t)dt = 0$ is crucial. In fact, for the case $\int_0^T b(t)dt > 0$, (1.1) and (1.8) will be dissipative, and therefore would be unstable for all $t \in \mathbb{R}$ (but would be stable for $t \geq 0$). For our case $\int_0^T b(t)dt = 0$, (1.1) becomes an conservative equation at $t = nT$, $n = 1, 2, \ldots$, although (1.1) is dissipative at the other time $t \neq nT$. This fact play an important role in our analysis.

3. Stability of the Nonlinear Damped Oscillator

Assume that (1.1) is elliptic and the Floquet multipliers of (1.1) are λ and $\overline{\lambda}$ with $\lambda = \exp(i\theta)$, $\theta > 0$, $\theta \neq n\pi$, $n = 1, 2, \ldots$. In general, the Poincaré Matrix $M(T)$ is conjugate, in the symplectic group

$$\mathrm{Symp}\left(\mathbb{R}^2\right) = \left\{M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \text{ is linear and } \det M = 1\right\} \tag{3.1}$$

to the rigid rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.2}$$

Definition 3.1. One says that (1.1) is *R-elliptic*, if (1.1) is elliptic and its Poincaré matrix is a rigid rotation.

Theorem 3.2. *Assume that (1.1) is elliptic. Then there always exist some $\alpha > 0$ and $t_0 \in \mathbb{R}$ such that under the transformation*

$$s = \alpha(t - t_0), \qquad y(s) = x\left(t_0 + \frac{s}{\alpha}\right), \tag{3.3}$$

the Poincaré matrix of the transformed equation

$$y''(s) + \alpha^{-1} b^*(s) y'(s) + \alpha^{-2} a^*(s) y(s) = 0 \tag{3.4}$$

is a rigid rotation R_θ, where $b^*(s) = b(t_0 + s/\alpha)$ and $a^*(s) = a(t_0 + s/\alpha)$.

Proof. The proof is similar to the proof of [24, Theorem 2.5] and [16, Proposition 8], here we omit it. □

Remark 3.3. When the linear oscillator is stable, we can assume that (1.1) satisfies the following condition:

$$\phi(t+T) = \overline{\lambda}\phi(t), \quad \forall t \in \mathbb{R}, \tag{3.5}$$

where $\lambda \in S^1$ is one of the eigenvalues of $M(T)$ and $\phi(t) = \phi_1(t) + i\phi_2(t)$. In fact, Theorem 3.2 guarantees that (3.5) holds for the case of ellipticity $|\lambda| = 1, \lambda \neq \pm 1$, after a temporal transformation. When (1.1) is parabolic and stable, condition (3.5) is always satisfied, because all solutions of (1.1) are either T-periodic or $2T$-periodic in this case.

The proof of the main results is based on the theory of stability of fixed points of area-preserving maps in the plane [17, 25]. Let $F : \Omega \subset \mathbb{C} \to \mathbb{C}$ be an area-preserving map defined in an open neighborhood of the origin, and $z = 0$ is a fixed point of F. It is assumed further that F is sufficiently smooth. For convenience, the complex notation $F = F(z, \overline{z})$ is used.

Lemma 3.4 (see [17, Lemma 3.1]). *Assume that for some $m \geq 3$,*

$$F(z, \overline{z}) = \lambda z + O\left(|z|^{m-1}\right), \quad z \longrightarrow 0, \quad \left(\lambda \in S^1\right). \tag{3.6}$$

Then there exists $H = H(z, \overline{z})$, a real-valued homogeneous polynomial of degree m such that

$$F(z, \overline{z}) = \lambda\left(z + 2i\partial_{\overline{z}} H(z, \overline{z}) + O(|z|^m)\right), \quad z \longrightarrow 0. \tag{3.7}$$

Now we assume that F satisfies the conditions of Lemma 3.4 with $m = 2n,\ n \geq 2$. The polynomial H given by the lemma can be expressed in the form

$$H(z, \overline{z}) = \beta|z|^{2n} + \sum_{k=0}^{n-1}\left(\alpha_k z^k \overline{z}^{2n-k} + \overline{\alpha}_k \overline{z}^k z^{2n-k}\right), \tag{3.8}$$

where $\beta \in \mathbb{R}$ and $\alpha_k \in \mathbb{C}, \quad k = 0, \ldots, n-1$.

Lemma 3.5 (see [17, Proposition 3.2]). *Assume that $\lambda \in S^1, F$ satisfies* (3.7) *with $m = 2n$ for some $n \geq 2, H$ is given by* (3.8)*, and one of the following conditions hold:*

(C_1) $\lambda^{2p} \neq 1$ *for each $p = 1, \ldots, n$ and $\beta \neq 0$;*

(C_2) $\lambda^{2p} = 1$ *for some $p = 1, \ldots, n$, and $H^{\#}(z, \overline{z}) \neq 0$ for each $z \in \mathbb{C} - \{0\}$, where*

$$H^{\#}(z, \overline{z}) = \frac{1}{2p}\sum_{r=0}^{2p-1} H\left(\lambda^r z, \overline{\lambda}^r \overline{z}\right). \tag{3.9}$$

Then $z = 0$ is stable with respect to F.

Before stating our result, we present the following simple result on the linear inhomogeneous system:

$$x'' + b(t)x' + a(t)x + f(t) = 0, \tag{3.10}$$

which is a consequence of the formula of variation of constants together with (3.5).

Lemma 3.6. *Let $x(t)$ be the solution of* (3.10) *with $x(0) = x'(0) = 0$. Assume that* (3.5) *is satisfied. Then one has*

$$x(T) + ix'(T) = -i\lambda \int_0^T \frac{\phi(t) f(t)}{F(t)} \mathrm{d}t, \tag{3.11}$$

where $F(t)$ is given by (2.4).

Proof. Using the formula of variation of constants, one may easily see that

$$\begin{aligned}\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} &= \int_0^t M(t)M^{-1}(\tau)\begin{pmatrix} 0 \\ -f(\tau) \end{pmatrix} d\tau \\ &= \int_0^t \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{pmatrix} \begin{pmatrix} \dfrac{\phi_2'(\tau)}{F(\tau)} & \dfrac{-\phi_2(\tau)}{F(\tau)} \\ \dfrac{-\phi_1'(\tau)}{F(\tau)} & \dfrac{\phi_1(\tau)}{F(\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ -f(\tau) \end{pmatrix} d\tau.\end{aligned} \tag{3.12}$$

By easy computations, we obtain

$$\begin{aligned} x(t) &= \int_0^t \left[\phi_1(t)\phi_2(\tau) - \phi_2(t)\phi_1(\tau)\right] \frac{f(\tau)}{F(\tau)} d\tau, \\ x'(t) &= \int_0^t \left[\phi_1'(t)\phi_2(\tau) - \phi_2'(t)\phi_1(\tau)\right] \frac{f(\tau)}{F(\tau)} d\tau. \end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned} x(T) + ix'(T) &= \int_0^T \left[\left(\phi_1(T)\phi_2(\tau) - \phi_2(T)\phi_1(\tau)\right) + i\left(\phi_1'(T)\phi_2(\tau) - \phi_2'(T)\phi_1(\tau)\right)\right] \frac{f(\tau)}{F(\tau)} d\tau \\ &= -i\lambda \int_0^T \frac{\phi(t) f(t)}{F(t)} \mathrm{d}t. \end{aligned} \tag{3.14}$$

□

The main result of this section reads as follows.

Theorem 3.7. *Assume that (A) holds and $\int_0^T |c(t)|dt \neq 0$. Furthermore, suppose that the following two conditions are satisfied:*

(H_1) *the linear oscillator* (1.1) *is stable;*

(H_2) $c(t) \leq 0$ *or* $c(t) \geq 0$.

Then the trivial solution of the nonlinear oscillator (1.8) *is stable.*

Proof. We prove the result assuming that condition (3.5) is satisfied. The result holds for the general case because we have Remark 3.3.

Let $x(t) = x(t,z,\overline{z})$ be the solution of the nonlinear system (1.8) with $x(0) = q$ and $x'(0) = p$, where $z = q + ip$. The theorem of differentiability with respect to initial conditions implies that

$$\begin{aligned} x(t,z,\overline{z}) &= \frac{\overline{\phi}(t)z + \phi(t)\overline{z}}{2} + O\left(|z|^2\right), \quad z \longrightarrow 0, \\ x'(t,z,\overline{z}) &= \frac{\overline{\phi}'(t)z + \phi'(t)\overline{z}}{2} + O\left(|z|^2\right), \quad z \longrightarrow 0. \end{aligned} \tag{3.15}$$

These two expansions are uniform in $t \in [0,T]$.

We look at the nonlinear oscillator (1.8) as one equation of the kind (3.10) with

$$f(t) = c(t)x^{2n-1}(t) + e(t,x(t)), \tag{3.16}$$

where

$$e(t,x(t)) = O\left(|z|^{2n}\right) \quad \text{as } |z| \longrightarrow 0. \tag{3.17}$$

Since we have assumed that (3.5) holds, we can apply Lemma 3.6 to (3.10) to obtain

$$P(z,\overline{z}) = x(T,z,\overline{z}) + ix'(T,z,\overline{z}) = \lambda z - i\lambda \int_0^T \frac{f(t)\phi(t)}{F(t)} \mathrm{d}t, \tag{3.18}$$

where f is given by (3.16).

Combining (3.15) and (3.18), we have the expansion

$$P(z,\overline{z}) = \lambda z - i\lambda \int_0^T c(t)\phi(t) \left(\frac{\overline{\phi}(t)z + \phi(t)\overline{z}}{2}\right)^{2n-1} / F(t)\mathrm{d}t + O\left(|z|^{2n}\right). \tag{3.19}$$

Then P satisfies (3.7) with H given by

$$H(z,\overline{z}) = -\frac{1}{2^{2n}} \int_0^T \frac{c(t)\left(\overline{\phi}(t)z + \phi(t)\overline{z}\right)^{2n}}{2nF(t)} \mathrm{d}t. \tag{3.20}$$

The coefficient β in (3.8) is given by

$$\beta = -\frac{1}{2^{2n+1}n}\binom{2n}{n}\int_0^T \frac{c(t)}{\exp\left(-\int_0^t b(s)ds\right)}\left|\phi(t)\right|^{2n}\mathrm{d}t. \tag{3.21}$$

Since $\phi_1(t)$ and $\phi_2(t)$ are linearly independent solutions of (1.1), we know that $|\phi(t)| \neq 0$ for all $t \in \mathbb{R}$.

Assume that condition (H_1) holds and $c(t) \geq 0, \int_0^T c(t)dt > 0$. Then $\beta < 0$ and $H(z, \overline{z}) < 0$ for all $z \in \mathbb{C} - \{0\}$. When $\lambda^{2p} \neq 1$ for each $p = 1, \ldots, n$, (C_1) is satisfied. If $\lambda^{2p} = 1$ for some $p = 1, \ldots, n$, then the definition of $H^\#$ and the negativity of H imply that $H^\#(z, \overline{z}) < 0$ for all $z \in \mathbb{C} - \{0\}$, and therefore (C_2) holds. When $c(t) \leq 0$ and $\int_0^T c(t)dt < 0$, the result holds by similar analysis. □

Every stability criterion for the linear oscillator (1.1) together with the assumption (H_2) produces a stability criterion for the nonlinear system (1.8). For example, we have the following.

Corollary 3.8. *Assume that (A), (H_2) holds and $\int_0^T |c(t)|dt \neq 0$. Furthermore, suppose that* (2.11) *or* (2.12) *holds. Then the trivial solution of* (1.8) *is stable.*

The following stability result for the swing with damping term (1.12) follows directly from Theorem 3.7.

Corollary 3.9. *Assume that (A) holds and $a(t)$ is a positive function. Then the trivial solution of* (1.12) *is stable if the linearized oscillator is stable.*

Proof. Equation (1.12) can be regarded as one of (1.8) with $n = 2$ and

$$c(t) = -\frac{a(t)}{6}. \tag{3.22}$$

Since $a(t)$ is a positive T-periodic function, the result is now a direct consequence of Theorem 3.7. □

Acknowledgments

The research of J. Chu is supported by the National Natural Science Foundation of China (Grant no. 10801044), Jiangsu Natural Science Foundation (Grant No. BK2008356), the Program for New Century Excellent Talents in University (Grant no. NCET-10-0325), and the Fundamental Research Funds for the Central Universities.

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