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Research Article

Multiple Positive Solutions of a Second Order Nonlinear Semipositone m-Point Boundary Value Problem on Time Scales

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In this paper, we study a general second-order m-point boundary value problem for nonlinear singular dynamic equation on time scales $u^{\Delta\nabla}(t)+a(t)u^{\Delta}(t)+b(t)u(t)+\lambda q(t)f(t,u(t))=0$, $t\in(0,1)_{\mathbb{T}}$, $u(\rho(0))=0$, $u(\sigma(1))=\sum_{i=1}^{m-2}\alpha_i u(\eta_i)$. This paper shows the existence of multiple positive solutions if f is semipositone and superlinear. The arguments are based upon fixed-point theorems in a cone.

1. Introduction

In this paper, we consider the following dynamic equation on time scales:

$$u^{\Delta\nabla}(t)+a(t)u^{\Delta}(t)+b(t)u(t)+\lambda q(t)f(t,u(t))=0,\quad t\in(0,1)_{\mathbb{T}},$$
$$u(\rho(0))=0,\qquad u(\sigma(1))=\sum_{i=1}^{m-2}\alpha_i u(\eta_i),\tag{1.1}$$

where $\alpha_i\geq 0$, $0<\eta_i<\eta_{i+1}<1$; for all $i=1,2,\ldots,m-2$; f,q,a and b satisfy

(C1) $q\in L$ is continuously and nonnegative function and there exists $t_0\in(\rho(0),\sigma(1))$ s.t. $q(t_0)>0$, $q(t)$ may be singular at $t=\rho(0),\sigma(1)$;

(C2) $a\in C([0,1],[0,+\infty))$, $b\in C([0,1],(-\infty,0])$.

In the past few years, the boundary value problems of dynamic equations on time scales have been studied by many authors (see [1–15] and references therein). Recently, multiple-point boundary value problems on time scale have been studied, for instance, see [1–9].

In 2008, Lin and Du [2] studied the m-point boundary value problem for second-order dynamic equations on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + f(t,u) = 0, \quad t \in (0,T) \in \mathbb{T}, \\ u(0) = 0, \qquad u(T) = \sum_{i=1}^{m-2} k_i u(\xi_i), \end{gathered} \tag{1.2}$$

where $\mathbb{T}$ is a time scale. This paper deals with the existence of multiple positive solutions for second-order dynamic equations on time scales. By using Green's function and the Leggett-Williams fixed point theorem in an appropriate cone, the existence of at least three positive solutions of the problem is obtained.

In 2009, Topal and Yantir [1] studied the general second-order nonlinear m-point boundary value problems (1.1) with no singularities and the case. The authors deal with the determining the value of λ; the existences of multiple positive solutions of (1.1) are obtained by using the Krasnosel'skii and Legget-William fixed point theorems.

Motivated by the abovementioned results, we continue to study the general second-order nonlinear m-point boundary value problem (1.1), but the nonlinear term may be singularity and semipositone.

In this paper, the nonlinear term f of (1.1) is suit to and semipositone and the superlinear case, we will prove our two existence results for problem (1.1) by using Krasnosel'skii fixed point theorem. This paper is organized as follows. In Section 2, starting with some preliminary lemmas, we state the Krasnosel'skii fixed point theorem. In Section 3, we give the main result which state the sufficient conditions for the m-point boundary value problem (1.1) to have existence of positive solutions.

2. Preliminaries

In this section, we state the preliminary information that we need to prove the main results. From Lemmas 2.1 and 2.3 in [1], we have the following lemma.

Lemma 2.1 (see [1]). *Assuming that (C2) holds. Then the equations*

$$\begin{gathered} \phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t) = 0, \quad t \in (0,1)_{\mathbb{T}}, \\ \phi_1(\rho(0)) = 0, \qquad \phi_1(\sigma(1)) = 1, \end{gathered} \tag{2.1}$$

$$\begin{gathered} \phi_2^{\Delta\nabla}(t) + a(t)\phi_2^{\Delta}(t) + b(t)\phi_2(t) = 0, \quad t \in (0,1)_{\mathbb{T}}, \\ \phi_2(\rho(0)) = 1, \qquad \phi_2(\sigma(1)) = 0 \end{gathered} \tag{2.2}$$

have unique solutions ϕ_1 and ϕ_2, respectively, and

(a) ϕ_1 *is strictly increasing on* $[\rho(0), \sigma(1)]$,

(b) ϕ_2 *is strictly decreasing on* $[\rho(0), \sigma(1)]$.

For the rest of the paper we need the following assumption:

(C3) $0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) < 1$.

Lemma 2.2 (see [1]). *Assuming that (C2) and (C3) hold. Let $y \in C[\rho(0), \sigma(1)]$. Then boundary value problem*

$$x^{\Delta\nabla}(t) + a(t)x^{\Delta}(t) + b(t)x(t) + y(t) = 0, \quad t \in (0,1)_{\mathbb{T}},$$
$$x(\rho(0)) = 0, \qquad x(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i) \tag{2.3}$$

is equivalent to integral equation

$$x(t) = \int_{\rho(0)}^{\sigma(1)} H(t,s)p(s)y(s)\nabla s + A\phi_1(t), \tag{2.4}$$

where

$$p(t) = e_a(\rho(t), \rho(0)), \qquad A = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s)p(s)y(s)\nabla s, \tag{2.5}$$

$$H(t,s) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \begin{cases} \phi_1(s)\phi_2(t), & s \le t, \\ \phi_1(t)\phi_2(s), & t \le s. \end{cases} \tag{2.6}$$

Proof. First we show that the unique solution of (2.3) can be represented by (2.4). From Lemma 2.1, we know that the homogenous part of (2.3) has two linearly independent solution ϕ_1 and ϕ_2 since

$$\begin{vmatrix} \phi_1(\rho(0)) & \phi_1^{\Delta}(\rho(0)) \\ \phi_2(\rho(0)) & \phi_2^{\Delta}(\rho(0)) \end{vmatrix} = -\phi_1^{\Delta}(\rho(0)) \neq 0. \tag{2.7}$$

Now by the method of variations of constants, we can obtain the unique solution of (2.3) which can be represented by (2.4) where A and H are as in (2.5) and (2.6), respectively. Next we check the function defined in (2.4) is the solution of the boundary value problem (2.3). For this purpose we first show that (2.4) satisfies (2.3). From the definition of Green's function (2.6), we get

$$x(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \left(\int_{\rho(0)}^{t} \phi_1(s)\phi_2(t)p(s)y(s)\nabla s + \int_{t}^{\sigma(1)} \phi_1(t)\phi_2(s)p(s)y(s)\nabla s \right) + A\phi_1(t). \tag{2.8}$$

Hence, the derivatives x^{Δ} and $x^{\Delta\nabla}$ are as follows:

$$x^{\Delta}(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\left(\phi_2^{\Delta}(t)\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s + \phi_1^{\Delta}(t)\int_{t}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s\right) + A\phi_1^{\Delta}(t),$$

$$\begin{aligned} x^{\Delta\nabla}(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\Bigg(&\phi_2^{\Delta\nabla}(t)\int_{\rho(0)}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s + \phi_2^{\Delta}(t)\phi_1(t)p(t)y(t) \\ &+\phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s + \phi_1^{\Delta}(t)\phi_2(t)p(t)y(t)\Bigg) + A\phi_1^{\Delta\nabla}(t). \end{aligned} \tag{2.9}$$

Replacing the derivatives in (2.3), we deduce that

$$\begin{aligned} &x^{\Delta\nabla}(t) + a(t)x^{\Delta}(t) + b(t)x(t) \\ &\quad= A\left(\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t)\right) \\ &\qquad+ \left(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s\right)\left(\phi_2^{\Delta\nabla}(t) + a(t)\phi_2^{\Delta}(t) + b(t)\phi_2(t)\right) \\ &\qquad+ \left(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{t}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s\right)\left(\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t)\right) \\ &\qquad+ \frac{1}{\phi_1^{\Delta}(\rho(0))}\left(\phi_2^{\Delta\nabla}(t)\int_{t}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s + \phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{t}\phi_2(s)p(s)y(s)\nabla s\right) \\ &\qquad+ \frac{1}{\phi_1^{\Delta}(\rho(0))}\left(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\right)p(t)y(t) \\ &\quad= \frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta\nabla}(t)(\rho(t)-t)\phi_1(t)p(t)y(t) - \phi_1^{\Delta\nabla}(t)\phi_2(t)p(t)y(t) \\ &\qquad\qquad+\phi_2^{\Delta}(t)\phi_1(t)p(t)y(t) - \phi_1^{\Delta}(t)\phi_2(t)p(t)y(t)\Big) \\ &\quad= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\left(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\right) \\ &\qquad+ \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)(\rho(t)-t)\left(\phi_2^{\Delta\nabla}(t)\phi_1(t) - \phi_1^{\Delta\nabla}(t)\phi_2(t)\right) \\ &\quad= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\left\{\left(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\right) + (\rho(t)-t)\left(\phi_2^{\Delta}(t)\phi_1(t) - \phi_1^{\Delta}(t)\phi_2(t)\right)^{\nabla}\right\} \\ &\quad= \frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\left(\phi_2^{\Delta}(\rho(t))\phi_1(\rho(t)) - \phi_1^{\Delta}(\rho(t))\phi_2(\rho(t))\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\phi_1^{\Delta}(\rho(0))} p(t)y(t)e_{\ominus a}(\rho(t),\rho(0))\left(-\phi_1^{\Delta}(\rho(0))\right)\\
&= -y(t).
\end{aligned}
\tag{2.10}
$$

Therefore the function defined in (2.4) satisfies (2.3). Further we obtain that the boundary value conditions are satisfied by (2.4). The first condition follows from (2.5) and (2.6) and Lemma 2.1. Now we verify the second boundary condition. Since

$$
H(\sigma(1),s) = \frac{1}{\phi_1^{\Delta}(\rho(0))}\phi_1(s)\phi_2(\sigma(1)) = 0,
\tag{2.11}
$$

we obtain that

$$
x(\sigma(1)) = \int_{\rho(0)}^{\sigma(1)} H(\sigma(1),s)p(s)y(s)\nabla s + A\phi_1(\sigma(1)) = A.
\tag{2.12}
$$

On the other hand, by using (2.5), we find that

$$
\begin{aligned}
\sum_{i=1}^{m-2}\alpha_i x(\eta_i) &= \sum_{i=1}^{m-2}\alpha_i\left(\int_{\rho(0)}^{\sigma(1)} H(\eta_i,s)p(s)y(s)\nabla s + A\phi_1(\eta_i)\right)\\
&= \sum_{i=1}^{m-2}\alpha_i\Bigg(\int_{\rho(0)}^{\sigma(1)} H(\eta_i,s)p(s)y(s)\nabla s\\
&\qquad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)\int_{\rho(0)}^{\sigma(1)} H(\eta_i,s)p(s)y(s)\nabla s\Bigg)\\
&= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i\int_{\rho(0)}^{\sigma(1)} H(\eta_i,s)p(s)y(s)\nabla s = A.
\end{aligned}
\tag{2.13}
$$

Combining the two equations above finishes the proof. □

Lemma 2.3. *Green's function $H(t,s)$ has the following properties:*

$$
H(t,s) \le H(t,t), \qquad \frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\|\phi_2\|}H(t,t)H(s,s) \le H(t,s) \le H(s,s), \quad H(t,t) \le \phi_1(t)\frac{\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}.
\tag{2.14}
$$

Lemma 2.4. *Assume that (C2) and (C3) hold. Let u be a solution of boundary value problem* (1.1) *if and only if u is a solution of the following integral equation:*

$$
u(t) = \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)f(s,u(s))\nabla s,
\tag{2.15}
$$

where

$$G(t,s) = H(t,s) + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,s)\phi_1(t). \tag{2.16}$$

The proofs of the Lemmas 2.3 and 2.4 can be obtained easily by Lemmas 2.1 and 2.2.

Lemma 2.5. *Green's function $G(t,s)$ defined by* (2.16) *has the following properties:*

$$C_2\phi_1(t)H(s,s) \le G(t,s) \le C_1 H(s,s), \quad G(t,s) \le C_3\phi_1(t), \tag{2.17}$$

where

$$\begin{aligned} C_1 &= 1 + \frac{\|\phi_1\|}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i, \\ C_2 &= \frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\|\phi_2\|}\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,\eta_i), \\ C_3 &= \frac{\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))} + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,\eta_i). \end{aligned} \tag{2.18}$$

Proof. From Lemma 2.3, we have

$$\begin{aligned} G(t,s) &\le H(s,s) + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(s,s)\|\phi_1\| \le C_1 H(s,s), \\ G(t,s) &\le \frac{\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}\phi_1(t) + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,\eta_i)\phi_1(t) \le C_3\phi_1(t), \\ G(t,s) &\ge \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,s)\phi_1(t) \\ &\ge \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i \frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\|\phi_2\|}H(\eta_i,\eta_i)H(s,s)\phi_1(t) \\ &\ge C_2\phi_1(t)H(s,s). \end{aligned} \tag{2.19}$$

The proof is complete. □

The following theorems will play major role in our next analysis.

Theorem 2.6 (see [16]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. Let Ω_1, Ω_2 be open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator, such that, either*

(1) $\|Sw\| \leq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_2$, *or*

(2) $\|Sw\| \geq \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \leq \|w\| w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap \overline{\Omega}_2 \setminus \Omega_1$.

3. Main Results

We make the following assumptions:

(H_1) $f(t,u) \in C([\rho(0),\sigma(1)] \times [0,+\infty),(-\infty,+\infty))$, moreover there exists a function $g(t) \in L^1([\rho(0),\sigma(1)],(0,+\infty))$ such that $f(t,u) \geq -g(t)$, for any $t \in (\rho(0),\sigma(1))$, $u \in [0,+\infty)$.

(H_1^*) $f(t,u) \in C((\rho(0),\sigma(1)) \times [0,+\infty),(-\infty,+\infty))$ may be singular at $t = \rho(0),\sigma(1)$, moreover there exists a function $g(t) \in L^1((\rho(0),\sigma(1)),(0,+\infty))$ such that $f(t,u) \geq -g(t)$, for any $t \in (\rho(0),\sigma(1))$, $u \in [0,+\infty)$.

(H_2) $f(t,0) > 0$, for $t \in [\rho(0),\sigma(1)]$.

(H_3) There exists $[\theta_1,\theta_2] \in (\rho(0),\sigma(1))$ such that $\lim_{u\uparrow+\infty}\min_{t\in[\theta_1,\theta_2]}(f(t,u)/u) = +\infty$.

(H_4) $\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)f(s,z)\nabla s < +\infty$ for any $z \in [0,m]$, $m > 0$ is any constant.

In fact, we only consider the boundary value problem

$$x^{\Delta\nabla}(t) + a(t)x^{\Delta}(t) + b(t)x(t) + \lambda q(t)\left[f\left(t,[x(t)-v(t)]^*\right) + g(t)\right] = 0, \quad \lambda > 0,$$
$$x(\rho(0)) = 0, \qquad x(\sigma(1)) = \sum_{i=1}^{m-2}\alpha_i x(\eta_i), \tag{3.1}$$

where

$$y(t)^* = \begin{cases} y(t), & y(t) \geq 0, \\ 0, & y(t) < 0, \end{cases} \tag{3.2}$$

and $v(t) = \lambda\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)g(s)\nabla s$, which is the solution of the boundary value problem

$$v^{\Delta\nabla}(t) + a(t)v^{\Delta}(t) + b(t)v(t) + \lambda q(t)g(t) = 0,$$
$$v(\rho(0)) = 0, \qquad v(\sigma(1)) = \sum_{i=1}^{m-2}\alpha_i v(\eta_i). \tag{3.3}$$

From Lemma 2.1, it is easy to verify that $v(t) \leq \lambda C_0\phi_1(t)$ and $C_0 = C_3\int_{\rho(0)}^{\sigma(1)} q(s)g(s)\nabla s$.

We will show that there exists a solution x for boundary value problem (3.1) with $x(t) \geq v(t),\ t \in [\rho(0), \sigma(1)]$. If this is true, then $u(t) = x(t) - v(t)$ is a nonnegative solution (positive on $(\rho(0), \sigma(1))$) of boundary value problem (3.1). Since for any $t \in (\rho(0), \sigma(1))$, from

$$(u(t) + v(t))^{\Delta\nabla} + a(t)(u(t) + v(t))^{\Delta} + b(t)(u(t) + v(t)) = -\lambda q(t)\left[f(t,u) + g(t)\right], \tag{3.4}$$

we have

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) = -\lambda q(t) f(t,u). \tag{3.5}$$

As a result, we will concentrate our study on boundary value problem (3.1).

We note that $x(t)$ is a solution of (3.1) if and only if

$$x(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left(f\left(t, [x(t) - v(t)]^*\right) + g(t)\right)\nabla s. \tag{3.6}$$

For our constructions, we will consider the Banach space $X = C[\rho(0), \sigma(1)]$ equipped with standard norm $\|x\| = \max_{0 \leq t \leq 1}|x(t)|,\ x \in X$. We define a cone P by

$$P = \left\{ x \in X \mid x(t) \geq \frac{C_2}{C_1}\phi_1(t)\|x\|,\ t \in \left[\rho(0), \sigma(1)\right] \right\}, \tag{3.7}$$

where ϕ_1 is defined by Lemma 2.1 (namely, ϕ_1 is solution (2.1)). Define an integral operator $T : P \to X$ by

$$Tx(t) = \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left(f\left(t, [x(t) - v(t)]^*\right) + g(t)\right)\nabla s. \tag{3.8}$$

Notice, from (3.8) and Lemma 2.5, we have $Tx(t) \geq 0$ on $[0,1]$ for $x \in P$ and

$$\begin{aligned} Tx(t) &= \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left(f\left(t, [x(t) - v(t)]^*\right) + g(t)\right)\nabla s \\ &\leq C_1\lambda \int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\left(f\left(t, [x(t) - v(t)]^*\right) + g(t)\right)\nabla s, \end{aligned} \tag{3.9}$$

then $\|Tx\| \leq C_1\lambda \int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)(f(t, [x(t) - v(t)]^*) + g(t))\nabla s$.

On the other hand, we have

$$\begin{aligned} Tx(t) &= \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left(f\left(t,[x(t)-v(t)]^*\right)+g(t)\right)\nabla s \\ &\geq \lambda \int_{\rho(0)}^{\sigma(1)} C_2\phi_1(t)H(s,s)p(s)q(s)\left(f\left(t,[x(t)-v(t)]^*\right)+g(t)\right)\nabla s \\ &\geq \frac{C_2}{C_1}\phi_1(t)\lambda \int_{\rho(0)}^{\sigma(1)} C_1H(s,s)p(s)q(s)\left(f\left(t,[x(t)-v(t)]^*\right)+g(t)\right)\nabla s \\ &\geq \frac{C_2}{C_1}\phi_1(t)\|Tx\|. \end{aligned} \tag{3.10}$$

Thus, $T(P) \subset P$. In addition, standard arguments show that $T(P) \subset P$ and T is a compact, and completely continuous.

Theorem 3.1. *Suppose that* (H_1)*-*(H_2) *hold. Then there exists a constant* $\overline{\lambda} > 0$ *such that, for any* $0 < \lambda \leq \overline{\lambda}$*, boundary value problem* (1.1) *has at least one positive solution.*

Proof. Fix $\delta \in (0,1)$. From (H_2), let $0 < \varepsilon < 1$ be such that

$$f(t,z) \geq \delta f(t,0), \quad \text{for } \rho(0) \leq t \leq \sigma(1), \quad 0 \leq z \leq \varepsilon. \tag{3.11}$$

Suppose that

$$0 < \lambda < \frac{\varepsilon}{2c\overline{f}(\varepsilon)} := \overline{\lambda} \tag{3.12}$$

where $\overline{f}(\varepsilon) = \max_{\rho(0)\leq t\leq\sigma(1),0\leq z\leq\varepsilon}\{f(t,z)+g(t)\}$ and $c = C_1\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\nabla s$. Since

$$\begin{gathered} \lim_{z\downarrow 0}\frac{\overline{f}(z)}{z} = +\infty, \\ \frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{2c\lambda}, \end{gathered} \tag{3.13}$$

there exists a $R_0 \in (0,\varepsilon)$ such that

$$\frac{\overline{f}(R_0)}{R_0} = \frac{1}{2c\lambda}. \tag{3.14}$$

Let $x \in P$ and $\nu \in (0,1)$ be such that $x = \nu T(x)$, we claim that $\|x\| \neq R_0$. In fact

$$\begin{aligned}
\|Tx(t)\| &\leq \nu\lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right)+g(s)\right]\nabla s \\
&\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right)+g(s)\right]\nabla s \\
&\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s) \max_{0\leq s\leq 1; 0\leq z\leq R_0}\left[f(s,z)+g(s)\right]\nabla s \\
&\leq \lambda \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s)\overline{f}(R_0)\nabla s \\
&\leq \lambda c \overline{f}(R_0),
\end{aligned} \tag{3.15}$$

that is,

$$\frac{\overline{f}(R_0)}{R_0} \geq \frac{1}{c\lambda} > \frac{1}{2c\lambda} = \frac{\overline{f}(R_0)}{R_0} \tag{3.16}$$

which implies that $\|x\| \neq R_0$. Let $U = \{x \in P : \|x\| < R_0\}$. By nonlinear alternative of Leray-Schauder type theorem, T has a fixed point $x \in \overline{U}$. Moreover, combing (3.8), (3.28), and $R_0 < \varepsilon$, we obtain that

$$\begin{aligned}
x(t) &= \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right)+g(s)\right]\nabla s \\
&\geq \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)\left[\delta f(s,0)+g(s)\right]\nabla s \\
&\geq \lambda \left[\delta \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)f(s,0)\nabla s + \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)g(s)\nabla s\right] \\
&> \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)g(s)\nabla s \\
&= v(t) \quad \text{for } t \in \left(\rho(0),\sigma(1)\right).
\end{aligned} \tag{3.17}$$

Let $u(t) = x(t) - v(t) > 0$. Then (1.1) has a positive solution u and $\|u\| \leq \|x\| \leq R_0 < 1$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose that* (H_1^*) *and* (H_3)-(H_4) *hold. Then there exists a constant* $\lambda^* > 0$ *such that, for any* $0 < \lambda \leq \lambda^*$, *boundary value problem* (1.1) *has at least one positive solution.*

Proof. Let $\Omega_1 = \{x \in P : \|x\| < R_1\}$, where $R_1 = \max\{1, r\}$ and $r = (C_1C_3/C_2)\int_{\rho(0)}^{\sigma(1)} q(s)g(s)\nabla s$. Choose

$$\lambda^* = \min\left\{1, R_1\left[C_1\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\left[\max_{0\leq z\leq R_1} f(s,z) + g(s)\right]\nabla s\right]^{-1}\right\}. \tag{3.18}$$

Then for any $x \in P \cap \partial\Omega_1$, then $\|x\| = R_1$ and $x(s) - v(s) \leq x(s) \leq \|x\|$, we have

$$\begin{aligned}
\|Tx(t)\| &\leq \lambda\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right) + g(s)\right]\nabla s \\
&\leq \lambda\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right) + g(s)\right]\nabla s \\
&\leq \lambda C_1\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\left[\max_{0\leq z\leq R_1} f(s,z) + g(s)\right]\nabla s \\
&\leq R_1 = \|x\|.
\end{aligned} \tag{3.19}$$

This implies that

$$\|Tx\| \leq \|x\|, \quad x \in P \cap \partial\Omega_1. \tag{3.20}$$

On the other hand, choose a constant $N > 0$ such that

$$\frac{\lambda C_2^2 N}{2C_1}\int_{\theta_1}^{\theta_2} H(s,s)\phi_1(s)p(s)q(s)\nabla s \min_{\theta_1\leq t\leq\theta_2}\phi_1(t) \geq 1. \tag{3.21}$$

By assumption (H_3), for any $t \in [\theta_1, \theta_2]$, there exists a constant $B > 0$ such that

$$\frac{f(t,z)}{z} > N, \quad \text{namely, } f(t,z) > Nz, \quad \text{for } z > B. \tag{3.22}$$

Choose $R_2 = \max\{R_1 + 1, 2\lambda r, 2C_1(B+1)/C_2\min_{\theta_1\le t\le\theta_2}\phi_1(t)\}$, and let $\Omega_2 = \{x \in P : \|x\| < R_2\}$, then for any $x \in P \cap \partial\Omega_2$, we have

$$\begin{aligned} x(t) - v(t) &= x(t) - \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)q(s)g(s)\nabla s \\ &\ge x(t) - \lambda C_3\phi_1(t)\int_{\rho(0)}^{\sigma(1)} p(s)q(s)g(s)\nabla s \\ &\ge x(t) - \frac{C_1 x(t)}{C_2\|x\|}\lambda C_3 \int_{\rho(0)}^{\sigma(1)} p(s)q(s)g(s)\nabla s \\ &\ge x(t) - \frac{x(t)}{R_2}\lambda r \\ &\ge \left(1 - \frac{\lambda r}{R_2}\right)x(t) \\ &\ge \frac{1}{2}x(t) \ge 0, \quad t \in [0,1]. \end{aligned} \tag{3.23}$$

Then,

$$\begin{aligned} \min_{\theta_1\le t\le\theta_2}\{x(t) - v(t)\} &\ge \min_{\theta_1\le t\le\theta_2}\left\{\frac{1}{2}x(t)\right\} \ge \min_{\theta_1\le t\le\theta_2}\left\{\frac{C_2}{2C_1}\phi_1(t)\|x\|\right\} \\ &= \frac{R_2C_2\min_{\theta_1\le t\le\theta_2}\phi_1(t)}{2C_1} \ge B + 1 > B. \end{aligned} \tag{3.24}$$

Now,

$$\begin{aligned} \|Tx(t)\| &\ge \max_{0\le t\le 1}\lambda \int_{\rho(0)}^{\sigma(1)} C_2\phi_1(t)H(s,s)p(s)q(s)\left[f\left(s,[x(s)-v(s)]^*\right) + g(s)\right]\nabla s \\ &\ge \max_{0\le t\le 1}\lambda C_2\phi_1(t)\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)f\left(s,[x(s)-v(s)]^*\right)\nabla s \\ &\ge \lambda C_2 \min_{\theta_1\le t\le\theta_2}\phi_1(t)\int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s)f(s,x(s)-v(s))\nabla s \\ &\ge \lambda C_2 \min_{\theta_1\le t\le\theta_2}\phi_1(t)\int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s)N(x(s)-v(s))\nabla s \\ &\ge \lambda C_2 \min_{\theta_1\le t\le\theta_2}\phi_1(t)\int_{\theta_1}^{\theta_2} H(s,s)p(s)q(s)\frac{N}{2}x(s)\nabla s \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\lambda C_2^2 N}{2C_1} \min_{\theta_1 \leq t \leq \theta_2} \phi_1(t) \int_{\theta_1}^{\theta_2} H(s,s) p(s) q(s) \phi_1(s) \|x\| \nabla s \\
&\geq \frac{\lambda C_2^2 N}{2C_1} \min_{\theta_1 \leq t \leq \theta_2} \phi_1(t) \int_{\theta_1}^{\theta_2} H(s,s) \phi_1(s) p(s) q(s) \nabla s \|x\| \\
&\geq \|x\|. \\
&\Longrightarrow \|Tx\| \geq \|x\|, \quad x \in P \cap \partial\Omega_2.
\end{aligned}
\tag{3.25}
$$

Condition (2.1) of Krasnosel'skii's fixed-point theorem is satisfied. So T has a fixed point x with $R_1 \leq \|x\| < R_2$ such that

$$
\begin{gathered}
x^{\Delta\nabla}(t) + a(t)x^{\Delta}(t) + b(t)x(t) = -\lambda q(s)\left(f\left(s, [x(s) - v(s)]^*\right) + g(s)\right), \quad 0 < t < 1, \\
x(\rho(0)) = 0, \qquad x(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i).
\end{gathered}
\tag{3.26}
$$

Since $r < \|x\|$,

$$
\begin{aligned}
x(t) - v(t) &\geq \frac{C_2}{C_1}\phi_1(t)\|x\| - \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s) p(s) q(s) g(s) \nabla s \\
&\geq \frac{C_2}{C_1}\phi_1(t)\|x\| - \phi_1(t)\lambda C_3 \int_{\rho(0)}^{\sigma(1)} p(s) q(s) g(s) \nabla s \\
&\geq \frac{C_2}{C_1}\phi_1(t)\|x\| - \frac{C_2}{C_1}\phi_1(t)\lambda r \\
&\geq \frac{C_2}{C_1}\phi_1(t) r - \frac{C_2}{C_1}\phi_1(t)\lambda r \\
&\geq (1-\lambda)\frac{C_2}{C_1} r \phi_1(t) > 0.
\end{aligned}
\tag{3.27}
$$

Let $u(t) = x(t) - v(t)$, then $u(t)$ is a positive solution of boundary value problem (1.1). This completes the proof of Theorem 3.2. □

Since condition (H_1) implies conditions (H_1^*) and (H_4), and from proof of Theorems 3.1 and 3.2, we immediately have the following theorem.

Theorem 3.3. *Suppose that (H_1)–(H_3) hold. Then boundary value problem* (1.1) *has at least two positive solutions for $\lambda > 0$ sufficiently small.*

Proof. On the hand, fix $\delta \in (0,1)$. From (H_2), let $0 < \varepsilon < \min\{1, r/2\}$ be such that

$$f(t,z) \geq \delta f(t,0), \quad \text{for } \rho(0) \leq t \leq \sigma(1),\ 0 \leq z \leq \varepsilon. \tag{3.28}$$

Choose

$$\frac{\varepsilon}{2c\overline{f}(\varepsilon)} := \overline{\lambda}, \tag{3.29}$$

where $\overline{f}(\varepsilon) = \max_{\rho(0)\leq t\leq\sigma(1),0\leq z\leq\varepsilon}\{f(t,z) + g(t)\}$, $c = C_1\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\nabla s$, and $r = (C_1C_3/C_2)\int_{\rho(0)}^{\sigma(1)} q(s)g(s)\nabla s$.

On the other hand, set $\Omega_1 = \{x \in P : \|x\| < R_1\}$, where $R_1 = 1 + r$. Choose

$$\lambda^* = \min\left\{1, R_1\left[C_1\int_{\rho(0)}^{\sigma(1)} H(s,s)p(s)q(s)\left[\max_{0\leq z\leq R_1} f(s,z) + g(s)\right]\nabla s\right]^{-1}\right\}. \tag{3.30}$$

So, let

$$0 < \lambda < \min\left\{\overline{\lambda}, \lambda^*\right\}. \tag{3.31}$$

From $0 < \lambda < \min\{\overline{\lambda}, \lambda^*\}$, we have $0 < \lambda < \overline{\lambda}$, from proof of Theorem 3.1, we know that (1.1) has a positive solution u_1 and $\|u_1\| \leq \|u_1\| \leq R_0 < r/2$. Further, also from $0 < \lambda < \min\{\overline{\lambda}, \lambda^*\}$, we have $0 < \lambda < \lambda^*$, from proof of Theorem 3.2, we know that (1.1) has a positive solution u_2 and $\|u_2\| \geq R_1/2 > r/2$. Then (1.1) has at least two positive solutions u_1 and u_2. This completes the proof of Theorem 3.3. □

4. Examples

To illustrate the usefulness of the results, we give some examples.

Example 4.1. Consider the boundary value problem

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda\left(u^{a}(t) + \frac{1}{(t-t^2)^{1/2}}\cos(2\pi u(t))\right) = 0, \quad \lambda > 0, \\ u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i), \end{gathered} \tag{4.1}$$

where $a > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.1) has a positive solution u with $u(t) > 0$ for $t \in (0,1)$.

To see this, we will apply Theorem 3.2 with

$$q(t)=1,\qquad f(t,u)=u^{a}(t)+\frac{1}{(t-t^{2})^{1/2}}\cos(2\pi u(t)),\qquad g(t)=\frac{1}{(t-t^{2})^{1/2}}. \tag{4.2}$$

Clearly

$$f(t,0)=\frac{1}{(t-t^{2})^{1/2}}>0,\qquad f(t,u)+g(t)\geq u^{a}(t)>0,\qquad \lim_{u\uparrow+\infty}\frac{f(t,u)}{u}=+\infty, \tag{4.3}$$

for $t\in(\rho(0),\sigma(1)),\ u>0$. Namely, (H_1^*) and (H_2)–(H_4) hold. From $\int_{\rho(0)}^{\sigma(1)}(1/((\sigma(1)-\rho(0))s-(s-\rho(0))^{2})^{1/2})\nabla s=\pi$, set $R_1=C_1\pi$ and $m=\max_{\rho(0)\leq t\leq\sigma(1)}\{p(t)\}+1$, we have

$$\begin{aligned}
&\int_{\rho(0)}^{\sigma(1)}C_1H(s,s)p(s)\left[\max_{0\leq z\leq R_1}f(s,z)+g(s)\right]\nabla s\\
&\quad\leq\int_{\rho(0)}^{\sigma(1)}\frac{mC_1\|\phi_1\|\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}\left[(C_1\pi)^{a}+\frac{1}{\left((\sigma(1)-\rho(0))s-(s-\rho(0))^{2}\right)^{1/2}}\right]\nabla s\\
&\quad\leq\frac{mC_1\|\phi_1\|\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}\left((C_1\pi)^{a}+\pi\right)
\end{aligned} \tag{4.4}$$

and $\lambda^*=\min\{1,\phi_1^{\Delta}(\rho(0))/m\|\phi_1\|\|\phi_2\|((C_1^{a}\pi^{a-1}+1)\}$. Now, if $\lambda<\lambda^*$, Theorem 3.2 guarantees that (4.1) has a positive solution u with $\|u\|\geq 2$.

Example 4.2. Consider the boundary value problem

$$\begin{gathered}
u^{\Delta\nabla}(t)+a(t)u^{\Delta}(t)+b(t)u(t)+\lambda\left(u^{2}(t)-\frac{9}{2}u(t)+2\right)=0,\quad 0<t<1,\ \lambda>0,\\
u(\rho(0))=0,\qquad u(\sigma(1))=\sum_{i=1}^{m-2}\alpha_i u(\eta_i).
\end{gathered} \tag{4.5}$$

Then, if $\lambda>0$ is sufficiently small, (4.5) has two solutions u_i with $u_i(t)>0$ for $t\in(0,1),\ i=1,2$.

To see this, we will apply Theorem 3.3 with

$$f(t,u)=u^{2}(t)-\frac{9}{2}u(t)+2,\quad g(t)=4. \tag{4.6}$$

Clearly

$$q(t)=0,\qquad f(t,0)=2>0,\qquad f(t,u)+g(t)\geq\frac{15}{16}>0,\qquad \lim_{u\uparrow+\infty}\frac{f(t,u)}{u}=+\infty. \tag{4.7}$$

Namely, (H_1)–(H_3) hold. Let $\delta = 1/4$, $\varepsilon = 1/4$ and $c = \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\nabla s$, then we may have

$$\overline{\lambda} = \frac{1}{8c\left(\max_{0\le z\le\varepsilon} f(t,z) + 4\right)} = \frac{1}{48c}. \tag{4.8}$$

Now, if $\lambda < \overline{\lambda}$, Theorem 3.2 guarantees that (4.5) has a positive solution u_1 with $\|u_1\| \le 1/4$.

Next, let $R_1 = p$, where $p = 4cC_3/C_2 + 1$ is a constant, then we have

$$\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\left[\max_{0\le z\le R_1} f(s,z) + g(s)\right]\nabla s = \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\left[\frac{9}{2} + 4\right]\nabla s = \frac{17c}{2} \tag{4.9}$$

and $\lambda^* = \min\{1, 2p/17c\}$. Now, if $0 < \lambda < \lambda^*$, Theorem 3.1 guarantees that (4.5) has a positive solution u_2 with $\|u_2\| \ge p$.

So, since all the conditions of Theorem 3.3 are satisfied, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.3 guarantees that (4.5) has two solutions u_i with $u_i(t) > 0$ $(i = 1,2)$.

Example 4.3. Consider the boundary value problem

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda\left(u^a(t) + \cos(2\pi u(t))\right) = 0, \quad 0 < t < 1,\ \lambda > 0, \\ u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{4.10}$$

where $a > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.10) has two solutions u_i with $u_i(t) > 0$ for $t \in (0,1)$, $i = 1,2$.

To see this we will apply Theorem 3.3 with

$$f(t,u) = u^a(t) + \cos(2\pi u(t)), \qquad g(t) = 2. \tag{4.11}$$

Clearly

$$f(t,0) = 1 > 0, \quad f(t,u) + g(t) \ge u^a(t) + 1 > 0, \quad \lim_{u\uparrow+\infty} \frac{f(t,u)}{u} = +\infty, \quad \text{for } t \in (\rho(0), \sigma(1)). \tag{4.12}$$

Namely, (H_1)–(H_3) hold. Let $\delta = 1/2$, $\varepsilon = 1/8$ and $c = \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\nabla s$, then we may have

$$\frac{\varepsilon}{2c\left(\max_{0\le x\le\varepsilon} f(t,x) + 2\right)} \ge \frac{1}{16c\left((1/8)^a + 3\right)} := \overline{\lambda}. \tag{4.13}$$

Now, if $0 < \lambda < \overline{\lambda}$ then $0 < \lambda < \varepsilon/2c(\max_{0\le x\le\varepsilon} f(t,x) + 2)$, Theorem 3.2 guarantees that (4.10) has a positive solution u_1 with $\|u_1\| \le 1/8$.

Next, let $R_1 = p$, where $p = 4cC_3/C_2 + 1$ is a constant, then we have

$$\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\left[\max_{0\le z\le R_1} f(s,z) + g(s)\right]\nabla s \le \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\left[p^a + 2\right]\nabla s = \left(p^a + 2\right)c \tag{4.14}$$

and $\lambda^* = \min\{1, p/(p^a + 2)c\}$. Now, if $0 < \lambda < \lambda^*$, then $0 < \lambda < p \times (\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)[\max_{0\le z\le R_1} f(s,z) + g(s)]\nabla s)^{-1}$, Theorem 3.1 guarantees that (4.10) has a positive solution u_2 with $\|u_2\| \ge 1$.

So, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.3 guarantees that (4.10) has two solutions u_i with $u_i(t) > 0$ $(i = 1, 2)$.

Example 4.4. Let $\mathbb{T} = \{0, 1/4, 2/4, 3/4, 1, 5/4, \ldots\}$. We consider the following four point boundary value problem:

$$\begin{gathered} u^{\Delta\nabla}(t) + \frac{12}{5}u^{\Delta}(t) - \frac{16}{5}u(t) + \lambda(u^a(t) + \cos(2\pi u(t))) = 0, \quad \lambda > 0, \\ u(0) = 0, \qquad u\left(\frac{5}{4}\right) = \frac{1}{2}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right), \end{gathered} \tag{4.15}$$

where $a(t) = 12/5$, $b(t) = -16/5$, and $q(t) = 1$. Then, if $\lambda > 0$ is sufficiently small, (4.15) has two solutions u_i with $u_i > 0$ $(i = 1, 2)$.

Let ϕ_1 and ϕ_2 be the solutions of the following linear boundary value problems, respectively,

$$\begin{gathered} u^{\Delta\nabla}(t) + \frac{12}{5}u^{\Delta}(t) - \frac{16}{5}u(t) = 0, \qquad u(0) = 0, \qquad u\left(\frac{5}{4}\right) = 1, \\ u^{\Delta\nabla}(t) + \frac{12}{5}u^{\Delta}(t) - \frac{16}{5}u(t) = 0, \qquad u(0) = 1, \qquad u\left(\frac{5}{4}\right) = 0. \end{gathered} \tag{4.16}$$

It is evident (form the Corollaries 4.24 and 4.25 and Theorem 4.28 of [17]) that

$$\phi_1(t) = \frac{(5/4)^{4t} - (1/2)^{4t}}{(5/4)^5 - (1/2)^5}, \qquad \phi_2(t) = \frac{(5/4)^5(1/2)^{4t} - (1/2)^5(5/4)^{4t}}{(5/4)^5 - (1/2)^5}. \tag{4.17}$$

Also ϕ_1 satisfies (C3). Green's function is

$$H(t,s) = \frac{1024}{9279}\begin{cases} \left(\frac{5}{4}\right)^{4s} - \left(\frac{1}{2}\right)^{4s}\left(\frac{5}{4}\right)^{5}\left(\frac{1}{2}\right)^{4t} - \left(\frac{1}{2}\right)^{5}\left(\frac{5}{4}\right)^{4t}, & s \le t, \\ \left(\frac{5}{4}\right)^{4t} - \left(\frac{1}{2}\right)^{4t}\left(\frac{5}{4}\right)^{5}\left(\frac{1}{2}\right)^{4s} - \left(\frac{1}{2}\right)^{5}\left(\frac{5}{4}\right)^{4s}, & t \le s, \end{cases} \tag{4.18}$$

and $p(t) = (2/5)^{4t-1}$ follows from $e_\alpha(t, t_0) = (1 + \alpha h)^{(t-t_0)/h}$ on $\mathbb{T} = h\mathbb{N}$.

To see this, we will apply Theorem 3.3 with

$$f(t,u) = u^a(t) + \cos(2\pi u(t)), \qquad g(t) = 2. \tag{4.19}$$

Clearly

$$f(t,0) = 1 > 0, \quad f(t,u) + g(t) \geq u^a(t) + 1 > 0, \quad \lim_{u\uparrow+\infty} \frac{f(t,u)}{u} = +\infty, \quad \text{for } t \in \big(\rho(0),\sigma(1)\big). \tag{4.20}$$

Namely, (H_1)–(H_3) hold. Let $\delta = 1/2$, $\varepsilon = 1/8$ and $c = \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\nabla s$, then we may have

$$\frac{\varepsilon}{2c\big(\max_{0\leq x\leq\varepsilon} f(t,x) + 2\big)} \geq \frac{1}{16c\big((1/8)^a + 3\big)} := \overline{\lambda}. \tag{4.21}$$

Now, if $0 < \lambda < \overline{\lambda}$ then $0 < \lambda < \varepsilon/2c(\max_{0\leq x\leq\varepsilon} f(t,x) + 2)$, Theorem 3.2 guarantees that (4.15) has a positive solution u_1 with $\|u_1\| \leq 1/8$.

Next, let $R_1 = 4cC_3/C_2 + 1$ is a constant, then we have

$$\int_{\rho(0)}^{\sigma(1)} \left[\max_{0\leq z\leq R_1} f(s,z) + g(s)\right] \nabla s \leq \int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)p(s)\left[R_1^a + 2\right] \nabla s = (p^a + 2)c \tag{4.22}$$

and $\lambda^* = \min\{1, R_1/(R_1^a + 2)c\}$. Now, if $0 < \lambda < \lambda^*$, then $0 < \lambda < R_1 \times (\int_{\rho(0)}^{\sigma(1)} C_1 H(s,s)[\max_{0\leq z\leq R_1} f(s,z) + g(s)]\nabla s)^{-1}$, Theorem 3.1 guarantees that (4.15) has a positive solution u_2 with $\|u_2\| \geq 1$.

So, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.3 guarantees that (4.15) has two solutions u_i with $u_i(t) > 0$ $(i = 1,2)$.

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References

[1] S. G. Topal and A. Yantir, "Positive solutions of a second order *m*-point BVP on time scales," *Nonlinear Dynamics and Systems Theory*, vol. 9, no. 2, pp. 185–197, 2009.

[2] X. Lin and Z. Du, "Positive solutions of *m*-point boundary value problem for second-order dynamic equations on time scales," *Journal of Difference Equations and Applications*, vol. 14, no. 8, pp. 851–864, 2008.

[3] H.-R. Sun and W.-T. Li, "Positive solutions for nonlinear three-point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.

[4] Y. Pang and Z. Bai, "Upper and lower solution method for a fourth-order four-point boundary value problem on time scales," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2243–2247, 2009.

[5] F. M. Atici and S. G. Topal, "The generalized quasilinearization method and three point boundary value problems on time scales," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 577–585, 2005.
[6] S. Liang and J. Zhang, "The existence of countably many positive solutions for nonlinear singular m-point boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 223, no. 1, pp. 291–303, 2009.
[7] S. Liang, J. Zhang, and Z. Wang, "The existence of three positive solutions of m-point boundary value problems for some dynamic equations on time scales," *Mathematical and Computer Modelling*, vol. 49, no. 7-8, pp. 1386–1393, 2009.
[8] L. Hu, "Positive solutions to singular third-order three-point boundary value problems on time scales," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 606–615, 2010.
[9] İ. Yaslan, "Multiple positive solutions for nonlinear three-point boundary value problems on time scales," *Computers & Mathematics with Applications*, vol. 55, no. 8, pp. 1861–1869, 2008.
[10] J. Li and J. Shen, "Existence results for second-order impulsive boundary value problems on time scales," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 4, pp. 1648–1655, 2009.
[11] J.-P. Sun, "Twin positive solutions of nonlinear first-order boundary value problems on time scales," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 6, pp. 1754–1758, 2008.
[12] R. P. Agarwal, M. Bohner, and D. O'Regan, "Time scale boundary value problems on infinite intervals," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 27–34, 2002.
[13] H. Chen, H. Wang, Q. Zhang, and T. Zhou, "Double positive solutions of boundary value problems for p-Laplacian impulsive functional dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 53, no. 10, pp. 1473–1480, 2007.
[14] R. Ma and H. Luo, "Existence of solutions for a two-point boundary value problem on time scales," *Applied Mathematics and Computation*, vol. 150, no. 1, pp. 139–147, 2004.
[15] D. R. Anderson, G. S. Guseinov, and Joan Hoffacker, "Higher-order self-adjoint boundary-value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 194, no. 2, pp. 309–342, 2006.
[16] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.
[17] *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.