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Research Article

Schauder Basis, Separability, and Approximation Property in Intuitionistic Fuzzy Normed Space

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We define and study the concepts of Schauder basis, separability, and approximation property in intuitionistic fuzzy normed spaces and establish some results related to these concepts. We also display here some interesting examples by using classical sequence spaces ℓ_p $(1 \le p \le \infty)$.

1. Introduction and Background

We will write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ℓ_∞, c, and c_0 denote the sets of all bounded, convergent, and null sequences, respectively. We write $\ell_p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, $e_n^{(n)} = 1$, and $e_k^{(n)} = 0$ $(k \neq n)$. Note that c_0, c, and ℓ_∞ are Banach spaces with the sup-norm $\|x\|_\infty = \sup_k |x_k|$, and ℓ_p $(1 \le p < \infty)$ are Banach spaces with the norm $\|x\|_p = (\sum |x_k|^p)^{1/p}$.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a normed linear space X is called a *Schauder basis* [1] if for every $x \in X$, there is a unique sequence $(\beta_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \beta_n b^{(n)}$, that is, $\lim_{m \to \infty} \|x - \sum_{n=0}^{m} \beta_n b^{(n)}\|_X = 0$.

Recently, the concept of intuitionistic fuzzy normed space has been introduced and studied by Saadati and Park [2] and further studied by Mursaleen and Mohiuddine [3–5]. The concept of Schauder basis and its applications have recently been studied by Palomares et al. [6] and by Yılmaz [7]. In this paper, we define and study the concept of Schauder basis, separability, and approximation property in intuitionistic fuzzy normed spaces and establish some results related to these concepts analogous to those of Yılmaz [7]. We also display here some interesting examples by using classical sequence spaces ℓ_p $(1 \le p \le \infty)$.

In this section, we recall some notations and basic definitions used in this paper.

Definition 1.1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative,

(b) $*$ is continuous,

(c) $a * 1 = a$ for all $a \in [0,1]$,

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 1.2. A binary operation $\diamond: [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a') $\diamond$ is associative and commutative,

(b') $\diamond$ is continuous,

(c') $a \diamond 0 = a$ for all $a \in [0,1]$,

(d') $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [2] have recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$,

(i) $\mu(x,t) + \nu(x,t) \leq 1$,

(ii) $\mu(x,t) > 0$,

(iii) $\mu(x,t) = 1$ if and only if $x = 0$,

(iv) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for each $\alpha \neq 0$,

(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(vi) $\mu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{t\to\infty}\mu(x,t) = 1$ and $\lim_{t\to 0}\mu(x,t) = 0$,

(viii) $\nu(x,t) < 1$,

(ix) $\nu(x,t) = 0$ if and only if $x = 0$,

(x) $\nu(\alpha x, t) = \nu(x, t/|\alpha|)$ for each $\alpha \neq 0$,

(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$,

(xii) $\nu(x, \cdot) : (0,\infty) \to [0,1]$ is continuous, and

(xiii) $\lim_{t\to\infty}\nu(x,t) = 0$ and $\lim_{t\to 0}\nu(x,t) = 1$.

In this case (μ, ν) is called an *intuitionistic fuzzy norm*.

Example 1.4. Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$, and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0; \end{cases} \qquad \nu(x,t) = \begin{cases} \dfrac{\|x\|}{t+|x|} & \text{if } t > 0, \\ 1 & \text{if } t \leq 0. \end{cases} \tag{1.1}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFNS.

Remark 1.5 (see [3]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNS with the condition

$$\mu(x,t) > 0, \qquad \nu(x,t) < 1 \quad \text{implies } x = 0 \ \ \forall t \in \mathbb{R}. \tag{1.2}$$

Let $\|x\|_\alpha = \inf\{t > 0 : \mu(x,t) \geq \alpha \text{ and } \nu(x,t) \leq 1-\alpha\}$, for all $\alpha \in (0,1)$. Then $\{\|\cdot\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. These norms are called α-norms on X corresponding to intuitionistic fuzzy norm (μ, ν).

2. Some Topological Concepts in IFNS

Recently, the strong and weak intuitionistic fuzzy convergence as well as strong and weak intuitionistic fuzzy limit were discussed by Mursaleen and Mohiuddine [3].

Definition 2.1. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence (x_k) is said to be

(i) *weakly intuitionistic fuzzy convergent* to $x \in X$ if and only if, for every $\epsilon > 0$ and $\alpha \in (0,1)$, there exists some $k_0 = k_0(\alpha, \epsilon)$ such that $\mu(x_k - x, \epsilon) \geq 1-\alpha$ and $\nu(x_k - x, \epsilon) \leq \alpha$ for all $k \geq k_0$. In this case we write $x_k \xrightarrow{\text{wif}} x$,

(ii) *strongly intuitionistic fuzzy convergent* to $x \in X$ if and only if, for every $\alpha \in (0,1)$, there exists some $k_0 = k_0(\alpha)$ such that $\mu(x_k - x, t) \geq 1-\alpha$ and $\nu(x_k - x, t) \leq \alpha$ for all $t > 0$. In this case we write $x_k \xrightarrow{\text{sif}} x$.

The following result characterizes the (wif)- and (sif)-limit through α-norms.

Proposition 2.2. *Let* $(X, \mu_1, \nu_1, *, \diamond)$ *and* $(Y, \mu_2, \nu_2, *, \diamond)$ *be two IFNS satisfying* (1.2) *and* $f : X \to Y$ *be a mapping. Then*

(i) (wif)-$\lim_{x \to x_0} f(x) = L$ *if and only if, for each* $\alpha \in (0,1)$

$$\lim_{\|x-x_0\|_\alpha^{(1)} \to 0} \left\| f(x) - L \right\|_\alpha^{(2)} = 0; \tag{2.1}$$

(ii) (sif)-$\lim_{x \to x_0} f(x) = L$ *if and only if*

$$\lim_{\|x-x_0\|_\alpha^{(1)} \to 0} \left\| f(x) - L \right\|_\alpha^{(2)} = 0 \quad \text{uniformly in } \alpha, \tag{2.2}$$

where $\|\cdot\|_\alpha^{(1)}$ *and* $\|\cdot\|_\alpha^{(2)}$ *are* α*-norms of the intuitionistic fuzzy norms* (μ_1, ν_1) *and* (μ_2, ν_2)*, respectively.*

Proof. Here we prove the case (ii). Suppose that (sif)-$\lim_{x\to x_0} f(x) = L$. For a given $\epsilon > 0$, there exists some $\delta = \delta(\epsilon) > 0$ such that

$$\mu_2(f(x) - L, \epsilon) \geq \mu_1(x - x_0, \delta), \qquad \nu_2(f(x) - L, \epsilon) \geq \nu_1(x - x_0, \delta), \tag{2.3}$$

for all $x \in X$. For each $\alpha \in (0,1)$, if

$$\|x - x_0\|_\alpha^{(1)} = \inf\{t > 0 : \mu_1(x - x_0, t) \geq \alpha,\ \nu_1(x - x_0, t) \leq 1 - \alpha\} < \delta, \tag{2.4}$$

then $\mu_1(x-x_0, \delta) \geq \alpha$ and $\nu_1(x-x_0, \delta) \leq 1-\alpha$. Hence $\mu_2(f(x)-L, \epsilon) \geq \alpha$ and $\nu_2(f(x)-L, \epsilon) \leq 1-\alpha$ and so that $\|f(x) - L\|_\alpha^{(2)} < \epsilon$. Since δ does not depend on α, we get

$$\lim_{\|x-x_0\|_\alpha^{(1)} \to 0} \|f(x) - L\|_\alpha^{(2)} = 0 \quad \text{uniformly in } \alpha. \tag{2.5}$$

Conversely, let $\lim_{\|x-x_0\|_\alpha^{(1)} \to 0} \|f(x) - L\|_\alpha^{(2)} = 0$ uniformly in α. Given $\epsilon > 0$, there exists some $\delta = \delta(\epsilon) > 0$ such that

$$\|x - x_0\|_\alpha^{(1)} < \delta \qquad \text{implies } \|f(x) - L\|_\alpha^{(2)} < \epsilon \tag{2.6}$$

for all $x \in X$ and $\alpha \in (0,1)$. Choose some $\lambda < \mu_1(x-x_0, \delta)$ and $\lambda > \nu_1(x-x_0, \delta)$ or $\nu_1(x-x_0, \delta) < \lambda < \mu_1(x - x_0, \delta)$. Since

$$\mu_1(x - x_0, \delta) = \sup\left\{\alpha \in (0,1) : \|x - x_0\|_\alpha^{(1)} < \delta\right\}, \tag{2.7}$$

there exists some $\alpha_0 \in (0,1)$ such that $\lambda < \alpha_0$ and $\|x - x_0\|_{\alpha_0}^{(1)} < \delta$. Hence $\|f(x) - L\|_{\alpha_0}^{(2)} < \epsilon$ by the hypothesis, that is,

$$\mu_2(f(x) - L, \epsilon) \geq \alpha_0 > \lambda, \qquad \nu_2(f(x) - L, \epsilon) \leq 1 - \alpha_0 < 1 - \lambda. \tag{2.8}$$

So, we get $\mu_2(f(x) - L, \epsilon) \geq \mu_1(x - x_0, \delta)$ and $\nu_2(f(x) - L, \epsilon) \leq \mu_1(x - x_0, \delta)$. □

Proposition 2.3. *Let* (x_k) *be a sequence in the IFNS* $(X, \mu_1, \nu_1, *, \diamond)$ *satisfying* (1.2)*. Then*

(i) $x_k \xrightarrow{\text{wif}} x$ *if and only if, for each* $\alpha \in (0,1)$

$$\lim_{k\to\infty} \|x_k - x\|_\alpha = 0. \tag{2.9}$$

(ii) $x_k \xrightarrow{\text{sif}} x$ *if and only if*

$$\lim_{k\to\infty} \|x_k - x\|_\alpha = 0 \quad \text{uniformly in } \alpha, \tag{2.10}$$

where $\|\cdot\|_\alpha$ *are* α*-norms of the intuitionistic fuzzy norms* (μ, ν)*.*

The proof of the above theorem directly follows from Propositions 2.2.

We define the following concepts analogous to that of Yılmaz [7].

Definition 2.4. The sif(wif)-*closure* of a subset B in IFNS $(X,\mu,\nu,*,\diamond)$ is the set of all $x \in X$ such that there exists a sequence $(x_n) \in B$ such that $x_n \xrightarrow{\text{sif(wif)}} x$. In this case, we write $\overline{B}^{\text{sif}}(\overline{B}^{\text{wif}})$. B is said to be sif(wif)-*closed* whenever $\overline{B}^{\text{sif}}(\overline{B}^{\text{wif}}) = B$.

It is easy to see that $\overline{B}^{\text{sif}} \subseteq (\overline{B}^{\text{wif}})$. The following example shows that inclusion is strict.

Example 2.5. Let $X = \mathbb{R}$ and

$$\mu(x,t) = \begin{cases} \dfrac{t-|x|}{t+|x|} & \text{if } t > |x|, \\ 0 & \text{if } t \le |x|; \end{cases} \qquad \nu(x,t) = \begin{cases} \dfrac{2|x|}{t+|x|} & \text{if } t > |x|, \\ 1 & \text{if } t \le |x|, \end{cases} \tag{2.11}$$

on X. Let $U_X = \{x \in X : |x| < 1\}$ and we show that $\overline{U_X}^{\text{wif}} = B_X = \{x \in X : |x| \le 1\}$. For every $x \in B_X$, there exists a sequence $(x_n) \subset U_X$ such that $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$, for each $\alpha \in (0,1)$. This is accomplished by taking $x_n = (1 - 1/(n+1))x$ since each $x_n \in U_X$ and

$$\|x_n - x\|_\alpha = \frac{1+\alpha}{1-\alpha}|x_n - x| < \left(\frac{1+\alpha}{1-\alpha}\right)\frac{1}{n+1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{2.12}$$

for each $\alpha \in (0,1)$. However, $\overline{U_X}^{\text{sif}} = U_X$. Indeed for $x \in \overline{U_X}^{\text{sif}}$, there exists $(x_n) \subset U_X$ such that $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$, uniformly in α. This means that, given $\epsilon > 0$, there exists an integer $n_\circ(\epsilon) > 0$ such that for every $\alpha \in (0,1)$ and $n \ge n_\circ$,

$$\|x_n - x\|_\alpha < \epsilon. \tag{2.13}$$

On the other hand,

$$|x| \le |x_n - x| + |x_n| < |x_n - x| + 1 = \left(\frac{1-\alpha}{1+\alpha}\right)\|x_n - x\|_\alpha + 1 < \left(\frac{1-\alpha}{1+\alpha}\right)\epsilon + 1, \tag{2.14}$$

for all $\alpha \in (0,1)$ and $n \ge n_\circ$. Letting $\epsilon \to 0$ (and hence $\alpha \to 1$), we get $|x| < 1$. Therefore, $\overline{U_X}^{\text{sif}} \subseteq U_X$.

Definition 2.6. A subset S of an IFNS $(X,\mu,\nu,*,\diamond)$ is said to be *dense* in $(X,\mu,\nu,*,\diamond)$ if and only if $\overline{S}^{(\text{sif})}(\overline{S}^{(\text{wif})}) = X$.

Definition 2.7. An IFNS $(X,\mu,\nu,*,\diamond)$ is said to *separable* if it contains a countable dense subset, that is, there is a countable set $\{x_k\}$ with the following property: for each $\epsilon > 0$ and each $x \in X$, there is at least one x_n with

$$\mu(x_n - x, \epsilon) \ge 1 - \alpha, \qquad \nu(x_n - x, \epsilon) \le \alpha \tag{2.15}$$

for $\alpha \in (0,1)$.

Theorem 2.8. *Every finite dimension IFNS is separable.*

Proof. Let $(X, \mu, \nu, *, \diamond)$ be a finite dimension normed linear space and $\{u_1, u_2, \ldots, u_n\}$ a basis of X. Since $Q_{\mathbb{K}}$ is a countable subset of $\mathbb{K}$, it follows that

$$D_n = \left\{ \sum_{j=1}^{n} \alpha_j u_j : \alpha_j \in Q_{\mathbb{K}} \right\} \tag{2.16}$$

is countable subset of $\operatorname{span}\{u_1, u_2, \ldots, u_n\} = X$. Also, D is dense in $(X, \mu, \nu, *, \diamond)$. To see this, let $x \in X$ and $\epsilon > 0$. Let $\beta_1, \ldots, \beta_n$ be scalars such that $x = \sum_{j=1}^{n} \beta_j u_j$. By the denseness of $Q_{\mathbb{K}}$ in $\mathbb{K}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $Q_{\mathbb{K}}$ such that

$$\mu\left(\beta_j - \alpha_j, \frac{\epsilon}{\left|\sum_{j=1}^{n} u_j\right|} \right) \geq 1 - \alpha, \qquad \nu\left(\beta_j - \alpha_j, \frac{\epsilon}{\left|\sum_{k=1}^{n} u_j\right|} \right) \leq \alpha, \tag{2.17}$$

for all $j \in \{1, 2, \ldots, n\}$. Then it follows that

$$\mu\left(x - \sum_{j=1}^{n} \alpha_j u_j, \epsilon \right) \geq \mu\left((\beta_j - \alpha_j) \sum_{j=1}^{n} u_j, \epsilon \right) = \mu\left(\beta_j - \alpha_j, \frac{\epsilon}{\left|\sum_{j=1}^{n} u_j\right|} \right) \geq 1 - \alpha. \tag{2.18}$$

Similarly,

$$\nu\left(x - \sum_{j=1}^{n} \alpha_j u_j, \epsilon \right) \leq \alpha. \tag{2.19}$$

This implies that D is dense in $(X, \mu, \nu, *, \diamond)$. □

Theorem 2.9. *Every IFNS having wif-basis is separable.*

Proof. Let Y be IFNS with wif-basis $\{u_1, u_2, \ldots\}$. Since $Y = \bigcup_{n=1}^{\infty} Y_n$ with $Y_n = \operatorname{span}\{u_1, u_2, \ldots, u_n\}$ for all $n \in \mathbb{N}$ is dense in Y, it is enough to show that Y has a countable dense subset. Let

$$D_n = \left\{ \sum_{j=1}^{n} \alpha_j u_j : \alpha_j \in Q_{\mathbb{K}} \right\}, \tag{2.20}$$

for all $n \in \mathbb{N}$. Then D_n will be a countable dense subset of Y_n (see Theorem 2.8). Thus $\bigcup_{n=1}^{\infty} D_n$ is a countable dense subset of Y. □

3. Intuitionistic Fuzzy Schauder Bases

In this section, we define strong and weak intuitionistic fuzzy Schauder bases.

Definition 3.1. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence (x_n) is said to be

(i) *strongly intuitionistic fuzzy (Schauder) basis* (for short, sif-basis) of X if and only if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that

$$\sum_{k=1}^{n} a_k x_k \xrightarrow{\text{sif}} x, \tag{3.1}$$

this means that for each $\alpha \in (0,1)$ there exists $n_\circ = n_\circ(\alpha)$ such that $n \geq n_\circ$ implies

$$\mu\left(x - \sum_{k=1}^{n} a_k x_k, t\right) \geq 1 - \alpha, \qquad \nu\left(x - \sum_{k=1}^{n} a_k x_k, t\right) \leq \alpha, \tag{3.2}$$

for all $t > 0$,

(ii) *weak intuitionistic fuzzy (Schauder) basis* (for short, wif-basis) of X if and only if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that

$$\sum_{k=1}^{n} a_k x_k \xrightarrow{\text{wif}} x, \tag{3.3}$$

this means, for each $\alpha \in (0,1)$ and $\epsilon > 0$, there exists $n_\circ = n_\circ(\alpha, \epsilon)$ such that $n \geq n_\circ$ implies

$$\mu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon\right) \geq 1 - \alpha, \qquad \nu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon\right) \leq \alpha. \tag{3.4}$$

Proposition 3.2. *Let (x_k) be a sequence in the IFNS $(X, \mu, \nu, *, \diamond)$ satisfying* (1.2). *Then*

(i) *(x_k) is a wif-basis of X if and only if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that for each $\alpha \in (0,1)$,*

$$\lim_{n \to \infty} \left\| x - \sum_{k=1}^{n} a_k x_k \right\|_\alpha = 0; \tag{3.5}$$

(ii) *(x_k) is a sif-basis of X if and only if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that*

$$\lim_{n \to \infty} \left\| x - \sum_{k=1}^{n} a_k x_k \right\|_\alpha = 0 \quad \text{uniformly in } \alpha, \tag{3.6}$$

where $\| \cdot \|_\alpha$ are α-norms of the intuitionistic fuzzy norms (μ, ν).

The proof of the above theorem is similar to Propositions 2.2.

Definition 3.3. By the notations in Definition 3.1., the mappings

$$g_n : X \longrightarrow \mathbb{R} \quad \text{such that } g_n(x) = g_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = a_n,$$
$$h_n : X \longrightarrow X \quad \text{such that } h_n(x) = h_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{n} a_k x_k, \quad n = 1, 2, \ldots \tag{3.7}$$

are called coordinate functionals and natural projections, respectively, associated to the sif(wif)-basis (x_n) in X.

Proposition 3.4. *Let (x_n) be a basis in wif-complete IFNS $(X, \mu, \nu, *, \diamond)$ satisfying* (1.2). *Then each g_n and h_n is wif-continuous.*

Proof. By Proposition 3.2, (x_n) is also a Schauder basis in the Banach space $(X, \|\cdot\|_\alpha)$ for each $\alpha \in (0, 1)$. Thus

$$g_n : (X, \|\cdot\|_\alpha) \longrightarrow \mathbb{R} \quad \text{such that } g_n(x) = g_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = a_n,$$
$$h_n : (X, \|\cdot\|_\alpha) \longrightarrow (X, \|\cdot\|_\alpha) \quad \text{such that } h_n(x) = h_n\left(\sum_{k=1}^{\infty} a_k x_k\right) = \sum_{k=1}^{n} a_k x_k \tag{3.8}$$

are continuous. Therefore, the mappings are wif-continuous for each n. □

Remark 3.5. It is obvious that, if (x_n) is an sif-basis of X then it is wif-basis of X, but not conversely. For the converse part, let us consider the following example.

Example 3.6. Let $X = \ell_p$ $(1 \le p < \infty)$, the Banach space of all absolutely p-summable sequences with the norm $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$, and consider the intuitionistic fuzzy norm

$$\mu(x,t) = \begin{cases} \dfrac{t - \|x\|_p}{t + \|x\|_p} & \text{if } t > \|x\|_p, \\ 0 & \text{if } t \le \|x\|_p; \end{cases} \qquad \nu(x,t) = \begin{cases} \dfrac{2\|x\|_p}{t + \|x\|_p} & \text{if } t > \|x\|_p, \\ 1 & \text{if } t \le \|x\|_p, \end{cases} \tag{3.9}$$

on X. We can find α-norms of intuitionistic fuzzy norm (μ, ν) since it satisfies condition (1.2). Thus

$$\mu(x,t) \ge \alpha \Longleftrightarrow \frac{t - \|x\|_p}{t + \|x\|_p} \ge \alpha \Longleftrightarrow \frac{1+\alpha}{1-\alpha}\|x\|_p \le t,$$
$$\nu(x,t) \le 1 - \alpha \Longleftrightarrow \frac{2\|x\|_p}{t + \|x\|_p} \le 1 - \alpha \Longleftrightarrow \frac{1+\alpha}{1-\alpha}\|x\|_p \le t. \tag{3.10}$$

This shows that

$$\|x\|_\alpha = \inf\{t > 0 : \mu(x,t) \geq \alpha,\ \nu(x,t) \leq 1-\alpha\} = \frac{1+\alpha}{1-\alpha}\|x\|_p. \tag{3.11}$$

Now, we show that the sequence $(e_k)_{k=1}^{\infty}$ is wif-basis but not sif-basis. Take any $x = (x_k) \in \ell_p$. Put

$$y_n = x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots), \tag{3.12}$$

then

$$y_n = (0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots). \tag{3.13}$$

Hence

$$\lim_n \|y_n\|_\alpha = \frac{1+\alpha}{1-\alpha}\lim_n \|y_n\|_p = \frac{1+\alpha}{1-\alpha}\lim_n\left(\sum_{k=n+1}^{\infty}|x_k|^p\right)^{1/p} = 0 \tag{3.14}$$

and by Proposition 3.2 (e_k) is wif-basis for ℓ_p. However, this convergence is not uniform in α since $(1+\alpha)/(1-\alpha)\epsilon \to \infty$ as $\alpha \to 1$.

However, if we put

$$\mu_1(x,t) = \begin{cases} 1 & \text{if } t > \|x\|_p, \\ 0 & \text{if } t \leq \|x\|_p; \end{cases} \qquad \nu_1(x,t) = \begin{cases} 0 & \text{if } t > \|x\|_p, \\ 1 & \text{if } t \leq \|x\|_p, \end{cases} \tag{3.15}$$

on X, then $(\ell_p, \mu_1, \nu_1, *, \diamond)$ is an IFNS satisfying (1.2), and (e_k) is a sif-basis for ℓ_p since $\|x\|_\alpha = \|x\|_p$ for each $\alpha \in (0,1)$.

Remark 3.7. In finite-dimensional spaces, the definition of basis is independent of the intuitionistic fuzzy norm and hence coincides with the definition of a classical vector space basis (Hamel basis).

We know that every intuitionistic fuzzy normed space induces a topology τ such that for some $A \subset X$, $A \in \tau$ if and only if for each $x \in A$ there exists some $t > 0$ and $\alpha \in (0,1)$ such that $B(x,\alpha,t) \subset A$, where

$$B(x,\alpha,t) := \{y : \mu(x-y,t) \geq 1-\alpha,\ \nu(x-y,t) \leq \alpha\}. \tag{3.16}$$

Proposition 3.8. *τ is a vector topology for X; that is, the vector space operations are continuous in this topology.*

Proof. Since the family $B(x, 1/n, 1/n) : n = 1, 2, \ldots$ is a countable local basis at x, τ is the first countable topology of X. Hence it is sufficient to show that the vector space operations are sequentially continuous in τ. Suppose $x_n \to x$ and $y_n \to y$ in the topological space (X, τ). This means $\mu(x_n - x, t/2), \mu(y_n - y, t/2), \nu(x_n - x, t/2)$, and $\nu(y_n - y, t/2) \to 1$ as $n \to \infty$, for all $t > 0$. Now

$$\begin{aligned}
\mu(x_n + y_n - (x + y), t) &\geq \mu\left(x_n - x, \frac{t}{2}\right) * \mu\left(y_n - y, \frac{t}{2}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \\
\nu(x_n + y_n - (x + y), t) &\leq \nu\left(x_n - x, \frac{t}{2}\right) \diamond \nu\left(y_n - y, \frac{t}{2}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.17}$$

for all $t > 0$. Further, if $\lambda_n \to \lambda$ in $\mathbb{R}$ or $\mathbb{C}$, the scalar field of X, then

$$\begin{aligned}
\mu(\lambda_n x_n - \lambda x, t) &= \mu(\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x, t) = \mu((\lambda_n - \lambda) x_n + \lambda(x_n - x), t) \\
&\geq \mu\left(x_n, \frac{t}{2|\lambda_n - \lambda|}\right) * \mu\left(x_n - x, \frac{t}{2|\lambda|}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \\
\nu(\lambda_n x_n - \lambda x, t) &= \nu(\lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x, t) = \nu((\lambda_n - \lambda) x_n + \lambda(x_n - x), t) \\
&\leq \nu\left(x_n, \frac{t}{2|\lambda_n - \lambda|}\right) \diamond \nu\left(x_n - \mathrm{x}, \frac{t}{2|\lambda|}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.18}$$

Analogous to the classical results, we prove here that a normed linear space having a Schauder basis is separable. □

Theorem 3.9. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS having wif-basis (x_n). Then the topological space (X, τ) is separable.*

Proof. Let E denotes the set of all finite linear combinations $\sum_{k=1}^{n} b_k x_k$, where each b_k is a (real or complex) rational number. Obviously, E is countable and let us show that it is dense in τ. Suppose $x \in X$ is arbitrary. There exists a unique sequence (a_n) of scalars such that for each $\epsilon > 0$ and $\alpha \in (0, 1)$, we can find some integer $n_\circ = n_\circ(\alpha, \epsilon)$ such that

$$\mu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon\right) \geq 1 - \alpha, \qquad \nu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon\right) \leq \alpha. \tag{3.19}$$

That is, for all $n \geq n_\circ$,

$$\sum_{k=1}^{n} a_k x_k \in B(x, \alpha, \epsilon). \tag{3.20}$$

On the other hand, one can constitute a sequence $(b_k^i)_{i=1}^{\infty}$ of scalars converging to a_k, for each k. Hence the sequence $(\sum_{k=1}^{n} b_k^i x_k)_{i=1}^{\infty}$ converges to $\sum_{k=1}^{n} a_k x_k$ in τ by the continuity of vector space operations. This implies that every x-centered τ-open sphere $B(x, \alpha, \epsilon)$ includes an element $\sum_{k=1}^{n} b_k^i x_k$ of E. □

Theorem 3.10. *Let* $(X, \|\cdot\|)$ *be a normed space and* (x_n) *a basis in* X*. Then* (x_n) *is a wif-basis for IFNS* $(X, \mu, \nu, *, \diamond)$*, where*

$$\mu(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0; \end{cases} \qquad \nu(x,t) = \begin{cases} \dfrac{\|x\|}{t+\|x\|} & \text{if } t > 0, \\ 1 & \text{if } t \leq 0. \end{cases} \tag{3.21}$$

Proof. By the hypothesis, for each $x \in X$, there exists a unique sequence (a_n) of scalars with $\sum_{k=1}^{n} a_k x_k \to 0$ in the norm topology as $n \to \infty$. Explicitly, for each $\delta > 0$, there exists an integer $n_\circ = n_\circ(\delta)$ such that $n \geq n_\circ$ implies

$$\left\| x - \sum_{k=1}^{n} a_k x_k \right\| \leq \delta. \tag{3.22}$$

Now, for each ϵ and $\alpha \in (0,1)$, take $\delta = \alpha\epsilon/(1-\alpha)$. So, there exists an integer $n_\circ = n_\circ(\delta) = n_\circ(\alpha, \epsilon)$ such that $n \geq n_\circ$ implies

$$\left\| x - \sum_{k=1}^{n} a_k x_k \right\| \leq \frac{\alpha\epsilon}{1-\alpha} \tag{3.23}$$

if and only if

$$\begin{gathered} \mu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon \right) = \frac{\epsilon}{\epsilon + \left\| x - \sum_{k=1}^{n} a_k x_k \right\|} \geq 1 - \alpha, \\ \nu\left(x - \sum_{k=1}^{n} a_k x_k, \epsilon \right) = \frac{\epsilon}{\epsilon + \left\| x - \sum_{k=1}^{n} a_k x_k \right\|} \leq \alpha. \end{gathered} \tag{3.24}$$

□

4. Intuitionistic Fuzzy Approximation Property

In this section, we define strong and weak intuitionistic fuzzy approximation property and prove some interesting results.

Definition 4.1. We say that sif-complete IFNS $(X, \mu, \nu, *, \diamond)$ is said to have *strong intuitionistic fuzzy approximation property* (for short, sif-AP) if for every sif-compact set $K \subset X$ and $\alpha \in (0,1)$ there exists an operator $T : X \to X$ of finite rank such that

$$\mu(T(x) - x, t) \geq 1 - \alpha, \qquad \nu(T(x) - x, t) \leq \alpha, \tag{4.1}$$

for all $x \in K$ and $t > 0$.

Definition 4.2. A wif-complete IFNS $(X, \mu, \nu, *, \diamond)$ is said to have *weak intuitionistic fuzzy approximation property* (for short, wif-AP) if for every wif-compact set $K \subset X$ and for each $\epsilon > 0$ and $\alpha \in (0,1)$ there exists an operator $T_\alpha : X \longrightarrow X$ of finite rank such that

$$\mu(T_\alpha(x) - x, \epsilon) \geq 1 - \alpha, \qquad \nu(T_\alpha(x) - x, \epsilon) \leq \alpha, \tag{4.2}$$

for all $x \in K$.

Remark 4.3. The operator T in wif-AP depends both on $\epsilon > 0$ and $\alpha \in (0,1)$ whereas it depends only on $\alpha \in (0,1)$ in sif-AP. T depends on the set K in both situations.

Proposition 4.4. (i) *A wif-complete IFNS* $(X, \mu, \nu, *, \diamond)$ *satisfying* (1.2) *has wif-AP if and only if for every wif-compact set* $K \in X$ *and for each* $\epsilon > 0$ *and* $\alpha \in (0,1)$ *there exists an operator* $T_\alpha : X \longrightarrow X$ *of finite rank such that*

$$\|T_\alpha(x) - x\|_\alpha < \epsilon, \tag{4.3}$$

for all $x \in K$.

(ii) *A sif-complete IFNS* $(X, \mu, \nu, *, \diamond)$ *satisfying* (1.2) *has sif-AP if and only if for every sif-compact set* $K \in X$ *and for each* $\epsilon > 0$ *there exists an operator* $T : X \longrightarrow X$ *of finite rank such that*

$$\|T(x) - x\|_\alpha < \epsilon, \tag{4.4}$$

for all $x \in K$.

The proof of the above theorem directly follows from Propositions 2.2.

Theorem 4.5. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS possessing a wif-basis* (x_n)*. Then X has the wif-AP.*

Proof. Let $K \subset X$ be a wif-compact subset of X. Let $\epsilon > 0$ and $\alpha \in (0,1)$ be arbitrary. By the hypothesis, for some $x \in K$, there exists a unique sequence (a_n) of scalars such that

$$P_n(x) = \sum_{k=1}^{n} a_k x_k \xrightarrow{\text{wif}} x \quad \text{as } n \longrightarrow \infty. \tag{4.5}$$

Then, there exists some $n_\circ(\alpha, \epsilon)$ such that

$$\mu(P_n(x) - x, \epsilon) \geq 1 - \alpha, \qquad \nu(P_n(x) - x, \epsilon) \leq \alpha \tag{4.6}$$

for all $n \geq n_\circ$. Further, each P_n has a finite rank in the linear space X since $\dim P_n(x) = n$. Hence, each P_n such that $n \geq n_\circ$ can be taken as a desired finite rank operator in the definition. □

Remark 4.6. Theorem 4.5 can also proved for sif-basis.

Example 4.7. Let $X = \ell_\infty$, the Banach space of all bounded sequence with sup-norm $\|x\|_\infty = \sup_n |x_n|$. Also, $\|x\|_\circ = \sup_n |x_n/n|$ is another norm on ℓ_∞. Define the function

$$\mu(x,t) = \begin{cases} 1 & \text{if } t > \|x\|_\infty, \\ \dfrac{1}{2} & \text{if } \|x\|_\circ < t \le \|x\|_\infty, \\ 0 & \text{if } t \le \|x\|_\circ; \end{cases} \qquad \nu(x,t) = \begin{cases} 0 & \text{if } t > \|x\|_\infty, \\ \dfrac{1}{2} & \text{if } \|x\|_\infty \le t < \|x\|_\circ, \\ 1 & \text{if } t \le \|x\|_\circ. \end{cases} \tag{4.7}$$

Then (μ, ν) is an intuitionistic fuzzy norm on ℓ_∞. We can find α-norms of intuitionistic fuzzy norm (μ, ν) since it satisfies (1.2) condition. Thus

$$\begin{aligned} \|x\|_\alpha &= \|x\|_\infty \quad \text{for } 1 > \alpha > \frac{1}{2}, \\ \|x\|_\alpha &= \|x\|_\circ \quad \text{for } 1 < \alpha \le \frac{1}{2}. \end{aligned} \tag{4.8}$$

IFNS $(\ell_\infty, \mu, \nu, *, \diamond)$ cannot have a wif and hence a sif-basis since $(\ell_\infty, \|\cdot\|_\alpha) = (\ell_\infty, \|\cdot\|_\infty)$ for $1 > \alpha > 1/2$ and the Banach space $(\ell_\infty, \|\cdot\|_\infty)$ is not separable. However, $(\ell_\infty, \mu, \nu, *, \diamond)$ has sif-AP. Recall that the set D of all partitions $p = (\beta_1, \beta_2, \ldots, \beta_n)$ of natural numbers is a directed set by the relation $p_a \ll p_b$ which means that each $\beta_i \in p_a$ is included in some $\beta'_j \in p_b$. Now, for each $p \in D$

$$\Lambda_p(x) = \sum_{i=1}^{n} x_{h_i} \chi_{\beta_i}, \tag{4.9}$$

where h_i is the distinguished point in β_i and χ_{β_i} is the characteristic function of β_i for $1 \le i \le n$. Then Λ_p is a projection on ℓ_∞ of finite rank. It is well known that the set $(\Lambda_p(x), D)$ converges to x in $(\ell_\infty, \|\cdot\|_\infty)$. Let $K \subseteq \ell_\infty$ be sif-compact. Given $\epsilon > 0$ and $x \in K$, then there exists a partition $p_\circ(\epsilon)$ such that for $p_\circ(\epsilon) \ll p$

$$\left\| \Lambda_p(x) - x \right\|_\infty < \epsilon. \tag{4.10}$$

But

$$\left\| \Lambda_p(x) - x \right\|_\circ < \epsilon, \tag{4.11}$$

for $p_\circ(\epsilon) \ll p$ since $\|x\|_\circ \le \|x\|_\infty$ for every $x \in \ell_\infty$. That is, for $p_\circ(\epsilon) \ll p$,

$$\left\| \Lambda_p(x) - x \right\|_\alpha < \epsilon, \tag{4.12}$$

for all $\alpha \in (0,1)$. Hence for some Λ_p $(p_\circ \ll p)$ meets all requirements for sif-AP in Proposition 4.4.

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