

Research Article

Existence of Positive Solutions to Singular p -Laplacian General Dirichlet Boundary Value Problems with Sign Changing Nonlinearity

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By using the well-known Schauder fixed point theorem and upper and lower solution method, we present some existence criteria for positive solution of an m -point singular p -Laplacian dynamic equation on time scales with the sign changing nonlinearity. These results are new even for the corresponding differential ($\mathbb{T} = \mathbb{R}$) and difference equations ($\mathbb{T} = \mathbb{Z}$), as well as in general time scales setting. As an application, an example is given to illustrate the results.

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1. Introduction

Initiated by Hilger in his Ph.D. thesis [1] in 1988, the theory of time scales has been improved greatly ever since, especially in the unification of the theory of differential equations in the continuous case and the theory of finite difference equations in the discrete case. For the time being, it remains active and attracts many distinguished researchers' attention. The reason is two sided. On the one hand, the calculus on time scales not only can unify differential and difference equations, but also can provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. On the other hand, it is also widely applied to the research of biology, heat transfer, stock market, wound healing and epidemic models [2–6], and so forth. For instance, Hoffacker et al. have used the theory to model how students suffering from the eating disorder bulimia are influenced by their college friends. With the theory on time scales, they can model how the number of sufferers changes during the continuous college term as well as during long breaks [5]. Hence, the dynamic equations on time scales are worth studying theoretically and practically [3, 5, 7].

Here and hereafter, we denote $\varphi_p(u)$ is p -Laplacian operator, that is, $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ and $(\varphi_p)^{-1} = \varphi_q$, where $1/p + 1/q = 1$. We make the blanket assumption that $0, T$ are points in \mathbb{T} , by an interval $(0, T)_{\mathbb{T}}$ we always mean $(0, T) \cap \mathbb{T}$. Other types of interval are defined similarly.

Recent research results indicate that considerable work has been made in the existence problems of solutions of boundary value problems on time scales, for details, see [8–16] and the references therein. In particular, some of them are considered the existence of positive solutions of p -Laplacian boundary value problems on time scales, see [17–22]. The main tools used in these papers are the various fixed point theorems in cones. Very recently, when the nonlinear term f is allowed to change sign, Su et al. [23–25] proved the existence of positive solutions to p -Laplacian dynamic equations *with sign changing nonlinearity on time scales*.

Motivated by references [23–25], we consider the following m -point singular p -Laplacian boundary value problem on time scales of the form

$$\left(\varphi_p\left(u^\Delta(t)\right)\right)^\nabla + q(t)f(t, u(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \quad (1.1)$$

$$u(0) = 0, \quad u(T) = \sum_{i=1}^m \varphi_i\left(u^\Delta(\xi_i)\right), \quad m \in \mathbb{N}, \quad (1.2)$$

where $f(t, u) : (0, T)_{\mathbb{T}} \times (0, \infty) \rightarrow \mathbb{R}$ is continuous and $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, nondecreasing and φ_i may be nonlinear, $0 \leq \xi_1 < \xi_2 < \dots < \xi_m \leq T$. The singularity may occur at $u = 0$, $t = 0$ and $t = T$, and the nonlinearity is allowed to change sign. In particular, the boundary condition (1.2) includes the Dirichlet boundary condition. We obtain some new existence criteria for positive solutions of the boundary value problem (1.1) and (1.2) by using the upper and lower method. Our results are new even for the corresponding differential ($\mathbb{T} = \mathbb{R}$) and difference equations ($\mathbb{T} = \mathbb{Z}$), as well as in general time scales setting. As an application, an example is given to illustrate these results. In particular, our results improve and generalize some known results of Agarwal et al. [26], O'Regan [27] ($p = 2$) and Lü et al. [28] when $\mathbb{T} = \mathbb{R}$; include the results of Lü et al. [29] when $\mathbb{T} = \mathbb{R}$; extend and include the results of Jiang et al. [30] in the case of $\mathbb{T} = \mathbb{Z}$.

For the convenience of statements, now we present some basic definitions and lemmas concerning the calculus on time scales that one needs to read this manuscript, which can be found in [3, 7]. One of other excellent sources on dynamical systems on time scales is from the book in [31].

Definition 1.1 (see [3, 7]). A time scale \mathbb{T} is a nonempty closed subset of \mathbb{R} . It follows that the jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \quad (1.3)$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If \mathbb{T} has a right-scattered minimum m , define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If \mathbb{T} has a left-scattered maximum M , define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.

Definition 1.2 (see [7]). We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-increasing at a point $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ provided the following conditions hold.

- (i) If t_0 is right-scattered, then $f(\sigma(t_0)) > f(t_0)$.
- (ii) If t_0 is right-dense, then there is a neighborhood U of t_0 such that $f(t) > f(t_0)$ for all $t \in U$ with $t > t_0$.

Similarly, we say that f is right-decreasing if above in (i), $f(\sigma(t_0)) < f(t_0)$ and (ii), $f(t) < f(t_0)$.

Definition 1.3 (see [3]). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called predifferentiable with (region of differential) D provided the following conditions hold:

- (i) f is continuous on \mathbb{T} ;
- (ii) $D \subset \mathbb{T}^\kappa$;
- (iii) $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of \mathbb{T} ;
- (iv) f is differentiable at each $t \in D$.

Next, we list some lemmas which will be used in the sequel.

Lemma 1.4 (see [3, 7]). Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$, then one has the following:

- (i) If f is differentiable at t , then f is continuous at t .
- (ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \quad (1.4)$$

- (iii) If f is right-dense, then f is differentiable at t if and only one the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} \quad (1.5)$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}. \quad (1.6)$$

- (iv) If f is differentiable at t , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t) = f(t) + (\sigma(t) - t)f^\Delta(t). \quad (1.7)$$

Lemma 1.5 (see [7]). Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at $t_0 \in \mathbb{T} \setminus \{\max \mathbb{T}\}$. If f assumes its local right-minimum at t_0 , then $f^\Delta(t_0) \geq 0$. If f assumes its local right-maximum at t_0 , then $f^\Delta(t_0) \leq 0$.

Lemma 1.6 ((Mean Value Theorem) [7]). *Let f be a continuous function on $[a, b]$ that is differentiable on $[a, b)$. Then there exist $\xi^*, \tau^* \in [a, b)$ such that*

$$f^\Delta(\tau^*) \leq \frac{f(b) - f(a)}{b - a} \leq f^\Delta(\xi^*). \quad (1.8)$$

Lemma 1.7 (see [3]). *Suppose f and g are pre-differential with D . If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then $|f(s) - f(r)| \leq \{\sup_{t \in U^* \cap D \cap \mathbb{T}} |f^\Delta(t)|\} |s - r|$.*

Now, we can obtain the following lemma which is similar to Lemma 1.7. The proofs are similar to the proofs of Lemma 1.7 by a slight modification and we omit the proofs.

Lemma 1.8. *Suppose $f(t, u)$ and $g(t, u)$ are predifferential with $D \times (0, +\infty)$. If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then $|f(s, u) - f(r, u)| \leq \{\sup_{t \in U^* \cap D \cap \mathbb{T}} |f^\Delta(t, u)|\} |s - r|$, here $D \subset \mathbb{T}^\kappa$.*

Throughout this paper, it is assumed that

- (H1) $f(t, u) : (0, T)_{\mathbb{T}} \times (0, \infty) \rightarrow \mathbb{R}$ is continuous;
- (H2) $q \in C((0, T)_{\mathbb{T}}, (0, \infty))$ and $q \in C_{id}^\nabla[0, T]_{\mathbb{T}}$;
- (H3) $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and nondecreasing, here $i = 1, 2, \dots, m$.

2. Existence Results

Define the Banach space $\mathbb{B} = C[0, T]_{\mathbb{T}}$ with the norm $\|y\| = \sup_{t \in [0, T]_{\mathbb{T}}} |y(t)|$.

To demonstrate existence of positive solutions to problem (1.1) and (1.2), we first approximate the singular problem by means of a sequence of nonsingular problems, and by using the lower and upper solution for nonsingular problem together with Schauders fixed point theorem, and then we establish the existence of solutions to each approximating problem. Our results are new even for the corresponding differential ($\mathbb{T} = \mathbb{R}$) and difference equations ($\mathbb{T} = \mathbb{Z}$), as well as in general time scales setting. If we consider the corresponding differential equation ($\mathbb{T} = \mathbb{R}$) of problem (1.1) and (1.2) in the method mentioned above, we obtain the same existence results to problem (1.1) and (1.2). In the same way, we consider the corresponding difference equation ($\mathbb{T} = \mathbb{R}$) of problem (1.1) and (1.2), we obtain the same existence results to problem (1.1) and (1.2). Here, the two same existence results are obtained in different settings by using the essentially same method. Naturally, it is quite necessary to consider the existence results to problem (1.1) and (1.2) in same setting. In this case, we need to solve the problem with the help of calculus on time scales, because it not only can unify differential and difference equations, but also can provide accurate information of phenomena that manifests themselves partly in continuous time and partly in discrete time. For example, we can consider the problem (1.1) and (1.2) on time scales

$$\mathbb{T} = \{0\} \cup \left\{ \left(\frac{1}{2} \right)^{\mathbb{N}} \right\} \cup \left[\frac{1}{2}, 1 \right] \cup [2, 3]. \quad (2.1)$$

However, if t is taken from (2.1), we cannot study the problem (1.1) and (1.2) only in differential case, neither can we study the problem (1.1) and (1.2) only in difference case.

Now we state and prove our main result.

Theorem 2.1. *Let $n_0 \in \{1, 2, \dots\}$ be fixed. Assume that (H1)–(H3) hold and the following conditions are satisfied.*

- (A1) *For each $n \in \{n_0, n_0 + 1, \dots\} \equiv \mathbb{N}_1$, there is a constant ρ_n such that $\{\rho_n\}$ is a strictly monotone decreasing sequence with $\lim_{n \rightarrow \infty} \rho_n = 0$, and $q(t)f(t, \rho_n) \geq 0$ for $t \in [1/2^{n+1}, T]_{\mathbb{T}}$;*
- (A2) *There exists a function $\alpha \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\alpha^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ with $\alpha(0) = 0$, $\alpha(T) - \sum_{i=1}^m \psi_i(\alpha^\Delta(\xi_i)) \leq 0$, $\alpha > 0$ on $(0, T]_{\mathbb{T}}$ and $-(\varphi_p(\alpha^\Delta))^\nabla \leq q(t)f(t, \alpha)$ for $t \in (0, T)_{\mathbb{T}}$;*
- (A3) *There exists a function $\beta \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\beta^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ with $\beta \geq \alpha$, $\beta \geq \rho_{n_0}$ for $t \in [0, T]_{\mathbb{T}}$ and $\beta(T) - \sum_{i=1}^m \psi_i(\beta^\Delta(\xi_i)) > 0$, with $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(t, \beta)$ for $t \in (0, T)_{\mathbb{T}}$, and $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(1/2^{n_0+1}, \beta)$ for $t \in (0, 1/2^{n_0+1})_{\mathbb{T}}$.*

Then the boundary value problem (1.1) and (1.2) has a positive solution $u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(u^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ with $u \geq \alpha$ for $t \in [0, T]_{\mathbb{T}}$.

Proof. It follows from the condition (A1) that $1/2^{n+1} \in (0, T]_{\mathbb{T}}$ for each $n \in \mathbb{N}_1$. That is, $(0, T]_{\mathbb{T}}$ is not empty. Without loss of generality, fix $n \in \mathbb{N}_1$. If $\xi_1 > 0$, then we can suppose that $\min_{t \in [\xi_1, T]_{\mathbb{T}}} \alpha(t) \geq \rho_n$, let $t_n \in (0, \xi_1)_{\mathbb{T}}$ be such that

$$\alpha(t_n) = \rho_n, \quad \alpha \leq \rho_n \quad \text{for } t \in [0, t_n]_{\mathbb{T}}. \tag{2.2}$$

If $\xi_1 = 0$, then we can suppose that $\min_{t \in [\xi_2, T]_{\mathbb{T}}} \alpha(t) \geq \rho_n$, let $t_n \in (0, \xi_2)_{\mathbb{T}}$ be such that (2.2) holds. Define

$$\alpha_n(t) = \begin{cases} \rho_n & \text{if } t \in [0, t_n]_{\mathbb{T}}, \\ \alpha & \text{if } t \in [t_n, T]_{\mathbb{T}}, \end{cases} \quad \text{here } \alpha(t_n) = \rho_n. \tag{2.3}$$

We denote $e_n = [1/2^{n+1}, T]_{\mathbb{T}}$, $\omega_n(t) = \max\{1/2^{n+1}, t\}$ for $t \in [0, T]_{\mathbb{T}}$ and

$$f_n(t, x) = \max\{f(t, x), f(\omega_n(t), x)\}. \tag{2.4}$$

Define a sequence $h_{n_0}(t, x) = f_{n_0}(t, x)$ and

$$h_n(t, x) = \min\{f_{n_0}(t, x), \dots, f_n(t, x)\}, \quad n = n_0 + 1, n_0 + 2, \dots \tag{2.5}$$

Then

$$\begin{aligned} f(t, x) &\leq \dots \leq h_{n+1}(t, x) \leq h_n(t, x) \leq \dots \leq h_{n_0}(t, x) \quad \text{for } (t, x) \in (0, T)_{\mathbb{T}} \times (0, \infty), \\ h_n(t, x) &= f(t, x) \quad \text{for } (t, x) \in e_n \times (0, \infty). \end{aligned} \tag{2.6}$$

Consider the p -Laplacian boundary value problem

$$\left(\varphi_p\left(u^\Delta(t)\right)\right)^\nabla + q(t)h_{n_0}^*(t, u(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \quad (2.7)$$

$$u(0) = \rho_{n_0}, \quad u(T) - \sum_{i=1}^m \psi_i^*\left(u^\Delta(\xi_i)\right) = \rho_{n_0}, \quad (2.8)$$

where

$$h_{n_0}^*(t, u(t)) = \begin{cases} h_{n_0}(t, \alpha_{n_0}(t)) + r(\alpha_{n_0}(t) - u(t)), & u(t) \leq \alpha_{n_0}(t), \\ h_{n_0}(t, u(t)), & \alpha_{n_0}(t) \leq u(t) \leq \beta, \\ h_{n_0}(t, \beta(t)) + r(\beta(t) - u(t)), & u(t) \geq \beta, \end{cases} \quad (2.9)$$

$$\psi_i^*(z_i) = \begin{cases} \psi_i(\alpha^\Delta(\xi_i)), & z_i \leq \alpha_{n_0}^\Delta(\xi_i) = \alpha^\Delta(\xi_i), \\ \psi_i(z_i), & \alpha_{n_0}^\Delta(\xi_i) \leq z_i \leq \beta^\Delta(\xi_i), \\ \psi_i(\beta^\Delta(\xi_i)), & z_i \geq \beta^\Delta(\xi_i), \end{cases} \quad i = 1, \dots, m, \quad (2.10)$$

and $r : \mathbb{R} \rightarrow [-1, 1]$ is the radial retraction function defined by

$$r(u) = \begin{cases} u, & |u| \leq 1, \\ \frac{u}{|u|}, & |u| > 1. \end{cases} \quad (2.11)$$

Suppose

$$C_0[0, T]_{\mathbb{T}} = \{u \in C[0, T]_{\mathbb{T}} : u(0) = 0\} \quad C_{\rho_{n_0}}^\Delta[0, T]_{\mathbb{T}} = \left\{u \in C^\Delta[0, T]_{\mathbb{T}} : u(0) = \rho_{n_0}\right\}. \quad (2.12)$$

We define the mappings $L_p, F : C_{\rho_{n_0}}^\Delta[0, T]_{\mathbb{T}} \rightarrow C_0[0, T]_{\mathbb{T}} \times \mathbb{R}$ be such that

$$L_p u(t) = \left(\varphi_p\left(u^\Delta(t)\right) - \varphi_p\left(u^\Delta(0)\right), u(T)\right), \quad (2.13)$$

$$F u(t) = \left(-\int_0^t q(x)h_{n_0}^*(x, u(x))\nabla x, \sum_{i=1}^m \psi_i^*\left(u^\Delta(\xi_i)\right) + \rho_{n_0}\right). \quad (2.14)$$

By using the Arzela-Ascoli theorem on time scales [2], we can show that F is continuous and compact. By using the (2.7), (2.8), (2.13) and (2.14), we obtain

$$\left(\varphi_p\left(u^\Delta(t)\right) - \varphi_p\left(u^\Delta(0)\right), u(T)\right) = \left(-\int_0^t q(x)h_{n_0}^*(x, u(x))\nabla x, \sum_{i=1}^m \psi_i^*\left(u^\Delta(\xi_i)\right) + \rho_{n_0}\right), \quad (2.15)$$

that is

$$L_p u(t) = Fu(t). \tag{2.16}$$

If

$$L_p v = (u, \gamma) \quad \text{for } u \in C_0[0, T]_{\mathbb{T}}, \quad \gamma = \rho_{n_0} + \int_0^T \varphi_q(u(x) - u(T)) \Delta x, \tag{2.17}$$

then $v(t) = \rho_{n_0} + \int_0^t \varphi_q(u(x) - u(T)) \Delta x$, hence L_p^{-1} exists and is continuous. So

$$u(t) = L_p^{-1} Fu(t). \tag{2.18}$$

It is clear that solving the boundary value problem (2.7) and (2.8) is equivalent to finding a fixed point of $u = L_p^{-1} Fu \equiv Nu$, where $N = L_p^{-1} F : C_{\rho_{n_0}}^\Delta[0, T]_{\mathbb{T}} \rightarrow C_{\rho_{n_0}}^\Delta[0, T]_{\mathbb{T}}$ is compact. Schauder's fixed point theorem guarantees that the boundary value problem (2.7) and (2.8) has a solution $u_{n_0}(t) \in C^\Delta[0, T]_{\mathbb{T}}$ with $\varphi_p(u_{n_0}^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$.

We first show that

$$\alpha_{n_0}(t) \leq u_{n_0}(t) \quad \text{for } t \in [0, T]_{\mathbb{T}}. \tag{2.19}$$

If (2.19) is not true, the function $u_{n_0}(t) - \alpha_{n_0}(t)$ has a negative minimum for some $\tau \in (0, T]_{\mathbb{T}}$. We consider two cases, namely, $\tau \in (0, T)_{\mathbb{T}}$ and $\tau = T$.

Case 1. Assume that $\tau \in (0, T)_{\mathbb{T}}$, then we claim

$$\left(\varphi_p(u_{n_0}^\Delta)\right)^\nabla(\tau) \geq \left(\varphi_p(\alpha_{n_0}^\Delta)\right)^\nabla(\tau). \tag{2.20}$$

Since $u_{n_0}(t) - \alpha_{n_0}(t)$ has a negative minimum for some $\tau \in (0, T)_{\mathbb{T}}$, in view of Definition 1.2, Lemmas 1.4 and 1.5, we have $u_{n_0}^\Delta(\tau) - \alpha_{n_0}^\Delta(\tau) \geq 0$ and there exists a δ with $\tau - \delta \in [0, \tau)_{\mathbb{T}}$ such that $u_{n_0}^\Delta(t) - \alpha_{n_0}^\Delta(t) \leq 0$ for $t \in [\tau - \delta, \tau)_{\mathbb{T}}$. Thus

$$\varphi_p(u_{n_0}^\Delta(t)) - \varphi_p(\alpha_{n_0}^\Delta(t)) \leq \varphi_p(u_{n_0}^\Delta(\tau)) - \varphi_p(\alpha_{n_0}^\Delta(\tau)) \quad \text{for } t \in [\tau - \delta, \tau)_{\mathbb{T}}, \tag{2.21}$$

which leads to

$$\frac{\varphi_p(u_{n_0}^\Delta(t)) - \varphi_p(u_{n_0}^\Delta(\tau))}{t - \tau} \geq \frac{\varphi_p(\alpha_{n_0}^\Delta(t)) - \varphi_p(\alpha_{n_0}^\Delta(\tau))}{t - \tau} \quad \text{for } t \in [\tau - \delta, \tau)_{\mathbb{T}}. \tag{2.22}$$

If τ is left-dense, in view of Lemma 1.4

$$\begin{aligned} \left(\varphi_p\left(u_{n_0}^\Delta\right)\right)^\nabla(\tau) &= \lim_{t \in [\tau-\delta, \tau) \rightarrow \tau} \frac{\varphi_p\left(u_{n_0}^\Delta(t)\right) - \varphi_p\left(u_{n_0}^\Delta(\tau)\right)}{t - \tau} \\ &\geq \lim_{t \in [\tau-\delta, \tau) \rightarrow \tau} \frac{\varphi_p\left(\alpha_{n_0}^\Delta(t)\right) - \varphi_p\left(\alpha_{n_0}^\Delta(\tau)\right)}{t - \tau} \\ &= \left(\varphi_p\left(\alpha_{n_0}^\Delta\right)\right)^\nabla(\tau). \end{aligned} \quad (2.23)$$

If τ is left-scattered, by Lemma 1.4 and (2.22) we obtain

$$\begin{aligned} \left(\varphi_p\left(u_{n_0}^\Delta\right)\right)^\nabla(\tau) &= \frac{\varphi_p\left(u_{n_0}^\Delta(\tau)\right) - \varphi_p\left(u_{n_0}^\Delta(\rho(\tau))\right)}{\tau - \rho(\tau)} \\ &\geq \frac{\varphi_p\left(\alpha_{n_0}^\Delta(\tau)\right) - \varphi_p\left(\alpha_{n_0}^\Delta(\rho(\tau))\right)}{\tau - \rho(\tau)} \\ &= \left(\varphi_p\left(\alpha_{n_0}^\Delta\right)\right)^\nabla(\tau). \end{aligned} \quad (2.24)$$

Hence, (2.20) is established.

However, by (2.3), (2.9) and $u_{n_0}(\tau) < \alpha_{n_0}(\tau)$, we obtain

$$\begin{aligned} &\left(\varphi_p\left(u_{n_0}^\Delta(\tau)\right)\right)^\nabla - \left(\varphi_p\left(\alpha_{n_0}^\Delta(\tau)\right)\right)^\nabla \\ &= -\left[q(\tau)h_{n_0}(\tau, \alpha_{n_0}(\tau)) + q(\tau)r(\alpha_{n_0}(\tau) - u_{n_0}(\tau)) + \left(\varphi_p\left(\alpha_{n_0}^\Delta(\tau)\right)\right)^\nabla\right] \\ &= \begin{cases} -\left[q(\tau)h_{n_0}(\tau, \alpha(\tau)) + q(\tau)r(\alpha(\tau) - u_{n_0}(\tau)) + \left(\varphi_p\left(\alpha^\Delta(\tau)\right)\right)^\nabla\right], & \tau \in [t_{n_0}, T]_{\mathbb{T}}, \\ -\left[q(\tau)h_{n_0}(\tau, \rho_{n_0}) + q(\tau)r(\rho_{n_0} - u_{n_0}(\tau))\right], & \tau \in (0, t_{n_0})_{\mathbb{T}}. \end{cases} \end{aligned} \quad (2.25)$$

Assume that $\tau \in [1/2^{n_0+1}, T]_{\mathbb{T}}$, then $g_{n_0}(\tau, x) = f(\tau, x)$ for $x \in (0, \infty)$, by (A1) and (A2), we have

$$\begin{aligned} &\left(\varphi_p\left(u_{n_0}^\Delta(\tau)\right)\right)^\nabla - \left(\varphi_p\left(\alpha_{n_0}^\Delta(\tau)\right)\right)^\nabla \\ &= \begin{cases} -\left[q(\tau)f(\tau, \alpha(\tau)) + q(\tau)r(\alpha(\tau) - u_{n_0}(\tau)) + \left(\varphi_p\left(\alpha^\Delta(\tau)\right)\right)^\nabla\right], & \tau \in [t_{n_0}, T]_{\mathbb{T}}, \\ -\left[q(\tau)f(\tau, \rho_{n_0}) + q(\tau)r(\rho_{n_0} - u_{n_0}(\tau))\right], & \tau \in (0, t_{n_0})_{\mathbb{T}}, \end{cases} \\ &< 0, \end{aligned} \quad (2.26)$$

which implies a contraction.

Assume that $\tau \in (0, 1/2^{n_0+1})_{\mathbb{T}}$, then $h_{n_0}(\tau, x) = \max\{f(1/2^{n_0+1}, x), f(\tau, x)\}$, in view of (A1), (A2) and $q(\tau) > 0$, we have

$$\begin{aligned} & \left(\varphi_p(u_{n_0}^\Delta(\tau))\right)^\nabla - \left(\varphi_p(\alpha_{n_0}^\Delta(\tau))\right)^\nabla \\ & \leq \begin{cases} -\left[q(\tau)f(\tau, \alpha(\tau)) + q(\tau)r(\alpha(\tau) - u_{n_0}(\tau)) + (\varphi_p(\alpha^\Delta(\tau)))^\nabla\right], & \tau \in [t_{n_0}, T]_{\mathbb{T}}, \\ -\left[q(\tau)f\left(\frac{1}{2^{n_0+1}}, \rho_{n_0}\right) + q(\tau)r(\rho_{n_0} - u_{n_0}(\tau))\right], & \tau \in (0, t_{n_0})_{\mathbb{T}}, \end{cases} \quad (2.27) \\ & < 0. \end{aligned}$$

which implies a contraction.

Case 2. Assume that $\tau = T$. That is, $\alpha_{n_0}(T) - u_{n_0}(T) > 0$, by (2.3), (2.8) and (2.10) together with $\alpha(T) \leq \sum_{i=1}^m \varphi_i(\alpha^\Delta(\xi_i))$, we have the following three subcases.

(a) If $u_{n_0}^\Delta(\xi_i) \leq \alpha^\Delta(\xi_i)$ for $i = 1, 2, \dots, m$, then

$$\begin{aligned} & 0 < \alpha_{n_0}(T) - u_{n_0}(T) \\ & = \alpha(T) - \sum_{i=1}^m \varphi_i^*(u_{n_0}^\Delta(\xi_i)) - \rho_{n_0} \\ & < \sum_{i=1}^m \varphi_i(\alpha^\Delta(\xi_i)) - \sum_{i=1}^m \varphi_i^*(u_{n_0}^\Delta(\xi_i)) \\ & = \sum_{i=1}^m \varphi_i(\alpha^\Delta(\xi_i)) - \sum_{i=1}^m \varphi_i(\alpha^\Delta(\xi_i)) = 0, \end{aligned} \quad (2.28)$$

this is a contradiction.

(b) If $\alpha^\Delta(\xi_i) < u_{n_0}^\Delta(\xi_i)$ for $i = 1, 2, \dots, m$. Assume that $u_{n_0}^\Delta(\xi_i) \leq \beta^\Delta(\xi_i)$ for $i = 1, 2, \dots, m$, then

$$\sum_{i=1}^m \varphi_i^*(u_{n_0}^\Delta(\xi_i)) = \sum_{i=1}^m \varphi_i(u_{n_0}^\Delta(\xi_i)). \quad (2.29)$$

Assume that $\beta^\Delta(\xi_i) < u_{n_0}^\Delta(\xi_i)$ for $i = 1, 2, \dots, m$, then

$$\sum_{i=1}^m \varphi_i^*(u_{n_0}^\Delta(\xi_i)) = \sum_{i=1}^m \varphi_i(\beta^\Delta(\xi_i)). \quad (2.30)$$

Assume that there exist sequences $\{i_{l_1}\} = \{1, 2, \dots, l_1\}$ and $\{i_{k_1}\} = \{1, 2, \dots, k_1\}$ such that $\beta^\Delta(\xi_{i_1}) < u_{n_0}^\Delta(\xi_i)$ and $u_{n_0}^\Delta(\xi_{i_{k_1}}) \leq \beta^\Delta(\xi_{i_{k_1}})$, here $l_1 + k_1 = m$, then

$$\sum_{i=1}^m \varphi_i^*(u_{n_0}^\Delta(\xi_i)) = \sum_{i_{k_1}=1}^{k_1} \varphi_{i_{k_1}}(u_{n_0}^\Delta(\xi_{i_{k_1}})) + \sum_{i_1=1}^{l_1} \varphi_{i_1}(\beta^\Delta(\xi_{i_1})). \quad (2.31)$$

Hence, by (2.29), (2.30) and (2.31) together with the monotonicity of ψ_i , we have

$$\begin{aligned} 0 &< \alpha_{n_0}(T) - u_{n_0}(T) \\ &= \alpha(T) - \sum_{i=1}^m \psi_i^*(u_{n_0}^\Delta(\xi_i)) - \rho_{n_0} \\ &\leq \sum_{i=1}^m \psi_i(\alpha^\Delta(\xi_i)) - \sum_{i=1}^m \psi_i^*(u_{n_0}^\Delta(\xi_i)) - \rho_{n_0} < 0, \end{aligned} \quad (2.32)$$

this is a contradiction.

(c) If there exist sequences $\{i_l\} = \{1, 2, \dots, l\}$ and $\{i_k\} = \{1, 2, \dots, k\}$ such that $\alpha^\Delta(\xi_{i_l}) < u_{n_0}^\Delta(\xi_{i_l})$ and $u_{n_0}^\Delta(\xi_{i_k}) \leq \alpha^\Delta(\xi_{i_k})$, here $l+k = m$. Essentially the same reasoning as before we have $0 < \alpha_{n_0}(T) - u_{n_0}(T) = \alpha(T) - \sum_{i=1}^m \psi_i^*(u_{n_0}^\Delta(\xi_i)) - \rho_{n_0} < 0$, this is a contradiction.

Thus, Cases 1–2 imply (2.19) is established. In particular, since $\alpha(t) \leq \alpha_{n_0}(t)$ for $t \in [0, T]_{\mathbb{T}}$, we obtain $\alpha(t) \leq \alpha_{n_0}(t) \leq u_{n_0}(t)$ for $t \in [0, T]_{\mathbb{T}}$.

Essentially the same reasoning as the proof of inequality (2.19) we obtain $u_{n_0}(t) \leq \beta$ for $t \in [0, T]_{\mathbb{T}}$.

Hence

$$\alpha(t) \leq \alpha_{n_0}(t) \leq u_{n_0}(t) \leq \beta(t) \quad \text{for } t \in [0, T]_{\mathbb{T}}. \quad (2.33)$$

Now, we discuss the boundary value problem

$$\begin{aligned} \left(\varphi_p \left(u^\Delta(t) \right) \right)^\nabla + q(t) h_{n_0+1}^*(t, u(t)) &= 0, \quad t \in (0, T)_{\mathbb{T}}, \\ u(0) = \rho_{n_0+1}, \quad u(T) - \sum_{i=1}^m \psi_i^*(u^\Delta(\xi_i)) &= \rho_{n_0+1}, \end{aligned} \quad (2.34)$$

where

$$\begin{aligned} h_{n_0+1}^*(t, u(t)) &= \begin{cases} h_{n_0+1}(t, \alpha_{n_0+1}(t)) + r(\alpha_{n_0+1}(t) - u(t)), & u(t) \leq \alpha_{n_0+1}(t), \\ h_{n_0+1}(t, u(t)), & \alpha_{n_0+1}(t) \leq u(t) \leq u_{n_0}(t), \\ h_{n_0+1}(t, u_{n_0}(t)) + r(u_{n_0}(t) - u(t)), & u(t) \geq u_{n_0}(t), \end{cases} \\ \psi_i^*(z_i) &= \begin{cases} \psi_i(\alpha^\Delta(\xi_i)), & z_i \leq \alpha_{n_0+1}^\Delta(\xi_i) = \alpha^\Delta(\xi_i), \\ \psi_i(z_i), & \alpha_{n_0+1}^\Delta(\xi_i) \leq z_i \leq u_{n_0}^\Delta(\xi_i), \quad i = 1, \dots, m. \\ \psi_i(u_{n_0}^\Delta(\xi_i)), & z_i \geq u_{n_0}^\Delta(\xi_i), \end{cases} \end{aligned} \quad (2.35)$$

Schauder's fixed point theorem guarantees that the boundary value problem (2.34) has a solution $u_{n_0+1}(t) \in C^\Delta[0, T]_{\mathbb{T}}$ with $\varphi_p(u_{n_0+1}^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$.

Essentially the same reasoning as the proof of inequality (2.33), we have

$$\alpha(t) \leq \alpha_{n_0+1}(t) \leq u_{n_0+1}(t) \leq u_{n_0}(t) \quad \text{for } t \in [0, T]_{\mathbb{T}}. \quad (2.36)$$

If there exists $u_k(t)$ for some $k \in \{n_0 + 1, n_0 + 2, \dots\}$ satisfying $\alpha_k(t) \leq u_k(t) \leq u_{k-1}(t)$ for $t \in [0, T]_{\mathbb{T}}$. Then we investigate the boundary value problem

$$\begin{aligned} (\varphi_p(u^\Delta(t)))^\nabla + q(t)h_{k+1}^*(t, u(t)) &= 0, \quad t \in (0, T)_{\mathbb{T}}, \\ u(0) = \rho_{k+1}, \quad u(T) - \sum_{i=1}^m \varphi_i^*(u^\Delta(\xi_i)) &= \rho_{k+1}, \end{aligned} \quad (2.37)$$

where

$$\begin{aligned} h_{k+1}^*(t, u(t)) &= \begin{cases} h_{k+1}(t, \alpha_{k+1}(t)) + r(\alpha_{k+1}(t) - u(t)), & u(t) \leq \alpha_{k+1}(t), \\ h_{k+1}(t, u(t)), & \alpha_{k+1}(t) \leq u(t) \leq u_k(t), \\ h_{k+1}(t, u_k(t)) + r(u_k(t) - u(t)), & u(t) \geq u_k(t), \end{cases} \\ \varphi_i^*(z_i) &= \begin{cases} \varphi_i(\alpha^\Delta(\xi_i)), & z_i \leq \alpha_{k+1}^\Delta(\xi_i) = \alpha^\Delta(\xi_i), \\ \varphi_i(z_i), & \alpha_{k+1}^\Delta(\xi_i) \leq z_i \leq u_k^\Delta(\xi_i), \quad i = 1, \dots, m. \\ \varphi_i^\Delta(u_k(\xi_i)), & z_i \geq u_k(\xi_i), \end{cases} \end{aligned} \quad (2.38)$$

It follows from Schauder's fixed point theorem that the boundary value problem (2.37) has a solution $u_{k+1}(t) \in C^\Delta[0, T]_{\mathbb{T}}$ with $\varphi_p(u_{k+1}^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$.

By using the similar arguments as above, we have

$$\alpha(t) \leq \alpha_{k+1}(t) \leq u_{k+1}(t) \leq u_k(t) \quad \text{for } t \in [0, T]_{\mathbb{T}}. \quad (2.39)$$

Hence, for each $n \in \{n_0, n_0 + 1, \dots\}$, the mathematical induction implies that

$$\alpha(t) \leq \alpha_n(t) \leq u_n(t) \leq u_{n-1}(t) \leq \dots \leq u_{n_0}(t) \leq \beta(t) \quad \text{for } t \in [0, T]_{\mathbb{T}}. \quad (2.40)$$

Denote

$$R_{n_0} = \sup \left\{ |f(t, y)| : t \in \left[\frac{1}{2^{n_0+1}}, T \right]_{\mathbb{T}}, \alpha(t) \leq y \leq u_{n_0}(t) \right\}. \quad (2.41)$$

It follows from Lemma 1.6 that there exist $\tau_1^*, \tau_2^* \in [1/2^{n_0+1}, T]_{\mathbb{T}}$ satisfy

$$u_n^\Delta(\tau_1^*) \leq \frac{u_n(T) - u_n(1/2^{n_0+1})}{T - 1/2^{n_0+1}} \leq u_n^\Delta(\tau_2^*). \quad (2.42)$$

From (2.42), we have

$$\begin{aligned} \varphi_p(u_n^\Delta(t)) &= \varphi_p(u_n^\Delta(\tau_1^*)) - \int_{\tau_1^*}^t q(s)f(s, u(s))\nabla s \\ &\leq \varphi_p\left(\frac{u_n(T) - u_n(1/2^{n_0+1})}{T - 1/2^{n_0+1}}\right) + R_{n_0} \int_{1/2^{n_0+1}}^T q(s)\nabla s \quad \text{for } t \in \left[\frac{1}{2^{n_0+1}}, T\right]_{\mathbb{T}}, \end{aligned} \quad (2.43)$$

$$\begin{aligned} \varphi_p(u_n^\Delta(t)) &= \varphi_p(u_n^\Delta(\tau_2^*)) - \int_{\tau_2^*}^t q(s)f(s, u(s))\nabla s \\ &\geq \varphi_p\left(\frac{u_n(T) - u_n(1/2^{n_0+1})}{T - 1/2^{n_0+1}}\right) - R_{n_0} \int_{1/2^{n_0+1}}^T q(s)\nabla s \quad \text{for } t \in \left[\frac{1}{2^{n_0+1}}, T\right]_{\mathbb{T}}. \end{aligned}$$

So there exists a positive number K_0 such that $|u_n^\Delta(t)| \leq K_0$. By Lemma 1.8, we have

$$\{u_n(t)\}_{n=n_0+1}^\infty \text{ is a bounded, equicontinuous family on } t \in \left[\frac{1}{2^{n_0+1}}, T\right]_{\mathbb{T}}. \quad (2.44)$$

The Arzela-Ascoli theorem on time scales [2] guarantees the existence of a subsequence \mathbb{N}_{n_0} of integers and a function $z_{n_0}(t) \in C[1/2^{n_0+1}, T]_{\mathbb{T}}$ with $u_n(t)$ converging uniformly to $z_{n_0}(t)$ on $[1/2^{n_0+1}, T]_{\mathbb{T}}$ as $n \rightarrow \infty$ through \mathbb{N}_{n_0} . Similarly

$$\{u_n(t)\}_{n=n_0+1}^\infty \text{ is a bounded, equicontinuous family on } t \in \left[\frac{1}{2^{n_0+2}}, T\right]_{\mathbb{T}}. \quad (2.45)$$

Thus there is a subsequence \mathbb{N}_{n_0+1} of \mathbb{N}_{n_0} and a function $z_{n_0+1}(t) \in C[1/2^{n_0+2}, T]_{\mathbb{T}}$ with $u_n(t)$ converging uniformly to $z_{n_0+1}(t)$ on $[1/2^{n_0+2}, T]_{\mathbb{T}}$ as $n \rightarrow \infty$ through \mathbb{N}_{n_0+1} . Since $\mathbb{N}_{n_0+1} \subseteq \mathbb{N}_{n_0}$, we have $z_{n_0+1}(t) = z_{n_0}(t)$ on $[1/2^{n_0+1}, T]_{\mathbb{T}}$. Proceed inductively to obtain subsequence of integers $\mathbb{N}_{n_0} \supseteq \mathbb{N}_{n_0+1} \supseteq \dots \supseteq \mathbb{N}_n \supseteq \dots$ and functions $z(t) \in C[1/2^{n+1}, T]_{\mathbb{T}}$ with $u_n(t)$ converging uniformly to $z_n(t)$ on $[1/2^{n+1}, T]_{\mathbb{T}}$ as $n \rightarrow \infty$ through \mathbb{N}_n and $z_n(t) = z_{n-1}(t)$ on $[1/2^n, T]_{\mathbb{T}}$.

Now, we define a function $u : [0, T] \rightarrow [0, \infty)$ with $u(t) = z_n(t)$ on $[1/2^{n+1}, T]_{\mathbb{T}}$ and $u(0) = 0$. Notice, $u(t)$ is well defined and $\alpha(t) \leq u(t) \leq u_{n_0}(t) \leq \beta$ for $t \in (0, T)_{\mathbb{T}}$. Next fix $t \in (0, T)_{\mathbb{T}}$ and let $l \in \{n_0, n_0 + 1, \dots\}$ be such that $t \in (1/2^{l+1}, T)_{\mathbb{T}}$, let $\mathbb{N}_l^* = \{n \in \mathbb{N}_l : n \geq l\}$, we have

$$\varphi_i^*(u_n(\xi_i)) = \varphi_i(u_n(\xi_i)), \quad h_n^*(t, u_n(t)) = h_n(t, u_n(t)) = f(t, u_n(t)) \quad \text{for } n \in \mathbb{N}_l^*. \quad (2.46)$$

Hence, for $n \in \mathbb{N}_l^*$, we have u_n which is the positive solution of the following boundary value problem

$$\begin{aligned} (\varphi_p(u_n^\Delta(t)))^\nabla + q(t)f(t, u_n(t)) &= 0, \quad t \in \left(\frac{1}{2^n}, T\right)_{\mathbb{T}}, \\ u_n(0) = \rho_n, \quad u_n(T) - \sum_{i=1}^m \psi_i(u_n^\Delta(\xi_i)) &= \rho_n. \end{aligned} \tag{2.47}$$

Let $n \rightarrow \infty$ through \mathbb{N}_l^* , we have $u(t)$ that satisfies

$$\begin{aligned} (\varphi_p(u^\Delta(t)))^\nabla + q(t)f(t, u(t)) &= 0, \quad t \in (0, T)_{\mathbb{T}}, \\ u(0) = 0, \quad u(T) - \sum_{i=1}^m \psi_i(u^\Delta(\xi_i)) &= 0. \end{aligned} \tag{2.48}$$

It remains to show that $u(t)$ is continuous at 0. Now by $\lim_{n \rightarrow \infty} u_n(0) = 0$, there exists $n_1 \in \{n_0, n_0 + 1, \dots\}$ with $u_{n_1}(0) < \varepsilon/2$. Since $u_{n_1}(t) \in C[0, T]_{\mathbb{T}}$ there exists $\delta_{n_1} \in (0, T)_{\mathbb{T}}$ with $u_{n_1}(t) < \varepsilon/2$ for $t \in [0, \delta_{n_1})_{\mathbb{T}}$. By the monotonicity of $\{u_n(t)\}_{n \in \mathbb{N}_0}$ for each $t \in [0, T]_{\mathbb{T}}$, we have $\alpha(t) \leq u_n(t) \leq u_{n_1}(t) < \varepsilon/2$ for $t \in [0, \delta_{n_1})_{\mathbb{T}}$ and $n \geq n_1$, which means $\alpha(t) \leq u(t) < \varepsilon/2$ for $t \in [0, \delta_{n_1})_{\mathbb{T}}$. So $u(t)$ is continuous at 0. \square

If we replace $t \in [1/2^{n+1}, T]_{\mathbb{T}}$ with $t \in [1/2^{n+1}, T - 1/2^{n+1}]_{\mathbb{T}}$, the singularity occurs at $u = 0$, $t = 0$ and $t = T$.

If we replace $t \in [1/2^{n+1}, T]_{\mathbb{T}}$ with $t \in [0, T - 1/2^{n+1}]_{\mathbb{T}}$, the singularity occurs at $u = 0$ and $t = T$.

If we replace $t \in [1/2^{n+1}, T]_{\mathbb{T}}$ with $t \in [0, T]_{\mathbb{T}}$, the singularity occurs at $u = 0$.

So it is easily obtain the analogue of Theorem 2.1 in this section. See the following remark.

Remark 2.2. If (A3) is appropriately adjusted, we can replace $t \in [1/2^{n+1}, T]_{\mathbb{T}}$ in (A1) by

$$t \in \left[\frac{1}{2^{n+1}}, T - \frac{1}{2^{n+1}}\right]_{\mathbb{T}}, \tag{2.49}$$

$$t \in \left[0, T - \frac{1}{2^{n+1}}\right]_{\mathbb{T}}, \tag{2.50}$$

or

$$t \in [0, T]_{\mathbb{T}}. \tag{2.51}$$

For example, if (2.49) occurs, (A3) is replaced by

(A3') There exists a function $\beta \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\beta^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ such that $\beta \geq \alpha$ and $\beta \geq \rho_{n_0}$ for $t \in [0, T]_{\mathbb{T}}$, $\beta(T) - \sum_{i=1}^m \psi_i(\beta^\Delta(\xi_i)) > 0$, $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(t, \beta)$ for $t \in (0, T)_{\mathbb{T}}$, $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(1/2^{n_0+1}, \beta)$ for $t \in (0, 1/2^{n_0+1})_{\mathbb{T}}$ and $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(T - 1/2^{n_0+1}, \beta)$ for $t \in (T - 1/2^{n_0+1}, T)_{\mathbb{T}}$.

Assume that (H1)–(H3), (A1) and (A2) hold, and in addition suppose the following conditions are satisfied:

$$(A4) \quad -(\varphi_p(\alpha^\Delta))^\nabla < q(t)f(t, u) \text{ for}$$

$$(t, u) \in (0, T]_{\mathbb{T}} \times \left\{ u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}} : 0 < u \leq \alpha \right\}; \quad (2.52)$$

(A5) There exists a function $\beta \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\beta^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ such that $\beta \geq \rho_{n_0}$ for $t \in [0, T]_{\mathbb{T}}$, $\beta(T) - \sum_{i=1}^m \varphi_i(\beta^\Delta(\xi_i)) > 0$, $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(t, \beta)$ for $t \in (0, T)_{\mathbb{T}}$ and $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(1/2^{n_0+1}, \beta)$ for $t \in (0, 1/2^{n_0+1})_{\mathbb{T}}$;

$$(A6) \quad \beta(T) \geq \alpha(T).$$

Then the result in Theorem 2.1 is also true. This follows immediately from Theorem 2.1 if we show (A3) holds. That is to say, if we show $\beta \geq \alpha$ for $t \in [0, T]_{\mathbb{T}}$, then the result holds. Assume it is not true, in view of (A6) we obtain $\beta - \alpha$ has a negative minimum for some $\tau_6 \in (0, T)_{\mathbb{T}}$, so $(\beta - \alpha)^\Delta(\tau_6) \geq 0$ and essentially the same reasoning as the proof of inequality (2.20), we have $(\varphi_p(\alpha^\Delta))^\nabla(\tau_6) \leq (\varphi_p(\beta^\Delta))^\nabla(\tau_6)$. However, by (A4), (A5) and $\alpha(\tau_6) > \beta(\tau_6) > 0$, we obtain $-(\varphi_p(\alpha^\Delta))^\nabla(\tau_6) < q(\tau_6)f(\tau_6, \beta(\tau_6))$. Hence $(\varphi_p(\alpha^\Delta))^\nabla(\tau_6) - (\varphi_p(\beta^\Delta))^\nabla(\tau_6) \geq (\varphi_p(\alpha^\Delta))^\nabla(\tau_6) + q(\tau_6)f(\tau_6, \beta(\tau_6)) > 0$, which implies a contradiction.

Corollary 2.3. *Let $n_0 \in \{1, 2, \dots\}$ be fixed, suppose (H1)–(H3), (A1), (A2) and (A4)–(A6) hold, then the boundary value problem (1.1) and (1.2) has a solution $u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(u^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ with $u \geq \alpha$ for $t \in [0, T]_{\mathbb{T}}$.*

3. Construction of α and β

In this section, we consider how to construct a lower solution α and an upper solution β in certain circumstances. In this section, we assume that

$$\sum_{i=1}^m \varphi_i(x_i) \geq 0 \quad \text{for } x_i \in \mathbb{R}. \quad (3.1)$$

Lemma 3.1. *Assume that there exists a nonincreasing positive sequence $\{\varepsilon_n\}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, then there exist a function $\lambda(t) \in C^\Delta[0, T]_{\mathbb{T}}$ satisfying*

$$(i) \quad \varphi_p(\lambda^\Delta(t)) \in C^\nabla[0, T]_{\mathbb{T}}, \lambda(t) > 0 \text{ for } t \in (0, T]_{\mathbb{T}} \text{ and } \max_{t \in [0, T]_{\mathbb{T}}} |(\varphi_p(\lambda^\Delta(t)))^\nabla| > 0;$$

$$(ii) \quad \lambda(0) = 0, \lambda(T) - \sum_{i=1}^m \varphi_i(\lambda^\Delta(\xi_i)) < 0 \text{ and } 0 < \lambda(t) \leq \varepsilon_n \text{ for } t \in (0, T]_{\mathbb{T}}.$$

Proof. Let $e_n = [1/2^{n+1}, T]_{\mathbb{T}}$ ($n \geq n_0$). Assume that $r : [0, T]_{\mathbb{T}} \rightarrow [0, \infty)$ be such that $r(0) = 0$, $r(t) = \varepsilon_n^{p-1}/(2T)^{p+1}$ for $t \in e_n \setminus e_{n-1}$, $n \geq n_0$ and $r(t) = \varepsilon_{n_0}^{p-1}/(2T)^{p+1}$ for $t \in [1/2^{n_0}, T]_{\mathbb{T}}$. Let $u(t) = \int_0^t r(s)\Delta s$, $v(t) = [\int_0^t u(s)\nabla s]^{1/(p-1)}$, $w(t) = \int_0^t v(s)\Delta s$. Suppose $\tau_7 \in e_n \setminus e_{n-1}$ for $n \geq n_0$, $\tau_8 \in (0, T]_{\mathbb{T}}$ satisfy $\tau_7 < \tau_8$ and $2\tau_8 - T \geq \tau_7$. It is easy to show that $u, v, w : [0, \tau_7]_{\mathbb{T}} \rightarrow [0, \infty)$ are continuous and increasing. Denote

$$a(t) = [c_0(\tau_8 - t) + c_1 t]^{1/(p-1)} \quad \text{for } t \in [\tau_7, T]_{\mathbb{T}}, \quad (3.2)$$

here

$$c_0 = -\frac{\tau_7}{\tau_8}u(\tau_7) + \frac{1}{\tau_8}(v(\tau_7))^{p-1}, \quad c_1 = \frac{\tau_8 - \tau_7}{\tau_8}u(\tau_7) + \frac{1}{\tau_8}(v(\tau_7))^{p-1}. \quad (3.3)$$

Hence, $a(t) > 0$ for $t \in [\tau_7, T]_{\mathbb{T}}$ and is nondecreasing. Define

$$b(t) = \int_{\tau_7}^t a(s)\Delta s + w(\tau_7) \quad \text{for } t \in [\tau_7, \tau_8]_{\mathbb{T}},$$

$$P(t) = \begin{cases} b(t), & t \in [\tau_7, \tau_8]_{\mathbb{T}}, \\ b(2\tau_8 - t), & t \in [\tau_8, T]_{\mathbb{T}}, \end{cases} \quad \lambda(t) = \begin{cases} w(t), & t \in [0, \tau_7]_{\mathbb{T}}, \\ P(t), & t \in [\tau_7, T]_{\mathbb{T}}. \end{cases} \quad (3.4)$$

We can easily prove $w(\tau_7) = P(\tau_7)$, $w^\Delta(\tau_7) = P^\Delta(\tau_7)$, $(\varphi_p(w^\Delta))^\nabla(\tau_7) = (\varphi_p(P^\Delta))^\nabla(\tau_7)$ and $w \in C^\Delta[0, \tau_7]_{\mathbb{T}}$, $P \in C^\Delta[\tau_7, T]_{\mathbb{T}}$, $\varphi_p(w^\Delta) \in C^\nabla[0, \tau_7]_{\mathbb{T}}$, $\varphi_p(P^\Delta) \in C^\nabla[\tau_7, T]_{\mathbb{T}}$. Thus, we have $\lambda \in C^\Delta[0, T]_{\mathbb{T}}$ and $\varphi_p(\lambda^\Delta) \in C^\nabla[0, T]_{\mathbb{T}}$ with $\max_{0 \leq t \leq T} |(\varphi_p(\lambda^\Delta))^\nabla(t)| > 0$. Now since $w(t) > 0$ for $t \in (0, \tau_7]_{\mathbb{T}}$ and $P(t) > 0$ for $t \in [\tau_7, T]_{\mathbb{T}}$, we have $\lambda(t) > 0$ for $t \in (0, T]_{\mathbb{T}}$. On the other hand,

$$u(\tau_7) = \int_0^{\tau_7} r(s)\Delta s \leq \tau_7 \frac{\varepsilon_n^{p-1}}{(2T)^{p+1}} < \frac{\varepsilon_n^{p-1}}{(2T)^p},$$

$$v(\tau_7) = \left[\int_0^{\tau_7} u(s)\nabla s \right]^{1/(p-1)} < \left(\tau_7 \frac{\varepsilon_n^{p-1}}{(2T)^p} \right)^{1/(p-1)} < \frac{\varepsilon_n}{2^{p/(p-1)}T}, \quad w(\tau_7) < \tau_7 \times \frac{\varepsilon_n}{2^{p/(p-1)}T} < \frac{\varepsilon_n}{2}, \quad (3.5)$$

by the monotonicity of $P(t)$ on $[\tau_7, \tau_8]_{\mathbb{T}}$, $[\tau_8, T]_{\mathbb{T}}$, respectively, we have

$$\begin{aligned} \lambda(\tau_8) &= \max_{t \in [\tau_7, T]_{\mathbb{T}}} \lambda(t) \\ &= \int_{\tau_7}^{\tau_8} a(s)\Delta s + w(\tau_7) \\ &\leq (\tau_8 - \tau_7) \max_{t \in [\tau_7, \tau_8]_{\mathbb{T}}} [c_0(\tau_8 - t) + c_1 t]^{1/(p-1)} + w(\tau_7) \\ &\leq (\tau_8 - \tau_7) \left[(\tau_8 - \tau_7)u(\tau_7) + (v(\tau_7))^{p-1} \right]^{1/(p-1)} + w(\tau_7) \\ &< T \left[T \frac{\varepsilon_n^{p-1}}{(2T)^p} + \frac{\varepsilon_n^{p-1}}{2(2T)^{p-1}} \right]^{1/(p-1)} + \frac{\varepsilon_n}{2} < \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2} \\ &= \varepsilon_n. \end{aligned} \quad (3.6)$$

Consequently, $0 < \lambda(t) \leq \varepsilon_n$, $t \in (0, T]_{\mathbb{T}}$.

Without loss of generality, $\sum_{i=1}^m \varphi_i(\lambda^\Delta(\xi_i)) \geq \varepsilon_n > \lambda(T)$. We have

$$\lambda(T) - \sum_{i=1}^m \varphi_i(\lambda^\Delta(\xi_i)) < 0. \quad (3.7)$$

□

Now we discuss how to construct a lower solution $\alpha(t)$ in (A2) and (A4).

(A7) For each $n \in \{1, 2, \dots\}$, there exist a constant k_0 and a strictly monotone decreasing sequence $\{\rho_n\}$ with $\lim_{n \rightarrow \infty} \rho_n = 0$, and $q(t)f(t, u) \geq k_0$ for $(t, u) \in [1/2^{n+1}, T]_{\mathbb{T}} \times \{u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}} : 0 < u \leq \rho_n\}$;

(A8) There exists a function $\beta \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\beta^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ such that $\beta \geq 0$ for $t \in [0, T]_{\mathbb{T}}$, $\beta(T) - \sum_{i=1}^m \varphi_i(\beta^\Delta(\xi_i)) > 0$, $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(t, \beta)$ for $t \in (0, T)_{\mathbb{T}}$ and $-(\varphi_p(\beta^\Delta))^\nabla \geq q(t)f(1/2^{n_0+1}, \beta)$ for $t \in (0, 1/2^{n_0+1})_{\mathbb{T}}$.

Theorem 3.2. *Let $n_0 \in \{3, 4, \dots\}$ be fixed. If (H1)–(H3), (3.1) and (A7)–(A8) hold, then boundary value problem (1.1) and (1.2) has a solution $u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$ with $\varphi_p(u^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ and $u(t) > 0$ for $t \in (0, T]_{\mathbb{T}}$.*

Proof. By Corollary 2.3, we need only show that conditions (A1), (A2), (A4)–(A6) are satisfied. Without loss of generality, suppose

$$\beta(t) > \rho_{n_0} \quad \text{for } t \in [0, T]_{\mathbb{T}}, \quad \beta(T) - \sum_{i=1}^m \varphi_i(\beta^\Delta(\xi_i)) > \rho_{n_0}, \quad (3.8)$$

by (A7), (A8) and (3.8), we obtain that (A1) and (A5) hold.

From Lemma 3.1 there exists a function $\lambda(t) \in C^\Delta[0, T]_{\mathbb{T}}$ satisfying

- (i) $\varphi_p(\lambda^\Delta(t)) \in C^\nabla[0, T]_{\mathbb{T}}$, $\lambda(t) > 0$ for $t \in (0, T]_{\mathbb{T}}$ and $R_1 = \max_{t \in [0, T]_{\mathbb{T}}} |(\varphi_p(\lambda^\Delta(t)))^\nabla| > 0$.
- (ii) $\lambda(0) = 0$, $\lambda(T) - \sum_{i=1}^m \varphi_i(\lambda^\Delta(\xi_i)) < 0$ and $0 < \lambda(t) \leq \rho_n$ for $t \in (0, T]_{\mathbb{T}}$.

Assume $m = \min\{1, (k_0/2R_1)^{1/(p-1)}, \rho_{n_0}/|\lambda|\}$. Let $\alpha(t) = m\lambda(t)$ for $t \in [0, T]_{\mathbb{T}}$. Then $\alpha(t) \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\alpha^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$, $\alpha(0) = 0$ with $0 < \alpha(t) \leq \lambda(t) \leq \rho_n$ for $t \in (1/2^{n+1}, T]_{\mathbb{T}}$. Without loss of generality, we have $\alpha(T) - \sum_{i=1}^m \varphi_i(\alpha^\Delta(\xi_i)) < 0$. For arbitrary $(t, u) \in (0, T]_{\mathbb{T}} \times \{u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}} : 0 < u \leq \alpha(t)\}$, there exists $n \in \{n_0, n_0 + 1, \dots\}$ such that $(t, u) \in [1/2^{n+1}, T]_{\mathbb{T}} \times \{u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}} : 0 < u \leq \alpha(t)\}$. We have

$$\begin{aligned} q(t)f(t, u) + \left(\varphi_p(\alpha^\Delta(t))\right)^\nabla &\geq k_0 + \left(\varphi_p(m\lambda^\Delta(t))\right)^\nabla \\ &= k_0 + m^{p-1} \left(\varphi_p(\lambda^\Delta(t))\right)^\nabla \\ &\geq k_0 - m^{p-1} \left| \left(\varphi_p(\lambda^\Delta(t))\right)^\nabla \right| \\ &\geq k_0 - \frac{k_0}{2R_1} \left| \left(\varphi_p(\lambda^\Delta(t))\right)^\nabla \right| \\ &\geq k_0 - \frac{k_0}{2R_1} \max_{t \in [0, T]_{\mathbb{T}}} \left| \left(\varphi_p(\lambda^\Delta(t))\right)^\nabla \right| = \frac{k_0}{2} > 0. \end{aligned} \quad (3.9)$$

Thus (A4) holds and (A2) is also true if $u(t) = \alpha(t)$. Also since $\alpha(T) \leq \sup_{t \in [0, T]_{\mathbb{T}}} |\alpha(t)| = m \sup_{t \in [0, T]_{\mathbb{T}}} |\lambda| \leq \rho_{n_0}$, we have $\beta(T) \geq \rho_{n_0} \geq \alpha(T)$, then (A6) is fulfilled. By Corollary 2.3, the boundary value problem (1.1) and (1.2) has a solution $u(t) \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(u^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$ with $u(t) \geq 0$ for $t \in (0, T]_{\mathbb{T}}$. \square

We can replace $t \in [1/2^{n+1}, T]_{\mathbb{T}}$ with $t \in [0, T - 1/2^{n+1}]_{\mathbb{T}}$ or $t \in [1/2^{n+1}, T - 1/2^{n+1}]_{\mathbb{T}}$. So it is easily obtain (see Remark 2.2) the analogue of Theorem 3.2 in this section.

Looking at Theorem 3.2, it is difficulty for us to discuss examples in constructing $\beta(t)$ in (A8). The following theorem removes (A8) and replaces it with an easy verified condition.

Theorem 3.3. *Let $n_0 \in \{1, 2, \dots\}$ be fixed. If (H1)–(H3), (A1) and (A2) hold, in addition suppose that the following conditions are satisfied:*

$$M_1 > 0, \quad M_2 > \max \left\{ \sup_{t \in [0, T]} \alpha(t), \rho_{n_0} \right\}, \quad \text{here } M_1, M_2 \text{ are constants,} \quad (3.10)$$

$$q(t)f(t, M_1 t + M_2) \leq 0 \quad \text{for } t \in (0, T)_{\mathbb{T}},$$

$$q(t)f\left(\frac{1}{2^{n_0+1}}, M_1 t + M_2\right) \leq 0 \quad \text{for } t \in \left(0, \frac{1}{2^{n_0+1}}\right)_{\mathbb{T}}, \quad (3.11)$$

$$M_1 T + M_2 - \sum_{i=1}^m \psi_i(M_1) > 0.$$

Then boundary value problem (1.1) and (1.2) has a solution $u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$ with $\varphi_p(u^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ and $u > 0$ for $t \in (0, T]_{\mathbb{T}}$.

Proof. Denote $\beta(t) = M_1 t + M_2$ for $t \in [0, T]_{\mathbb{T}}$, then $\beta(t) \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\beta^\Delta(t)) \in C^\nabla(0, T)_{\mathbb{T}}$ with $\beta(t) \geq \alpha(t)$ and $\beta(t) \geq \rho_{n_0}$ for $t \in [0, T]_{\mathbb{T}}$,

$$\beta(T) - \sum_{i=1}^m \psi_i(\beta^\Delta(\xi_i)) > 0, \quad (3.12)$$

with

$$-\left(\varphi_p(\beta^\Delta(t))\right)^\nabla \geq q(t)f(t, \beta) \quad \text{for } t \in (0, T)_{\mathbb{T}},$$

$$-\left(\varphi_p(\beta(t)^\Delta)\right)^\nabla \geq q(t)f\left(\frac{1}{2^{n_0+1}}, \beta(t)\right) \quad \text{for } t \in \left(0, \frac{1}{2^{n_0+1}}\right)_{\mathbb{T}}, \quad (3.13)$$

then (A3) holds. By Theorem 2.1 the result holds. \square

From Theorems 3.2 and 3.3 we have the following theorem.

Theorem 3.4. *Let $n_0 \in \{1, 2, \dots\}$ be fixed. If (H1)–(H3), (3.1) and (A7) hold, in addition suppose there exist constants $M_1, M_2 > 0$ such that (3.11) and (3.5) are true. Then the problem (1.1) and (1.2) has a solution $u \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$ with $\varphi_p(u^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ and $u > 0$ for $t \in (0, T]_{\mathbb{T}}$.*

Proof. Without loss of generality suppose $\rho_{n_0} < M_2$, by (A7) we have (A1) which holds and

$$M_2 > \rho_{n_0} > \rho_{n_0+1} > \dots, \quad \lim_{n \rightarrow \infty} \rho_n = 0. \quad (3.14)$$

By the similar way as the proof of the Theorem 3.2, there exists a function $\alpha \in C[0, T]_{\mathbb{T}} \cap C^\Delta(0, T]_{\mathbb{T}}$, $\varphi_p(\alpha^\Delta) \in C^\nabla(0, T)_{\mathbb{T}}$ with $\alpha(0) = 0$, $\alpha(T) - \sum_{i=1}^m \psi_i(\alpha^\Delta(\xi_i)) < 0$, $\alpha(t) > 0$ for $t \in (0, T]_{\mathbb{T}}$, such that $-(\varphi_p(\alpha^\Delta(t)))^\nabla \leq q(t)f(t, \alpha(t))$ for $t \in (0, T)_{\mathbb{T}}$ and $\alpha(t) \leq \sup_{t \in [0, T]_{\mathbb{T}}} |\alpha| \leq \rho_{n_0}$. This together with (3.14) we have $M_2 > \max\{\sup_{t \in [0, T]_{\mathbb{T}}} \alpha(t), \rho_{n_0}\}$. Thus all the conditions of the Theorem 3.3 are fulfilled. \square

4. An Example

In this section, we present an example to illustrate our results. Let

$$\mathbb{T} = \{0\} \cup \left\{ \left(\frac{1}{2} \right)^{\mathbb{N}} \right\} \cup \left[\frac{1}{2}, 1 \right]. \quad (4.1)$$

Consider the following boundary value problem

$$\begin{aligned} - \left(\left| u^\Delta(t) \right|^2 u^\Delta(t) \right)^\nabla &= q(t)f(t, u(t)) \quad \text{for } t \in (0, 1)_{\mathbb{T}}, \\ u(0) = 0, \quad u(1) - \frac{1}{5}u^\Delta\left(\frac{1}{8}\right) - \frac{1}{10}u^\Delta\left(\frac{1}{4}\right) - \frac{1}{5}u^\Delta\left(\frac{3}{4}\right) - \frac{1}{10}u^\Delta(1) &= 0. \end{aligned} \quad (4.2)$$

It is obvious that $T = 1$, $p = 4$, $m = 4$, $\varphi_1(x) = \varphi_3(x) = (1/5)x$, $\varphi_2(x) = \varphi_4(x) = (1/10)x$. Denote $q(t) = t^8 + 5$ and $f(t, u(t)) = t/u^7(t) + u^7(t) - \lambda^2$, here $\lambda^2 \geq 2$ is constant. Let $n_0 \in \{1, 2, \dots\}$, $\rho_n = (1/2^{n+1}(\lambda^2 + a_1))^{1/7}$ and $k_0 = a_1 > 0$ is a constant. We have $\rho_{n_0} \leq 1$. Note that (H1)–(H3) and (3.1) hold. For $n \in \{1, 2, \dots\}$, $t \in [1/2^{n+1}, 1]_{\mathbb{T}}$ and $0 < u \leq \rho_n$, we have

$$q(t)f(t, u) \geq (t^8 + 5) \left(\frac{1}{2^{n+1}\rho_n^7} - \lambda^2 \right) \geq (t^8 + 5)(\lambda^2 + a_1 - \lambda^2) > a_1, \quad (4.3)$$

which implies (A7) is satisfied.

Now we show that (A8) holds with $\beta = t^{1/7}$.

Notice that if $t \in (1/2, 1]_{\mathbb{T}}$, then $\beta^\Delta(t) = \beta'(t) = (1/7)t^{-6/7}$,

$$\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) = \frac{1}{343} t^{-18/7}, \quad \left(\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) \right)^\nabla = -\frac{18}{2401} t^{-25/7} \leq 0. \quad (4.4)$$

If $t = 1/2$, then $\beta^\Delta(t) = (1/7)t^{-6/7}$ and $\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) = (1/343)t^{-18/7}$,

$$\left(\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) \right)^\nabla = \frac{2}{343} \left(\frac{1}{2} \right)^{-18/7} - \frac{2}{343} \left(\frac{1}{4} \right)^{-18/7} \approx -0.1714 \leq 0. \quad (4.5)$$

If $t = 1/2^n (n = 2, 3, \dots)$, then $\sigma(t) = 2t$, $\rho(t) = t/2$, $\mu(t) = t$, $\nu(t) = t/2$, we have

$$\beta^\Delta(t) = \frac{1}{t} \left[(2t)^{1/7} - t^{1/7} \right], \quad \left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) = \frac{1}{t^3} \left[(2t)^{1/7} - t^{1/7} \right]^3, \quad (4.6)$$

by induction, one gets

$$\left(\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) \right)^\nabla = 2^{4n+1} \left[\left(\frac{1}{2^{n-1}} \right)^{1/7} - \left(\frac{1}{2^n} \right)^{1/7} \right]^3 - 2^{4n+4} \left[\left(\frac{1}{2^n} \right)^{1/7} - \left(\frac{1}{2^{n+1}} \right)^{1/7} \right]^3 \leq 0. \quad (4.7)$$

Thus, for $t \in (0, 1)_{\mathbb{T}}$, we have

$$\begin{aligned} \left(\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) \right)^\nabla + q(t) f(t, \beta) &\leq (t^8 + 5) \left(\frac{t}{t} + (t^{1/7})^7 - \lambda^2 \right) \\ &\leq (t^8 + 5) (2 - \lambda^2) \leq 0, \\ \left(\left| \beta^\Delta(t) \right|^2 \beta^\Delta(t) \right)^\nabla + q(t) f\left(\frac{1}{2^{n_0+1}}, \beta\right) &\leq (t^8 + 5) \left(\frac{1}{2^{n_0+1} (t^{1/7})} + (t^{1/7})^7 - \lambda^2 \right) \\ &\leq (t^8 + 5) (2 - \lambda^2) \leq 0 \quad \text{for } t \in \left(0, \frac{1}{2^{n_0+1}}\right)_{\mathbb{T}}. \end{aligned} \quad (4.8)$$

Now

$$\beta(1) - \psi_1\left(\beta^\Delta\left(\frac{1}{8}\right)\right) - \psi_2\left(\beta\left(\frac{1}{4}\right)\right) - \psi_3\left(\beta^\Delta\left(\frac{3}{4}\right)\right) - \psi_4(\beta(1)) > 0. \quad (4.9)$$

Hence, all conditions of the Theorem 3.2 are satisfied. As a result, the problem (4.2) has a positive solution.

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