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Research Article

Inclusion Properties for Certain Subclasses of Analytic Functions Defined by a Linear Operator

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The purpose of the present paper is to investigate some inclusion properties of certain subclasses of analytic functions associated with a family of linear operators, which are defined by means of the Hadamard product (or convolution). Some integral preserving properties are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that $f(z) = g(w(z))$. We denote by $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{C}$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike, convex, and close-to-convex in $\mathbb{U}$.

Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\mathrm{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$.

Making use of the principle of subordination between analytic functions, many authors investigated the subclasses $\mathcal{S}^*(\phi)$, $\mathcal{K}(\phi)$, and $\mathcal{C}(\phi, \psi)$ of the class $\mathcal{A}$ for $\phi, \psi \in \mathcal{N}$ (cf. [1, 2]), which are defined by

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \text{ in } \mathbb{U} \right\},$$

$$\mathcal{K}(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \text{ in } \mathbb{U} \right\},$$
$$\mathcal{C}(\phi, \psi) := \left\{ f \in \mathcal{A} : \exists g \in \mathcal{S}^*(\phi) \text{ s.t. } \frac{zf'(z)}{g(z)} \prec \psi(z) \text{ in } \mathbb{U} \right\}. \tag{1.2}$$

For $\phi(z) = \psi(z) = (1+z)/(1-z)$ in the definitions defined above, we have the well-known classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{C}$, respectively. Furthermore, for the function classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ investigated by Janowski [3] (also see [4]), it is easily seen that

$$\mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right) = \mathcal{S}^*[A,B] \quad (-1 \le B < A \le 1),$$
$$\mathcal{K}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{K}[A,B] \quad (-1 \le B < A \le 1). \tag{1.3}$$

We now define the function $h(a,c)(z)$ by

$$h(a,c)(z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad \left(z \in \mathbb{U};\ a \in \mathbb{R};\ c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \dots\}\right), \tag{1.4}$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k = 0,\ \nu \in \mathbb{C} \setminus \{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k \in \mathbb{N} := \{1, 2, \dots\},\ \nu \in \mathbb{C}. \end{cases} \tag{1.5}$$

We also denote by $L(a,c)$: $\mathcal{A} \to \mathcal{A}$ the operator defined by

$$L(a,c)f(z) = h(a,c)(z) * f(z) \quad (z \in \mathbb{U};\ f \in \mathcal{A}), \tag{1.6}$$

where the symbol $(*)$ stands for the Hadamard product (or convolution). Then it is easily observed from definitions (1.4) and (1.6) that $L(2,1)f(z) = zf'(z)$ and

$$z\big(L(a,c)f(z)\big)' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z). \tag{1.7}$$

Furthermore, we note that $L(n+1,1)f(z) = D^n f(z)$ $(n > -1)$, where the symbol D^n denotes the familiar Ruscheweyh derivative [5] (also, see [6]) for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The operator $L(a,c)$ was introduced and studied by Carlson and Shaffer [7] which has been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see also [8]).

By using the operator $L(a,c)$, we introduce the following classes of analytic functions for $\phi, \psi \in \mathcal{N}$, $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$:

$$\mathcal{S}_{a,c}(\phi) := \{ f \in \mathcal{A} : L(a,c)f(z) \in \mathcal{S}^*(\phi) \},$$
$$\mathcal{K}_{a,c}(\phi) := \{ f \in \mathcal{A} : L(a,c)f(z) \in \mathcal{K}(\phi) \},$$
$$\mathcal{C}_{a,c}(\phi, \psi) := \{ f \in \mathcal{A} : L(a,c)f(z) \in \mathcal{C}(\phi, \psi) \}. \tag{1.8}$$

We also note that

$$f(z) \in \mathcal{K}_{a,c}(\phi) \iff zf'(z) \in \mathcal{S}_{a,c}(\phi). \tag{1.9}$$

In particular, we set

$$\begin{gathered}\mathcal{S}_{a,c}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{S}_{a,c}[A,B] \quad (-1 \le B < A \le 1),\\ \mathcal{K}_{a,c}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{K}_{a,c}[A,B] \quad (-1 \le B < A \le 1).\end{gathered} \tag{1.10}$$

In this paper, we investigate several inclusion properties of the classes $\mathcal{S}_{a,c}(\phi)$, $\mathcal{K}_{a,c}(\phi)$, and $\mathcal{C}_{a,c}(\phi,\psi)$. The integral preserving properties in connection with the operator $L(a,c)$ are also considered. Furthermore, relevant connections of the results presented here with those obtained in earlier works are pointed out.

2. Inclusion properties involving the operator $L(a,c)$

The following lemmas will be required in our investigation.

Lemma 2.1 (see [9, pages 60-61]). *Let $a_2 \ge a_1 > 0$. If $a_2 \ge 2$ or $a_1 + a_2 \ge 3$, then the function $h(a_1,a_2)(z)$ defined by* (1.4) *belongs to the class $\mathcal{K}$.*

Lemma 2.2 (see [10]). *Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then for every analytic function Q in $\mathbb{U}$,*

$$\frac{(f*Qg)}{(f*g)}(\mathbb{U}) \subset \overline{\text{co}}Q(\mathbb{U}), \tag{2.1}$$

where $\overline{\text{co}}Q(\mathbb{U})$ denote the closed convex hull of $Q(\mathbb{U})$.

Theorem 2.3. *Let $a_2 \ge a_1 > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, and $\phi \in \mathcal{N}$. If $a_2 \ge 2$ or $a_1 + a_2 \ge 3$, then*

$$\mathcal{S}_{a_2,c}(\phi) \subset \mathcal{S}_{a_1,c}(\phi). \tag{2.2}$$

Proof. Let $f \in \mathcal{S}_{a_2,c}(\phi)$. Then there exists an analytic function w in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$ such that

$$\frac{z(L(a_2,c)f(z))'}{L(a_2,c)f(z)} = \phi(w(z)) \quad (z \in \mathbb{U}). \tag{2.3}$$

By using (1.6) and (2.3), we have

$$\begin{aligned}\frac{z(L(a_1,c)f(z))'}{L(a_1,c)f(z)} &= \frac{z(h(a_1,c)(z)*f(z))'}{h(a_1,c)(z)*f(z)}\\ &= \frac{z(h(a_2,c)(z)*h(a_1,a_2)(z)*f(z))'}{h(a_2,c)(z)*h(a_1,a_2)(z)*f(z)}\\ &= \frac{h(a_1,a_2)(z)*z(L(a_2,c)f(z))'}{h(a_1,a_2)(z)*L(a_2,c)f(z)}\\ &= \frac{h(a_1,a_2)(z)*\phi(w(z))L(a_2,c)f(z)}{h(a_1,a_2)(z)*L(a_2,c)f(z)}.\end{aligned} \tag{2.4}$$

It follows from (2.3) and Lemma 2.1 that $L(a_2,c)f(z) \in \mathcal{S}^*$ and $h(a_1,a_2)(z) \in \mathcal{K}$, respectively. Then by applying Lemma 2.2 to (2.4), we obtain

$$\frac{\{h(a_1,a_2)(z)*\phi(w)L(a_2,c)f\}}{\{h(a_1,a_2)(z)*L(a_2,c)f\}}(\mathbb{U}) \subset \overline{\text{co}}(\phi(\mathbb{U})) \subset \phi(\mathbb{U}), \tag{2.5}$$

since ϕ is convex univalent. Therefore, from the definition of subordination and (2.5), we have

$$\frac{z(L(a_1,c)f(z))'}{L(a_1,c)f(z)} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{2.6}$$

or, equivalently, $f \in \mathcal{S}_{a_1,c}(\phi)$, which completes the proof of Theorem 2.3. □

Theorem 2.4. *Let $a \in \mathbb{R}$, $c_2 \geq c_1 > 0$ and $\phi \in \mathcal{N}$. If $c_2 \geq 2$ or $c_1 + c_2 \geq 3$, then*

$$\mathcal{S}_{a,c_1}(\phi) \subset \mathcal{S}_{a,c_2}(\phi). \tag{2.7}$$

Proof ($f \in \mathcal{S}_{a,c_1}(\phi)$). Using a similar argument as in the proof of Theorem 2.3, we obtain

$$\frac{z(L(a,c_2)f(z))'}{L(a,c_2)f(z)} = \frac{h(a_1,a_2)(z)*\phi(w(z))L(a,c_1)f(z)}{h(a_1,a_2)(z)*L(a,c_1)f(z)}, \tag{2.8}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. Applying Lemma 2.1 and the fact that $L(a,c_1)f(z) \in \mathcal{S}^*$, we see that

$$\frac{\{h(a_1,a_2)*h(w)L(a_2,c)f\}}{\{h(a_1,a_2)*L(a,c_1)f\}}(\mathbb{U}) \subset \overline{\text{co}}(\phi(\mathbb{U})) \subset \phi(\mathbb{U}), \tag{2.9}$$

since ϕ is convex univalent. Thus the proof of Theorem 2.3 is completed. □

Corollary 2.5. *Let $a_2 \geq a_1 > 0$, $c_2 \geq c_1 > 0$, and $\phi \in \mathcal{N}$. If $a_2 \geq \min\{2, 3-a_1\}$ and $c_2 \geq \min\{2, 3-c_1\}$, then*

$$\mathcal{S}_{a_2,c_1}(\phi) \subset \mathcal{S}_{a_2,c_2}(\phi) \subset \mathcal{S}_{a_1,c_2}(\phi). \tag{2.10}$$

Theorem 2.6. *Let $a_2 \geq a_1 > 0$, $c_2 \geq c_1 > 0$ and $\phi \in \mathcal{N}$. If $a_2 \geq \min\{2, 3-a_1\}$ and $c_2 \geq \min\{2, 3-c_1\}$, then*

$$\mathcal{K}_{a_2,c_1}(\phi) \subset \mathcal{K}_{a_2,c_2}(\phi) \subset \mathcal{K}_{a_1,c_2}(\phi). \tag{2.11}$$

Proof. Applying (1.9) and Corollary 2.5, we observe that

$$\begin{aligned}
f(z)\in\mathcal{K}_{a_2,c_1}(\phi) &\Longleftrightarrow L(a_2,c_1)f(z)\in\mathcal{K}(\phi)\\
&\Longleftrightarrow z\big(L(a_2,c_1)f(z)\big)'\in\mathcal{S}^*(\phi)\\
&\Longleftrightarrow L(a_2,c_1)\big(zf'(z)\big)\in\mathcal{S}^*(\phi)\\
&\Longleftrightarrow zf'(z)\in\mathcal{S}_{a_2,c_1}(\phi)\\
&\Longrightarrow zf'(z)\in\mathcal{S}_{a_2,c_2}(\phi)\\
&\Longleftrightarrow L(a_2,c_2)\big(zf'(z)\big)\in\mathcal{S}^*(\phi)\\
&\Longleftrightarrow z\big(L(a_2,c_2)f(z)\big)'\in\mathcal{S}^*(\phi)\\
&\Longleftrightarrow L(a_2,c_2)f(z)\in\mathcal{K}(\phi)\\
&\Longleftrightarrow f(z)\in\mathcal{K}_{a_2,c_2}(\phi),\\
f(z)\in\mathcal{K}_{a_2,c_2}(\phi) &\Longleftrightarrow L(a_2,c_2)f(z)\in\mathcal{K}(\phi)\\
&\Longleftrightarrow L(a_2,c_2)\big(zf'(z)\big)\in\mathcal{S}^*(\phi)\\
&\Longrightarrow zf'(z)\in\mathcal{S}_{a_1,c_2}(\phi)\\
&\Longleftrightarrow z\big(L(a_1,c_2)f(z)\big)'\in\mathcal{S}^*(\phi)\\
&\Longleftrightarrow f(z)\in\mathcal{K}_{a_1,c_2}(\phi),
\end{aligned}\tag{2.12}$$

which evidently proves Theorem 2.6. □

Taking $\phi(z)=(1+Az)/(1+Bz)$ $(-1\le B<A\le 1;\ z\in\mathbb{U})$ in Corollary 2.5 and Theorem 2.6, we have the following corollary.

Corollary 2.7. *Let $a_2\ge a_1>0$ and $c_2\ge c_1>0$. If $a_2\ge\min\{2,3-a_1\}$ and $c_2\ge\min\{2,3-c_1\}$, then*

$$\begin{aligned}
&\mathcal{S}_{a_2,c_1}[A,B]\subset\mathcal{S}_{a_2,c_2}[A,B]\subset\mathcal{S}_{a_1,c_2}[A,B]\quad(-1\le B<A\le 1),\\
&\mathcal{K}_{a_2,c_1}[A,B]\subset\mathcal{K}_{a_2,c_2}[A,B]\subset\mathcal{K}_{a_1,c_2}[A,B]\quad(-1\le B<A\le 1).
\end{aligned}\tag{2.13}$$

To prove the theorems below, we need the following lemma.

Lemma 2.8. *Let $\phi\in\mathcal{N}$. If $f\in\mathcal{K}$ and $q\in\mathcal{S}^*(\phi)$, then $f*q\in\mathcal{S}^*(\phi)$.*

Proof. Let $q\in\mathcal{S}^*(\phi)$. Then

$$zq'(z)=q(z)\phi\big(\omega(z)\big)\quad(z\in\mathbb{U}),\tag{2.14}$$

where ω is an analytic function in $\mathbb{U}$ with $|w(z)|<1$ $(z\in\mathbb{U})$ and $w(0)=0$. Thus we have

$$\frac{z\big(f(z)*q(z)\big)'}{f(z)*q(z)}=\frac{f(z)*zq'(z)}{f(z)*q(z)}=\frac{f(z)*\phi\big(\omega(z)\big)q(z)}{f(z)*q(z)}\quad(z\in\mathbb{U}).\tag{2.15}$$

By using similar arguments to those used in the proof of Theorem 2.3, we conclude that (2.15) is subordinated to ϕ in $\mathbb{U}$ and so $f*q\in\mathcal{S}^*(\phi)$. □

Theorem 2.9. *Let $a_2 \geq a_1 > 0$, $c_2 \geq c_1 > 0$ and $\phi, \psi \in \mathcal{N}$. If $a_2 \geq \min\{2, 3 - a_1\}$ and $c_2 \geq \min\{2, 3 - c_1\}$, then*

$$\mathcal{C}_{a_2,c_1}(\phi,\psi) \subset \mathcal{C}_{a_2,c_2}(\phi,\psi) \subset \mathcal{C}_{a_1,c_2}(\phi,\psi). \tag{2.16}$$

Proof. First of all, we show that

$$\mathcal{C}_{a_2,c_1}(\phi,\psi) \subset \mathcal{C}_{a_2,c_2}(\phi,\psi). \tag{2.17}$$

Let $f \in \mathcal{C}_{a_2,c_1}(\phi,\psi)$. Then there exists a function $q_2 \in \mathcal{S}^*(\phi)$ such that

$$\frac{z\left(L(a_2,c_1)f(z)\right)'}{q_2(z)} \prec \psi(z) \quad (z \in \mathbb{U}). \tag{2.18}$$

From (2.18), we obtain

$$z\left(L(a_2,c_1)f(z)\right)' = \psi(w(z)) \quad (z \in \mathbb{U}), \tag{2.19}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. By virtue of Lemmas 2.1 and 2.8, we see that $h(a_1,a_2)(z) * q_2(z) \equiv q_1(z)$ belongs to $\mathcal{S}^*(\phi)$. Then we have

$$\begin{aligned}\frac{z\left(L(a_2,c_2)f(z)\right)'}{q_1(z)} &= \frac{h(c_1,c_2)(z) * z\left(L(a_2,c_1)f(z)\right)'}{h(c_1,c_2)(z) * q_2(z)} \\ &= \frac{h(c_1,c_2)(z) * \psi(w(z))q_2(z)}{h(c_1,c_2)(z) * q_2(z)} \\ &\prec \psi(z) \quad (z \in \mathbb{U}),\end{aligned} \tag{2.20}$$

which implies that $f \in \mathcal{C}_{a_1,c}(\phi,\psi)$.

Moreover, the proof of the second part is similar to that of the first part and so we omit the details involved. □

3. Inclusion properties involving various operators

The next theorem shows that the classes $\mathcal{S}_{a,c}(\phi)$, $\mathcal{K}_{a,c}(\phi)$, and $\mathcal{C}_{a,c}(\phi,\psi)$ are invariant under convolution with convex functions.

Theorem 3.1. *Let $a > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\phi, \psi \in \mathcal{N}$ and let $g \in \mathcal{K}$. Then*

(i) $f \in \mathcal{S}_{a,c}(\phi) \Rightarrow g * f \in \mathcal{S}_{a,c}(\phi)$,
(ii) $f \in \mathcal{K}_{a,c}(\phi) \Rightarrow g * f \in \mathcal{K}_{a,c}(\phi)$,
(iii) $f \in \mathcal{C}_{a,c}(\phi,\psi) \Rightarrow g * f \in \mathcal{C}_{a,c}(\phi,\psi)$.

Proof. (i) Let $f \in \mathcal{S}_{a,c}(\phi)$. Then we have

$$\frac{z\left(L(a,c)(g*f)(z)\right)'}{L(a,c)(g*f)(z)} = \frac{g(z) * z\left(L(a,c)f(z)\right)'}{g(z) * L(a,c)f(z)}. \tag{3.1}$$

By using the same techniques as in the proof of Theorem 2.3, we obtain (i).

(ii) Let $f \in \mathcal{K}_{a,c}(\phi)$. Then, by (1.9), $zf'(z) \in \mathcal{S}_{a,c}(\phi)$ and hence from (i), $g(z)*zf'(z) \in \mathcal{S}_{a,c}(\phi)$. Since

$$g(z)*zf'(z) = z(g*f)'(z), \tag{3.2}$$

we have (ii) applying (1.9) once again.

(iii) Let $f \in \mathcal{C}_{a,c}(\phi,\psi)$. Then there exists a function $q \in \mathcal{S}^*(\phi)$ such that

$$z\big(L(a,c)f(z)\big)' = \psi\big(w(z)\big)q(z) \quad (z \in \mathbb{U}), \tag{3.3}$$

where w is an analytic function in $\mathbb{U}$ with $|w(z)| < 1\,(z \in \mathbb{U})$ and $w(0) = 0$. From Lemma 2.8, we have that $g*q \in \mathcal{S}^*(\phi)$. Since

$$\frac{z\big(L(a,c)(g*f)(z)\big)'}{(g*q)(z)} = \frac{g(z)*z\big(L(a,c)f(z)\big)'}{g(z)*q(z)} = \frac{g(z)*\psi\big(w(z)\big)q(z)}{g(z)*q(z)} \prec \psi(z) \quad (z \in \mathbb{U}), \tag{3.4}$$

we obtain (iii). □

Now we consider the following operators [5, 11] defined by

$$\begin{aligned}\Psi_1(z) &= \sum_{k=1}^{\infty} \frac{1+c}{k+c} z^k \quad \big(\mathrm{Re}\{c\} \geq 0;\ z \in \mathbb{U}\big),\\ \Psi_2(z) &= \frac{1}{1-x}\log\left[\frac{1-xz}{1-z}\right] \quad \big(\log 1 = 0;\ |x| \leq 1,\ x \neq 1;\ z \in \mathbb{U}\big).\end{aligned} \tag{3.5}$$

It is well known ([12], see also [5]) that the operators Ψ_1 and Ψ_2 are convex univalent in $\mathbb{U}$. Therefore, we have the following result, which can be obtained from Theorem 3.1 immediately.

Corollary 3.2. *Let $a > 0$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\phi, \psi \in \mathcal{N}$ and let Ψ_i $(i = 1, 2)$ be defined by* (3.5). *Then*

(i) $f \in \mathcal{S}_{a,c}(\phi) \Rightarrow \Psi_i * f \in \mathcal{S}_{a,c}(\phi)$,
(ii) $f \in \mathcal{K}_{a,c}(\phi) \Rightarrow \Psi_i * f \in \mathcal{K}_{a,c}(\phi)$,
(iii) $f \in \mathcal{C}_{a,c}(\phi,\psi) \Rightarrow \Psi_i * f \in \mathcal{C}_{a,c}(\phi,\psi)$.

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