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Research Article

Compact Weighted Composition Operators and Multiplication Operators between Hardy Spaces

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We estimate the essential norm of a compact weighted composition operator uC_φ acting between different Hardy spaces of the unit ball in $\mathbb{C}^N$. Also we will discuss a compact multiplication operator between Hardy spaces.

1. Introduction

Let N be a fixed integer. Let B_N denote the unit ball of $\mathbb{C}^N$ and let $H(B_N)$ denote the space of all holomorphic functions in B_N. For each p, $1 \le p < \infty$, the *Hardy space* $H^p(B_N)$ is defined by

$$H^p(B_N) = \left\{ f \in H(B_N) : \sup_{0<r<1} \int_{\partial B_N} \left| f(r\zeta) \right|^p d\sigma(\zeta) < \infty \right\},$$
$$\|f\|_p = \left[\int_{\partial B_N} \left| f^*(\zeta) \right|^p d\sigma(\zeta) \right]^{1/p}, \tag{1.1}$$

where $d\sigma$ is the normalized Lebesgue measure on the boundary ∂B_N of B_N.

For a given holomorphic self-map φ of B_N and holomorphic function u in B_N, *the weighted composition operator* uC_φ is defined by $uC_\varphi f = u(f \circ \varphi)$. In particular, if u is the constant function 1, then uC_φ becomes the *composition operator* C_φ. In the special case that φ is the identity mapping of B_N, uC_φ is called *the multiplication operator* and is denoted by M_u.

Let X and Y be Banach spaces. For a bounded linear operator $T : X \to Y$, the *essential norm* $\|T\|_{e,X\to Y}$ is defined to be the distance from T to the set of the compact operators $\mathcal{K}$, namely,

$$\|T\|_{e,X\to Y} = \inf\{\|T - \mathcal{K}\| : \mathcal{K} \text{ is compact from } X \text{ into } Y\}, \tag{1.2}$$

where $\|\cdot\|$ denotes the usual operator norm. Clearly, T is compact if and only if $\|T\|_{e,X\to Y} = 0$. Thus, the essential norm is closely related to the compactness problem of concrete operators. Many mathematicians have studied the essential norm of various concrete operators. For these studies about composition operators on Hardy spaces of the unit disc, refer to [1–4]. In this paper, our objects are weighted composition operators between Hardy spaces of the unit ball B_N. Several authors have also studied weighted composition operators on various analytic function spaces. For more information about weighted composition operators, refer to [5–10].

Recently, Contreras and Hernández-Díaz [11, 12] have characterized the compactness of uC_φ from $H^p(B_1)$ into $H^q(B_1)$ $(1 < p \le q < \infty)$ in terms of the pull-back measure. Here, B_1 denotes the open unit disc in the complex plane. But they have not given the estimate for the essential norm of uC_φ. The essential norm of $uC_\varphi : H^p(B_1) \to H^q(B_1)$ has been studied by Čučković and Zhao [13, 14]. In the higher-dimensional case, Ueki [15] characterized the boundedness and compactness of $uC_\varphi : H^p(B_N) \to H^q(B_N)$, in terms of the pull-back measure and the integral operator. The purpose of this paper is to estimate the essential norm of $uC_\varphi : H^p(B_N) \to H^q(B_N)$. The following theorem is our main result.

Main Theorem. *Let $1 < p \le q < \infty$. If uC_φ is a bounded weighted composition operator from $H^p(B_N)$ into $H^q(B_N)$, then*

$$\begin{aligned}\|uC_\varphi\|^q_{e,H^p\to H^q} &\sim \limsup_{|w|\to 1^-} \int_{\partial B_N} |u^*(\zeta)|^q \left\{ \frac{1-|w|^2}{|1-\langle \varphi^*(\zeta), w\rangle|^2} \right\}^{qN/p} d\sigma(\zeta) \\ &\sim \limsup_{t\to 0^+} \sup_{\zeta\in\partial B_N} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}},\end{aligned} \tag{1.3}$$

where $\mu_{\varphi,u}$ is the pull-back measure induced by φ and u, $S(\zeta,t)$ is the Carleson set of $\overline{B_N}$, and the notation $\sim$ means that the ratios of two terms are bounded below and above by constants dependent upon the dimension N and other parameters.

The one variable case of the first estimate for $\|uC_\varphi\|_e$ in above theorem may be found in the work [14] by Čučković and Zhao. In the case $p = q = 2$ and $u = 1$, Choe [1] and Luo [16] showed that the essential norm $\|C_\varphi\|_e$ is comparable to the value $\limsup_{t\to 0^+} \sup_{\zeta\in\partial B_N} (\mu_\varphi(S(\zeta,t))/t^N)$.

We give the proof of main theorem in Section 3. The ideas of our proofs are based on the method which Choe or Luo used in their papers. In Section 4, we have a discussion on the compact multiplication operator between different Hardy spaces.

Throughout the paper, the symbol C denotes a positive constant, possibly different at each occurrence, but always independent of the function f and other parameters r or t.

2. Carleson-type measures

For each $u \in H^q(B_N)$, we can define a finite positive Borel measure $\mu_{\varphi,u}$ on $\overline{B_N}$ by

$$\mu_{\varphi,u}(E) = \int_{\varphi^{*-1}(E)} |u^*|^q \, d\sigma \quad (\forall \text{ Borel sets } E \text{ of } \overline{B_N}), \tag{2.1}$$

where φ^* denotes the *radial limit map* of the mapping φ considered as a map of $\partial B_N \to \overline{B_N}$. A change-of-variable formula from measure theory shows that

$$\int_{\overline{B_N}} g\, d\mu_{\varphi,u} = \int_{\partial B_N} |u^*|^q (g \circ \varphi^*)\, d\sigma, \tag{2.2}$$

for each nonnegative measurable function g on $\overline{B_N}$. This type of pull-back measure played an important role in past studies of composition operators on Hardy spaces of B_N.

For each $\zeta \in \partial B_N$ and $t > 0$, let

$$\begin{gathered} S(\zeta,t) = \{z \in \overline{B_N} : |1 - \langle z,\zeta\rangle| < t\}, \\ B(\zeta,t) = S(\zeta,t) \cap B_N, \qquad Q(\zeta,t) = S(\zeta,t) \cap \partial B_N. \end{gathered} \tag{2.3}$$

It is well known that $\sigma(Q(\zeta,t))$ is comparable to t^N ([17, page 67]).

The proof of the following lemma is essentially the same as that of Power's theorem in [18].

Lemma 2.1. *Let $1 \le \alpha < \infty$. Suppose that μ is a positive Borel measure on B_N and that there exists a constant $C > 0$ such that*

$$\mu(B(\zeta,t)) \le Ct^{\alpha N} \quad (\zeta \in \partial B_N,\ t > 0). \tag{2.4}$$

Then there exists a constant $K > 0$ such that

$$\left[\int_{B_N} |f|^{p\alpha} d\mu\right]^{1/p\alpha} \le K\|f\|_{H^p} \quad (f \in H^p(B_N)). \tag{2.5}$$

Proof. Fix $f \in H^p(B_N)$ and $s > 0$. By the same argument as in the proof of theorem in [18, pages 14-15], it follows from (2.4) that there exists a constant $C > 0$ such that

$$\mu(\{z \in B_N : |f(z)| \ge s\}) \le C[\sigma(\{\zeta \in \partial B_N : Mf(\zeta) \ge s\})]^\alpha, \tag{2.6}$$

where Mf is *the admissible maximal function* of f which is defined by

$$Mf(\zeta) = \sup\{|f(z)| : z \in \mathbb{C}^n, |1 - \langle z,\zeta\rangle| < 1 - |z|^2\}, \tag{2.7}$$

for $\zeta \in \partial B_N$. By (2.6), we have

$$\int_{B_N} |f|^{p\alpha} d\mu = p\alpha \int_0^\infty \mu\{|f| > s\} s^{p\alpha-1} ds \le Cp\alpha \int_0^\infty \sigma\{Mf \ge s\}^\alpha s^{p\alpha-1} ds. \tag{2.8}$$

Since $f \in H^p(B_N)$, it follows from [17, Theorem 5.6.5] that

$$\sigma\{Mf \ge s\}^{\alpha-1} s^{p\alpha-p} \le \left[\int_{\partial B_N} \{Mf(\zeta)\}^p d\sigma(\zeta)\right]^{\alpha-1} \le C\|f\|_{H^p}^{p(\alpha-1)}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned} \int_{B_N} |f|^{p\alpha} d\mu &\le C\|f\|_{H^p}^{p(\alpha-1)} p \int_0^\infty \sigma\{Mf \ge s\} s^{p-1} ds \\ &\le C\|f\|_{H^p}^{p(\alpha-1)} \int_{\partial B_N} \{Mf(\zeta)\}^p d\sigma(\zeta) \le C\|f\|_{H^p}^{p\alpha}. \end{aligned} \tag{2.10}$$

This completes the proof. □

Lemma 2.2. *Let $1 \le \alpha < \infty$. Suppose that μ is a positive Borel measure on ∂B_N such that*

$$\mu(Q(\zeta,t)) \le Ct^{\alpha N} \quad (\zeta \in \partial B_N,\ t > 0), \tag{2.11}$$

for some constant $C > 0$.

(a) *If $\alpha = 1$, then there exist a $g \in L^\infty(\partial B_N)$ and a constant $C' > 0$ (C' is the product of C and a constant depending only on the dimension N) such that $d\mu = gd\sigma$ and $\|g\|_{L^\infty} \le C'$.*

(b) *If $\alpha > 1$, then $\mu \equiv 0$ for all Borel sets of ∂B_N.*

Proof. Part (a) is completely analogous to [19, page 238, Lemma 1.3]. So we only prove part (b). Combining $\sigma(Q(\zeta,t))\sim t^N$ with (2.11), we have

$$\frac{\mu(Q(\zeta,t))}{\sigma(Q(\zeta,t))} \le Ct^{N(\alpha-1)} \tag{2.12}$$

for all $\zeta \in \partial B_N$ and $t > 0$. Hence we see that *the maximal function* $M\mu$ of the positive measure μ satisfies $M\mu(\zeta) < \infty$ for all $\zeta \in \partial B_N$. By [17, page 70, Theorem 5.2.7], we obtain $d\mu = gd\sigma$ for some $g \in L^1(\partial B_N)$. By (2.12), we have

$$0 \le \frac{1}{\sigma(Q(\zeta,t))}\int_{Q(\zeta,t)} gd\sigma = \frac{\mu(Q(\zeta,t))}{\sigma(Q(\zeta,t))} \le Ct^{N(\alpha-1)} \tag{2.13}$$

for all $\zeta \in \partial B_N$ and $t > 0$. Letting $t \to 0^+$, we see that $g = 0$ a.e. on ∂B_N, and so $\mu \equiv 0$. This completes the proof of part (b). □

Combining Lemma 2.1 with Lemma 2.2 and using the same argument as in [19, page 239], we obtain the following lemma.

Lemma 2.3. *Let $1 < p \le q < \infty$. Suppose that μ is a positive Borel measure on $\overline{B_N}$ such that*

$$\mu(S(\zeta,t)) \le Ct^{qN/p} \quad (\zeta \in \partial B_N,\ t > 0), \tag{2.14}$$

for some constant $C > 0$. Then, there exists a constant $K > 0$ such that

$$\left[\int_{\overline{B_N}} |f^*|^q d\mu\right]^{1/q \le K\|f\|_{H^p}}, \tag{2.15}$$

for all $f \in H^p(B_N)$. Here, the notation f^* denotes the function defined on $\overline{B_N}$ by $f^* = f$ in B_N and $f^* = \lim_{r\to 1^-} f_r$ a.e. $[\sigma]$ on ∂B_N.

Remark 2.4. In Lemma 2.3 (or in Lemma 2.1), we see that the constant K of (2.15) (or (2.5)) can be chosen to be the product of C and a positive constant depending only on p, q, and the dimension N.

In order to prove the main theorem, we need some results. These are the extensions of [19, Corollary 1.4 and Lemma 1.6] to the case of weighted composition operators uC_φ.

Proposition 2.5. *Let $1 < p \leq q < \infty$. Suppose that $\varphi : B_N \to B_N$ is a holomorphic map and $u \in H^q(B_N) \setminus \{0\}$ such that $uC_\varphi : H^p(B_N) \to H^q(B_N)$ is bounded. Then φ^* cannot carry a set of positive σ-measure in ∂B_N into a set of σ-measure 0 in ∂B_N.*

Proof. Suppose that $E, F \subset \partial B_N$ and $\varphi^*(E) \subset F$ with $\sigma(E) > 0$ and $\sigma(F) = 0$. Put $\mu = \mu_{\varphi,u}|_{\partial B_N}$. As in the case of composition operators, it is well known that the boundedness of $uC_\varphi : H^p(B_N) \to H^q(B_N)$ implies

$$\mu(S(\zeta,t)) \leq Ct^{qN/p} \quad (\zeta \in \partial B_N,\ t > 0), \tag{2.16}$$

for some positive constant C (see [15]). By Lemma 2.2, we see that $\mu \equiv 0$ (if $p < q$) or μ is absolutely continuous with respect to $d\sigma$ (if $p = q$). Thus we have

$$0 \geq \mu(\varphi^*(E)) \equiv \int_{\varphi^{*-1}(\varphi^*(E))} |u^*|^q d\sigma \geq \int_E |u^*|^q d\sigma. \tag{2.17}$$

That is, $u^* = 0$ a.e. on E. Hence [17, page 83, Theorem 5.5.9] gives that $u \equiv 0$ in B_N. This contradicts $u \not\equiv 0$. □

Lemma 2.6. *Let $1 < p \leq q < \infty$ and $f \in H^p(B_N)$. Suppose that $\varphi : B_N \to B_N$ is a holomorphic map and $u \in H^q(B_N) \setminus \{0\}$ such that $uC_\varphi : H^p(B_N) \to H^q(B_N)$ is bounded. Then $u^*(f \circ \varphi)^* = u^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on ∂B_N. Here the notation f^* is used as in Lemma 2.3.*

Proof (cf. [19, Lemma 1.6]). Since φ^* cannot carry a set of positive measure in ∂B_N into a set of measure 0 in ∂B_N (by Proposition 2.5) and since the radial limit of φ, f and ψ exist on a set of full measure in ∂B_N, we have $\lim_{r \to 1^-} u^*(f_r \circ \varphi^*) = u^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on ∂B_N.

On the other hand, since f_r is in the ball algebra $A(B_N)$ and $f_r \to f$ as $r \to 1^-$ in $H^p(B_N)$, the boundedness of uC_φ shows that

$$\begin{aligned} 0 &\leq \int_{\partial B_N} \left| u^*(\zeta)(f \circ \varphi)^*(\zeta) - u^*(\zeta)(f^* \circ \varphi^*)(\zeta) \right|^q d\sigma(\zeta) \\ &\leq \liminf_{r \to 1^-} \int_{\partial B_N} \left| u^*(\zeta)(f \circ \varphi)^*(\zeta) - u^*(\zeta)(f_r \circ \varphi)^*(\zeta) \right|^q d\sigma(\zeta) \\ &= \liminf_{r \to 1^-} \left\| uC_\varphi f - uC_\varphi f_r \right\|_{H^q}^q = 0. \end{aligned} \tag{2.18}$$

This implies that $u^*(f \circ \varphi)^* = u^*(f^* \circ \varphi^*)$ a.e. $[\sigma]$ on ∂B_N. □

3. Weighted composition operators between Hardy spaces

Theorem 3.1. *Let $1 < p \leq q < \infty$. If uC_φ is a bounded weighted composition operator from $H^p(B_N)$ into $H^q(B_N)$, then*

$$\begin{aligned} \left\| uC_\varphi \right\|_{e, H^p \to H^q}^q &\sim \limsup_{|w| \to 1^-} \int_{\partial B_N} |u^*(\zeta)|^q \left\{ \frac{1 - |w|^2}{\left|1 - \langle \varphi^*(\zeta), w \rangle\right|^2} \right\}^{qN/p} d\sigma(\zeta) \\ &\sim \limsup_{t \to 0^+} \sup_{\zeta \in \partial B_N} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}}. \end{aligned} \tag{3.1}$$

Proof of the lower estimates. For each $w \in B_N$, we define the function f_w on $\overline{B_N}$ by

$$f_w(z) = \left\{ \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right\}^{N/p}. \tag{3.2}$$

Then the functions $\{f_w : w \in B_N\}$ belong to the ball algebra $A(B_N)$ and form a bounded sequence of $H^p(B_N)$. Take a compact operator $\mathcal{K} : H^p(B_N) \to H^q(B_N)$ arbitrarily. Since the bounded sequence $\{f_w\}$ converges to 0 uniformly on compact subsets of B_N as $|w| \to 1^-$, we have $\|\mathcal{K} f_w\|_{H^q} \to 0$ as $|w| \to 1^-$. Thus we obtain

$$\left\| uC_\varphi - \mathcal{K} \right\|_{H^p \to H^q} \geq C \limsup_{|w| \to 1^-} \left\| (uC_\varphi - \mathcal{K}) f_w \right\|_{H^q} \geq C \limsup_{|w| \to 1^-} \left\| uC_\varphi f_w \right\|_{H^q}. \tag{3.3}$$

By the definition of f_w, we also see that

$$\left\| uC_\varphi f_w \right\|_{H^q}^q = \int_{\partial B_N} \left| u^*(\zeta) \right|^q \left\{ \frac{1 - |w|^2}{\left| 1 - \langle \varphi^*(\zeta), w \rangle \right|^2} \right\}^{qN/p} d\sigma(\zeta). \tag{3.4}$$

Combining this with (3.3), we get

$$\left\| uC_\varphi - \mathcal{K} \right\|_{H^p \to H^q}^q \geq C \limsup_{|w| \to 1^-} \int_{\partial B_N} \left| u^*(\zeta) \right|^q \left\{ \frac{1 - |w|^2}{\left| 1 - \langle \varphi^*(\zeta), w \rangle \right|^2} \right\}^{qN/p} d\sigma(\zeta). \tag{3.5}$$

Since this holds for every compact operator $\mathcal{K}$, it follows that

$$\left\| uC_\varphi \right\|_{e, H^p \to H^q}^q \geq C \limsup_{|w| \to 1^-} \int_{\partial B_N} \left| u^*(\zeta) \right|^q \left\{ \frac{1 - |w|^2}{\left| 1 - \langle \varphi^*(\zeta), w \rangle \right|^2} \right\}^{qN/p} d\sigma(\zeta). \tag{3.6}$$

Furthermore, we put $w = (1 - t)\zeta$ for each t, $0 < t < 1$ and $\zeta \in \partial B_N$ in the definition of f_w. Since we see that $|f_{(1-t)\zeta}(z)| \geq C t^{-qN/p}$ for all $z \in S(\zeta, t)$, we have

$$C \sup_{\zeta \in \partial B_N} \frac{\mu_{\varphi,u}(S(\zeta, t))}{t^{qN/p}} \leq \sup_{\zeta \in \partial B_N} \int_{S(\zeta,t)} \left| f_{(1-t)\zeta} \right|^q d\mu_{\varphi,u} \leq \sup_{\zeta \in \partial B_N} \left\| uC_\varphi f_{(1-t)\zeta} \right\|_{H^q}^q. \tag{3.7}$$

Letting $t \to 0^+$, we get

$$C \limsup_{t \to 0^+} \sup_{\zeta \in \partial B_N} \frac{\mu_{\varphi,u}(S(\zeta, t))}{t^{qN/p}} \leq \limsup_{t \to 0^+} \sup_{\zeta \in \partial B_N} \left\| uC_\varphi f_{(1-t)\zeta} \right\|_{H^q}^q. \tag{3.8}$$

Combining this with (3.6), we obtain

$$C \limsup_{t \to 0^+} \sup_{\zeta \in \partial B_N} \frac{\mu_{\varphi,u}(S(\zeta, t))}{t^{qN/p}} \leq \left\| uC_\varphi \right\|_{e, H^p \to H^q}^q, \tag{3.9}$$

completing the proof of the lower estimates. □

To prove the upper estimates, we need some technical results about the polynomial approximation of $f \in H^p(B_N)$. Recall that a holomorphic function f in B_N has the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^{\gamma}, \tag{3.10}$$

where $\gamma = (\gamma_1, \dots, \gamma_N)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_N$, and $z^{\gamma} = z_1{}^{\gamma_1} \cdots z_N{}^{\gamma_N}$. For the homogeneous expansion of f and any integer $n \geq 1$, let

$$R_n f(z) = \sum_{k=n}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^{\gamma}, \tag{3.11}$$

and $K_n = I - R_n$, where $If = f$ is the identity operator.

Proposition 3.2. *Suppose that X is a Banach space of holomorphic functions in B_N with the property that the polynomials are dense in X. Then $\|K_n f - f\|_X \to 0$ as $n \to \infty$ if and only if* $\sup\{\|K_n\| : n \geq 1\} < \infty$.

Proof. We see that [20, Proposition 1] also holds if we replace the unit disc with the unit ball. So we omit the proof of this proposition. □

Proposition 3.3. *If $1 < p < \infty$, then $\|K_n f - f\|_{H^p} \to 0$ as $n \to \infty$ for each $f \in H^p(B_N)$.*

Proof. For each $f \in H^p(B_N)$ and r, $0 < r < 1$, the slice function $(f_r)_{\zeta}$ ($\zeta \in \partial B_N$) of f_r is in the disc algebra $A(\mathbb{D})$. Here, f_r denotes the dilated function of f, that is $f_r(z) = f(rz)$. Hence [20, Corollary 3 and Proposition 1] implies that there is a positive constant C_p such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| K_n (f_r)_{\zeta} (e^{i\theta}) \right|^p d\theta \leq C_p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| (f_r)_{\zeta} (e^{i\theta}) \right|^p d\theta, \tag{3.12}$$

for every integer $n \geq 1$. Since $K_n (f_r)_{\zeta} (e^{i\theta}) = K_n f(re^{i\theta}\zeta)$, integration by slices (see [17, page 15, Proposition 1.4.7.]) shows

$$\int_{\partial B_N} \left| K_n f(r\zeta) \right|^p d\sigma(\zeta) \leq C_p \int_{\partial B_N} \left| f(r\zeta) \right|^p d\sigma(\zeta), \tag{3.13}$$

that is, $\|K_n\| \leq C_p^{1/p}$ for every integer $n \geq 1$. By Proposition 3.2, we see $\|K_n f - f\|_{H^p} \to 0$ as $n \to \infty$. This completes the proof of the proposition. □

Corollary 3.4. *If $1 < p < \infty$, then R_n converges to 0 pointwise in $H^p(B_N)$ as $n \to \infty$. Moreover,* $\sup\{\|R_n\| : n \geq 1\} < \infty$.

Proof. Since $R_n f = f - K_n f$, Proposition 3.3 shows that $\|R_n f\|_p \to 0$ as $n \to \infty$. Furthermore, the principle of uniform boundedness implies that $\sup_{n \geq 1} \|R_n\| < \infty$. □

Lemma 3.5. *Let $1 < p < \infty$. For each $f \in H^p(B_N)$ and $n \geq 1$,*

$$\left| R_n f(z) \right| \leq \|f\|_{H^p} \sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k! \Gamma(N)} |z|^k. \tag{3.14}$$

Proof. Let K_w be the reproducing kernel for $H^2(B_N)$ and let $C[f]$ be the Cauchy-Szegö projection. Then, the orthogonality of monomials ζ^α implies that

$$R_n f(z) = C[R_n f](z) = \int_{\partial B_N} R_n f(\zeta)\overline{K_z(\zeta)}\,d\sigma(\zeta) = \int_{\partial B_N} f(\zeta)\overline{R_n K_z(\zeta)}\,d\sigma(\zeta). \tag{3.15}$$

Hölder's inequality and the expansion of $K_z(w)$ give

$$\begin{aligned}|R_n f(z)| &\le \int_{\partial B_N} |f(\zeta)||R_n K_z(\zeta)|\,d\sigma(\zeta)\\ &\le \left\{\int_{\partial B_N} |f(\zeta)|^p\,d\sigma(\zeta)\right\}^{1/p}\left\{\int_{\partial B_N} |R_n K_z(\zeta)|^q\,d\sigma(\zeta)\right\}^{1/q}\\ &\le \|f\|_{H^p}\sum_{k=n}^{\infty}\frac{\Gamma(N+k)}{k!\Gamma(N)}|z|^k.\end{aligned} \tag{3.16}$$

This completes the proof. □

The following lemma is well known in the case of functional Hilbert spaces (cf. [4, 21]). As in the proof of [21, Lemma 3.16], an elementary argument verifies Lemma 3.6.

Lemma 3.6. *Let $1 < p \le q < \infty$. If uC_φ is bounded from $H^p(B_N)$ into $H^q(B_N)$, then*

$$\|uC_\varphi\|_{e,H^p\to H^q} \le \liminf_{n\to\infty}\|uC_\varphi R_n\|_{H^p\to H^q}. \tag{3.17}$$

Let us prove the upper estimates for the essential norm of uC_φ.

Proof of the upper estimates. For the sake of convenience, we set

$$M_1 = \limsup_{|w|\to 1^-}\int_{\partial B_N} |u^*(\zeta)|^q\left\{\frac{1-|w|^2}{|1-\langle\varphi^*(\zeta),w\rangle|^2}\right\}^{(qN/p)} d\sigma(\zeta), \tag{3.18}$$

$$M_2 = \limsup_{t\to 0^+}\sup_{\zeta\in\partial B_N}\frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}}, \tag{3.19}$$

$$D(\zeta,t) = \left\{z\in\overline{B_N} : |z| > 1-t,\ \frac{z}{|z|}\in Q(\zeta,t)\right\}. \tag{3.20}$$

By the notation (3.18), for given $\varepsilon > 0$, we can choose an R_1, $0 < R_1 < 1$ such that

$$\int_{\partial B_N} |u^*(\zeta)|^q\left\{\frac{1-|w|^2}{|1-\langle\varphi^*(\zeta),w\rangle|^2}\right\}^{qN/p} d\sigma(\zeta) < M_1 + \varepsilon, \tag{3.21}$$

for $w \in B_N$ with $|w| \ge R_1$. For each $\zeta \in \partial B_N$ and t, $0 < t \le 1 - R_1 \equiv t_1$, we put $w_1 = (1-t)\zeta$. Since the function $f_{w_1}(z) = \{(1-|w_1|^2)/(1-\langle z,w_1\rangle)^2\}^{N/p}$ satisfies $|f_{w_1}(z)|^p > 4^{-N}t^{-N}$ for all

$z \in S(\zeta, t)$, the inequality (3.21) implies that

$$\frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}} < C\int_{S(\zeta,t)} \left|f_{w_1}(z)\right|^q d\mu_{\varphi,u}(z) < C(M_1+\varepsilon) \tag{3.22}$$

for all $\zeta \in \partial B_N$ and all $t,\ 0 < t \le t_1$.

By the notation (3.19), we can also choose a $t_2,\ 0 < t_2 < 1$, so that

$$\sup_{\zeta\in\partial B_N} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}} < M_2 + \varepsilon \tag{3.23}$$

for all $t,\ 0 < t \le t_2$. Let μ_1 and μ_2 be the restrictions of $\mu_{\varphi,u}$ to $\overline{B_N} \setminus (1-t_1)\overline{B_N}$ and $\overline{B_N} \setminus (1-t_2)\overline{B_N}$, respectively. We claim that $\mu_j\ (j = 1, 2)$ also satisfies the Carleson measure condition

$$\mu_j(S(\zeta,t)) \le C(M_j+\varepsilon)t^{qN/p} \tag{3.24}$$

for all $\zeta \in \partial B_N$ and $t > 0$. By (3.22) or (3.23), these conditions are true for all $t,\ 0 < t \le t_j$. Hence, we assume that $t > t_j$. For a finite cover $\{Q(w_k, t_j/3)\}$, where $w_k \in Q(\zeta, t)$ of the set $\overline{Q}(\zeta,t) = \{z \in \partial B_N : |1 - \langle z, \zeta\rangle| \le t\}$, the covering property implies that there exists a disjoint subcollection Γ of $\{Q(w_k, t_j/3)\}$ so that

$$Q(\zeta,t) \subset \bigcup_{\Gamma} Q(w_k, t_j). \tag{3.25}$$

Furthermore, we obtain $\operatorname{card}(\Gamma) \le C(t/t_j)^N$. By the notation (3.20), we have

$$\begin{aligned} \mu_j(S(\zeta,t)) &\le \mu_j(D(\zeta,t)) \le \sum_{\Gamma} \mu_j(D(w_k,t_j)) \\ &\le \sum_{\Gamma} \mu_j(S(w_k,2t_j)) \le C\left(\frac{t}{t_j}\right)^N (M_j+\varepsilon) {t_j}^{qN/p} \\ &= C(M_j+\varepsilon)t^N {t_j}^{(q/p-1)N} \le C(M_j+\varepsilon)t^{qN/p}, \end{aligned} \tag{3.26}$$

where the constant C depends only on p, q, and the dimension N.

Now, we take a function $f \in H^p(B_N)$ with $\|f\|_{H^p} \le 1$. By Lemma 2.6, we have

$$\begin{aligned} \|uC_\varphi R_n f\|_{H^q}^q &= \int_{\partial B_N} \left|u^*(R_n f^* \circ \varphi^*)\right|^q d\sigma \\ &= \int_{\overline{B_N}} \left|R_n f^*\right|^q d\mu_{\varphi,u} \\ &= \int_{\overline{B_N}} \left|R_n f^*\right|^q d\mu_j + \int_{(1-t_j)\overline{B_N}} \left|R_n f\right|^q d\mu_{\varphi,u} \end{aligned} \tag{3.27}$$

for all integers $n \ge 1$. Condition (3.24) and Lemma 2.3 implies that

$$\int_{\overline{B_N}} \left|R_n f^*\right|^q d\mu_j \le C(M_j+\varepsilon)\|R_n f\|_{H^p}^q \le C \sup_{n\ge1}\|R_n\|^q (M_j+\varepsilon). \tag{3.28}$$

On the other hand, by Lemma 3.5, we have

$$\int_{(1-t_j)\overline{B_N}} \left|R_n f\right|^q d\mu_{\varphi,u} \le \|f\|_{H^p}^q \left\{ \sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k!\Gamma(N)} \left|1-t_j\right|^k \right\}^q \|u\|_{H^q}^q. \tag{3.29}$$

The boundedness of uC_φ implies that $u \in H^q(B_N)$ and the convergence of the series $\sum(\Gamma(N+k)/k!\Gamma(N))|1-t_j|^k$ implies that

$$\sum_{k=n}^{\infty} \frac{\Gamma(N+k)}{k!\Gamma(N)} \left|1-t_j\right|^k \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

So we obtain

$$\int_{(1-t_j)\overline{B_N}} \left|R_n f\right|^q d\mu_{u,\varphi} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.31}$$

Combining (3.27), (3.28), and (3.31) with Lemma 3.6, we have

$$\left\|uC_\varphi\right\|_{e,H^p\to H^q}^q \le \liminf_{n\to\infty} \left\|uC_\varphi R_n\right\|_{H^p\to H^q}^q \le C \sup_{n\ge 1} \left\|R_n\right\|^q \left(M_j + \varepsilon\right). \tag{3.32}$$

Since Corollary 3.4 implies that $\sup_{n\ge1}\|R_n\| < \infty$, and $\varepsilon > 0$ was arbitrary, we conclude that

$$\|uC_\varphi\|_{e,H^p\to H^q}^q \le \begin{cases} C \limsup\limits_{|w|\to 1^-} \displaystyle\int_{\partial B_N} \left|u^*(\zeta)\right|^q \left\{ \frac{1-|w|^2}{\left|1-\langle \varphi^*(\zeta), w\rangle\right|^2} \right\}^{qN/p} d\sigma(\zeta), \\ C \limsup\limits_{t\to 0^+} \sup\limits_{\zeta\in\partial B_N} \dfrac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}}, \end{cases} \tag{3.33}$$

which were to be proved. □

Corollary 3.7 (see [15]). *Suppose that* $1 < p \le q < \infty$. *For the bounded weighted composition operator* $uC_\varphi : H^p(B_N) \to H^q(B_N)$, *the following conditions are equivalent:*

(a) $uC_\varphi : H^p(B_N) \to H^q(B_N)$ *is compact;*

(b) u *and* φ *satisfy*

$$\lim_{|w|\to 1^-} \int_{\partial B_N} \left|u^*(\zeta)\right|^q \left\{ \frac{1-|w|^2}{\left|1-\langle \varphi^*(\zeta), w\rangle\right|^2} \right\}^{qN/p} d\sigma(\zeta) = 0; \tag{3.34}$$

(c) u *and* φ *satisfy*

$$\lim_{t\to 0^+} \sup_{\zeta\in\partial B_N} \frac{\mu_{\varphi,u}(S(\zeta,t))}{t^{qN/p}} = 0. \tag{3.35}$$

4. Multiplication operators between Hardy spaces

In this section, we consider the compact multiplication operator M_u between Hardy spaces. As a consequence of Theorem 3.1, we obtain the following results.

Corollary 4.1. *Suppose that $1 < p \leq q < \infty$. For the bounded multiplication operator $M_u : H^p(B_N) \to H^q(B_N)$, the following inequality holds:*

$$\|M_u\|^q_{e,H^p \to H^q} \sim \limsup_{|w| \to 1^-} \int_{\partial B_N} |u^*(\zeta)|^q \left\{ \frac{1-|w|^2}{|1-\langle \zeta, w\rangle|^2} \right\}^{qN/p} d\sigma(\zeta). \tag{4.1}$$

Furthermore, $M_u : H^p(B_N) \to H^q(B_N)$ is compact if and only if

$$\lim_{|w| \to 1^-} \int_{\partial B_N} |u^*(\zeta)|^q \left\{ \frac{1-|w|^2}{|1-\langle \zeta, w\rangle|^2} \right\}^{qN/p} d\sigma(\zeta) = 0. \tag{4.2}$$

By using Corollary 4.1, we can completely characterize the compactness of a multiplication operator M_u from $H^p(B_N)$ into $H^q(B_N)$.

Theorem 4.2. *Suppose that $1 < p \leq q < \infty$. Then $M_u : H^p(B_N) \to H^q(B_N)$ is compact if and only if $u = 0$ in B_N.*

Proof. If $u \equiv 0$, then M_u is compact. Thus, we only prove that the compactness of M_u implies $u \equiv 0$. The boundedness of M_u implies that $u \in H^q(B_N)$. Hence, the Poisson representation for u gives that

$$u(w) = \int_{\partial B_N} u^*(\zeta) P(w,\zeta)\, d\sigma(\zeta) \quad (w \in B_N), \tag{4.3}$$

where $P(w,\zeta)$ is the Poisson kernel. Hölder's inequality shows that

$$\begin{aligned} |u(w)| &\leq \int_{\partial B_N} |u^*(\zeta)| P(w,\zeta)\, d\sigma(\zeta) \\ &\leq \left\{ \int_{\partial B_N} |u^*(\zeta)|^q P(w,\zeta)^{q/p}\, d\sigma(\zeta) \right\}^{1/q} \left\{ \int_{\partial B_N} P(w,\zeta)^{(1-1/p)q'}\, d\sigma(\zeta) \right\}^{1/q'}, \end{aligned} \tag{4.4}$$

where $1/q + 1/q' = 1$. By the assumption $1 < p \leq q < \infty$, we see that

$$s \equiv \left(1 - \frac{1}{p}\right) q' = \frac{q(p-1)}{p(q-1)} \leq \frac{pq-p}{p(q-1)} = 1, \tag{4.5}$$

and so we have

$$\int_{\partial B_N} P(w,\zeta)^{(1-1/p)q'}\, d\sigma(\zeta) \leq \left\{ \int_{\partial B_N} \{P(w,\zeta)^s\}^{1/s}\, d\sigma(\zeta) \right\}^s = 1. \tag{4.6}$$

Inequality (4.4) and Corollary 4.1 give that $\lim_{|w|\to 1^-}|u(w)| = 0$. Since $u \in H^q(B_N)$, this implies that u has a K-limit 0 on a set of positive σ-measure in ∂B_N. Hence [17, page 83, Theorem 5.5.9] shows that $u \equiv 0$. This completes the proof. □

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