Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 62852, 17 pages
doi:10.1155/2007/62852

Research Article

On Weighted Hadamard-Type Singular Integrals and Their Applications

Yong Jia Xu

Received 18 September 2006; Accepted 16 January 2007

Recommended by Allan C. Peterson

By means of an expression with a kind of integral operators, some properties of the weighted Hadamard-type singular integrals are revealed. As applications, the solution for certain strongly singular integral equations is discussed and illustrated.

Copyright © 2007 Yong Jia Xu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The concept of Hadamard-type singular integrals was first introduced by Hadamard [1], and then developed and adopted in applications by many authors (see [2–13]). This type of integrals is expressed as

$$\text{f.p.} \int_\Gamma \frac{f(\tau)}{(\tau-t)^{m+1}}d\tau, \quad t \in \Gamma^0 \tag{1.1}$$

and its general definition can be found in Lu [11], where $\Gamma = \overset{\frown}{ab}$ is an open smooth curve on a complex plane, $f \in C^{m+1}(\Gamma)$, and m is a positive integer. Although from

$$\text{f.p.} \int_\Gamma \frac{f(\tau)d\tau}{(\tau-t)^{m+1}} = \frac{1}{m!}\text{p.v.} \int_\Gamma \frac{f^{(m)}(\tau)}{\tau-t}d\tau + \sum_{r=0}^{m-1} \frac{(m-r-1)!}{m!}\left[\frac{f^{(r)}(a)}{(a-t)^{m-r}} - \frac{f^{(r)}(b)}{(b-t)^{m-r}}\right] \tag{1.2}$$

we would get some characteristics of this type of integrals, many of them such as mapping properties still need to have a further investigation, especially for its "weighted type":

$$\text{f.p.} \int_\Gamma \frac{w(\tau)f(\tau)}{(\tau-t)^{m+1}}d\tau, \quad t\in\Gamma^0, \tag{1.3}$$

where $w(t)$ is an integrable function. In many cases, $w(t)$ is a fundamental function derived from some mixed boundary problem and therefore may not be smooth enough or even have certain singularities (see [6, 10, 14]).

It is found that Hadamard-type singular integrals can be expressed effectively by a kind of integral operators which we will define and discuss in the next section. So, in Section 3 of the present paper we directly use this expression as the definition of Hadamard-type singular integrals and it appears that the definition is more advantageous than the traditional one. In this paper, some useful results are developed and then in the final section we use them for the solution of certain strongly singular integral equations. Meanwhile, we illustrate some examples as well.

Throughout the paper we always assume that m is a nonnegative integer; Γ is an open smooth curve on the complex plane oriented from the point a to the point b, and c_0 is a fixed positive constant such that for all $t_1,t_2\in\Gamma$, the arc length $|\overset{\frown}{t_1t_2}|\le c_0|t_1-t_2|$; as usual, $C(\Gamma)$ and $C^m(\Gamma)$ denote the spaces of continuous and m-times continuously differentiable complex-valued functions on Γ, respectively, $\|\psi\|\equiv\max_{t\in\Gamma}|\psi(t)|$, $\|\psi\|_{C^m}\equiv\sum_{k=0}^m\|\psi^{(k)}\|$, and the modulus of continuity for $\psi\in C(\Omega)$ is denoted by $\omega(\psi,x)$, where $\Omega=\Gamma$ or $\Gamma\times\Gamma$; for convenience, each absolute constant is denoted by c but takes different values in different places. And, if there is no confusion, we will omit the symbol Γ in some notations of function classes such as $C(\Gamma)$ and $C^m(\Gamma)$, and so forth.

2. Some integral inequalities

It is clear that the kind of integral operators introduced in [15] has a close relation to Hadamard-type integrals. Here we restate their definition as follows.

Definition 2.1. Let w and φ be integrable functions on Γ and assume φ is m times differentiable at $t_0\in\Gamma$. If the integral

$$k!\int_\Gamma w(\tau)\frac{\varphi(\tau)-P_k(\varphi;\tau,t_0)}{(\tau-t_0)^{k+1}}d\tau \tag{2.1}$$

exists, then we denote it by $T_k(w,\varphi)(t_0)$, where

$$P_k(\varphi;\tau,t_0)=\varphi(t_0)+\varphi'(t_0)(\tau-t_0)+\frac{\varphi''(t_0)}{2!}(\tau-t_0)^2+\cdots+\frac{\varphi^{(k)}t_0}{k!}(\tau-t_0)^k \tag{2.2}$$

and $k=0,1,\dots,m$. If $k=0$, T_k is written as T.

In this section, we will mainly discuss this kind of operators in some smooth function classes, which are given by the following definition.

Definition 2.2. Let γ be an oriented open smooth curve and let $\Lambda_n(\gamma)$ denote the function class

$$\left\{f \in C(\gamma) : \int_0^1 \frac{\omega_\gamma(f,x)}{x} \ln^{n-1}\frac{1}{x}dx < \infty\right\}, \tag{2.3}$$

where $\omega_\gamma(\varphi,x) = \max\{|\varphi(t') - \varphi(t'')| : |t' - t''| \le x,\ t',t'' \in \gamma\}$ and n is a positive integer. For differentiable function classes, we let

$$\Lambda_n^m(\gamma) = \{f \in C^m(\gamma) : f^{(m)} \in \Lambda_n(\gamma)\}. \tag{2.4}$$

For $t_0 \in \Gamma$, if we say $f \in C^m(t_0,\Gamma)$ or $f \in \Lambda_n^m(t_0,\Gamma)$, it means that f is an integrable function on Γ and there is a neighborhood $\mathbb{O} \subset \Gamma$ of t_0 such that $f \in C^m(\mathbb{O})$ or $f \in \Lambda_n^m(\mathbb{O})$.

Some properties of modulus of continuity will be used repeatedly and we list them in the following lemma. Their proofs are trivial (cf. [16, Chapter 3]).

Lemma 2.3. *Let $\omega(x)$ be a modulus of continuity. Then*

$$\omega(x) \le \frac{2}{\ln 2}\int_0^x \frac{\omega(y)}{y}dy, \quad x > 0, \tag{2.5}$$

$$\omega(x)\ln\frac{1}{x} \le 2\int_x^{\sqrt{x}} \frac{\omega(y)}{y}dy, \quad 0 < x \le 1, \tag{2.6}$$

$$x\int_x^l \frac{\omega(y)}{y^2}dy \le 2\omega(x)\ln\frac{l}{x}, \quad 0 < x \le l, \tag{2.7}$$

$$\int_0^l \frac{\omega(y)}{y}dy \le (l+1)\int_0^1 \frac{\omega(y)}{y}dy, \quad l > 1. \tag{2.8}$$

Now we suppose $\varphi \in C(\Gamma) \cap C^m(\gamma)$, and set $\Psi_k(\tau,t) = k!(\varphi(\tau) - P_k(\varphi;\tau,t))/(\tau - t)^{k+1}$, where γ is a subarc of Γ and $(\tau,t) \in \Gamma \times \gamma$. Then it is easy to verify that

$$\frac{\partial}{\partial t}\Psi_k(\tau,t) = \Psi_{k+1}(\tau,t), \quad k = 0,1,\dots,m-1 \tag{2.9}$$

for $(\tau,t) \in \Gamma \times \gamma$ but $\tau \ne t$, and

$$|\Psi_k(\tau,t)| \le c\begin{cases} \max\limits_{\tau\in\gamma} |\varphi^{(k+1)}(\tau)|, & 0 \le k \le m-1, \\ \dfrac{\omega_\gamma(\varphi^{(m)}, |\tau - t|)}{|\tau - t|}, & k = m, \end{cases} \tag{2.10}$$

for $(\tau,t) \in \gamma \times \gamma$ (cf. [15]). So, if $\varphi \in \Lambda_1^m(\Gamma)$, $T_k\varphi = T_k(w,\varphi)$ is differentiable when $k = 0,1,\dots$ or $m-1$ and $T_m\varphi$ is integrable. Furthermore, we have the following theorem.

Theorem 2.4. *Let w be an integrable function on Γ, $\varphi \in \Lambda_1^m(\Gamma)$, and $0 < x \le 1$. If w is bounded, then*

$$\omega(T_m\varphi,x) \le c\|w\|\left(\omega(\varphi^{(m)},x)\ln\frac{1}{x} + \int_0^x \frac{\omega(\varphi^{(m)},y)}{y}dy\right), \tag{2.11}$$

and if $w \in \Lambda_1(\Gamma)$ *satisfying* $w(a) = w(b) = 0$, *then*

$$\omega(T_m\varphi, x) \leq c\|w\|_{\Lambda_1}\left(\int_0^x \frac{\omega(\varphi^{(m)}, y)}{y}dy + x\int_x^1 \frac{\omega(\varphi^{(m)}, y)}{y^2}dy\right), \tag{2.12}$$

where $\|w\|_{\Lambda_1} = \|w\| + \int_0^1(\omega(w,y)/y)dy$ *and* c *is a positive number related to* m *and* Γ.

Proof. It is equivalent to prove that for $t_1, t_2 \in \Gamma$

$$|T_m(\varphi)(t_1) - T_m(\varphi)(t_2)| \leq c\|w\|\left(\omega(\varphi^{(m)}, \delta)\ln\frac{L}{\delta} + \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y}dy\right) \tag*{(2.9)$'$}$$

if w is bounded and

$$|T_m(\varphi)(t_1) - T_m(\varphi)(t_2)| \leq c\|w\|_{\Lambda_1}\left(\int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y}dy + \delta\int_\delta^L \frac{\omega(\varphi^{(m)}, y)}{y^2}dy\right) \tag*{(2.10)$'$}$$

if $w \in \Lambda_1(\Gamma)$ satisfying $w(a) = w(b) = 0$, where $\delta = |t_2 - t_1|$ and $L = |\Gamma|$. For convenience, we assume $0 < \delta < 1$ and $a \prec t_1 \prec t_2 \prec b$. Here $t_1 \prec t_2$ means that t_1 precedes t_2.

(i) If $|t_1 - a| > \delta$ and $|b - t_2| \leq \delta$ or $|t_1 - a| \leq \delta$ and $|b - t_2| > \delta$, we let

$$T_m(\varphi)(t_1) - T_m(\varphi)(t_2) = m!\left(\int_{\widehat{at_1}} + \int_{\widehat{t_1b}}\right)w(\tau)[\Psi_m(\tau,t_2) - \Psi_m(\tau,t_1)]d\tau = I_1 + I_2. \tag{2.13}$$

Because of the similarity, we assume $|t_1 - a| > \delta$ and $|b - t_2| \leq \delta$. In this case, from (2.10) and $|\widehat{t_1b}| = |\widehat{t_1t_2}| + |\widehat{t_2b}| \leq 2c_0\delta$, we have

$$\begin{aligned} |I_2| &= m!\left|\int_{\widehat{t_1b}} w(\tau)[\Psi_m(\tau,t_2) - \Psi_m(\tau,t_1)]d\tau\right| \\ &\leq c\|w\|\int_{\widehat{t_1b}} \frac{\omega(\varphi^{(m)}, |\tau - t_2|)}{|\tau - t_2|} + \frac{\omega(\varphi^{(m)}, |\tau - t_1|)}{|\tau - t_1|}|d\tau| \\ &\leq c\|w\|\int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y}dy. \end{aligned} \tag{2.14}$$

If we let

$$h(\tau) = P_m(\varphi;\tau,t_1) - P_m(\varphi;\tau,t_2), \quad \tau \in \Gamma, \tag{2.15}$$

then

$$h(\tau) = h(t_2) + \frac{h'(t_2)}{1!}(\tau - t_2) + \cdots + \frac{h^{(m)}(t_2)}{m!}(\tau - t_2)^m, \tag{2.16}$$

$$\frac{m!}{k!}|h^{(k)}(t_2)| \leq c\binom{m}{k}\omega(\varphi^{(m)}, \delta)\delta^{m-k} \tag{2.17}$$

for $k = 1, 2, \ldots, m$, therefore

$$
\begin{aligned}
i_1 &\equiv m! \left| \int_{\widehat{at_1}} w(\tau) \frac{P_m(\varphi;\tau,t_2) - P_m(\varphi;\tau,t_1)}{(\tau - t_2)^{m+1}} d\tau \right| \\
&\leq m! \sum_{k=0}^{m} \frac{|h^{(k)}|(t_2)}{k!} \int_{\widehat{at_1}} \frac{|w(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau| \\
&\leq c\omega(\varphi^{(m)}, \delta) \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k} \int_{\widehat{at_1}} \frac{|w(\tau)|}{|\tau - t_2|^{m-k+1}} |d\tau|.
\end{aligned}
\tag{2.18}
$$

For $\tau \in \widehat{at_1}$, $c_0|\tau - t_2| \geq |\widehat{\tau t_2}| = |\widehat{\tau t_1}| + |\widehat{t_1 t_2}| \geq |t_2 - t_1| = \delta$, that is, $\delta/|\tau - t_2| \leq c_0$, so that the above inequality becomes

$$
i_1 \leq c\omega(\varphi^{(m)}, \delta) \int_{\widehat{at_1}} \frac{|w(\tau)|}{|\tau - t_2|} |d\tau|. \tag{2.19}
$$

If w is bounded, then, by some computation, we have

$$
i_1 \leq c\|w\| \omega(\varphi^{(m)}, \delta) \ln \frac{L}{\delta}. \tag{2.20}
$$

Since $|b - t_2| \leq |t_2 - t_1| \leq c_0|\tau - t_2|$, $|\tau - b| \leq |\tau - t_2| + |t_2 - b| \leq (1 + c_0)|\tau - t_2|$ and it follows that, if $w \in \Lambda_1$ and $w(b) = 0$,

$$
\begin{aligned}
\int_{\widehat{at_1}} \frac{|w(\tau)|}{|\tau - t_2|} |d\tau| &= \int_{\widehat{at_1}} \frac{|w(\tau) - w(b)|}{|\tau - t_2|} |d\tau| \\
&\leq \int_{\widehat{at_1}} \frac{\omega(w, (1 + c_0)|\tau - t_2|)}{|\tau - t_2|} |d\tau| \leq c \int_0^1 \frac{\omega(w, y)}{y} dy,
\end{aligned}
\tag{2.21}
$$

or

$$
i_1 \leq c \int_0^1 \frac{\omega(w, y)}{y} dy \, \omega(\varphi^{(m)}, \delta). \tag{2.22}
$$

On the other hand, by (2.10),

$$
\begin{aligned}
i_2 &\equiv m! \left| \int_{\widehat{at_1}} w(\tau)[\varphi(\tau) - P_m(\varphi;\tau,t_1)] \left[\frac{1}{(\tau - t_2)^{m+1}} - \frac{1}{(\tau - t_1)^{m+1}} \right] d\tau \right| \\
&\leq c\|w\| |t_1 - t_2| \sum_{k=0}^{m} \int_{\widehat{at_1}} \frac{\omega(\varphi^{(m)}, |\tau - t_1|) |\tau - t_1|^k}{|\tau - t_1| |\tau - t_2|^{k+1}} |d\tau| \\
&\leq c(m+1)\|w\| |t_1 - t_2| \int_{\widehat{at_1}} \frac{\omega(\varphi^{(m)}, |\tau - t_1|)}{|\tau - t_1| |\tau - t_2|} |d\tau| \\
&\leq c\|w\| \delta \int_0^L \frac{\omega(\varphi^{(m)}, y)}{y(y + \delta)} dy \\
&\leq c\|w\| \left(\int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} dy + \delta \int_\delta^L \frac{\omega(\varphi^{(m)}, y)}{y^2} dy \right),
\end{aligned}
\tag{2.23}
$$

where $L = |\Gamma|$. Now from

$$\begin{aligned} I_1 &= \left| m! \int_{\widehat{at_1}} w(\tau) \left[\frac{\varphi(\tau) - P_m(\varphi;\tau,t_2)}{(\tau-t_2)^{m+1}} - \frac{\varphi(\tau) - P_m(\varphi;\tau,t_1)}{(\tau-t_1)^{m+1}} \right] d\tau \right| \\ &= \left| m! \int_{\widehat{at_1}} w(\tau) \left\{ \frac{P_m(\varphi;\tau,t_1) - P_m(\varphi;\tau,t_2)}{(\tau-t_2)^{m+1}} \right. \right. \\ &\qquad \left. \left. + [\varphi(\tau) - P_m(\varphi:\tau,t_1)] \left(\frac{1}{(\tau-t_2)^{m+1}} - \frac{1}{(\tau-t_1)^{m+1}} \right) \right\} d\tau \right| \le i_1 + i_2, \end{aligned} \tag{2.24}$$

we have

$$I_1 \le c\|w\| \left(\omega(\varphi^{(m)},\delta) \ln \frac{L}{\delta} + \int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y} dy \right) \tag{2.25}$$

if w is bounded, where we have used the inequality (2.7), or

$$I_1 \le c\|w\|_{\Lambda_1} \left(\int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y} dy + \delta \int_\delta^L \frac{\omega(\varphi^{(m)},y)}{y^2} dy \right) \tag{2.26}$$

if $w \in \Lambda_1$ and $w(b) = 0$, where we have used the inequality (2.5), and together with (2.14), we obtain (2.9)$'$ and (2.10)$'$.

(ii) If $|t_1 - a| > \delta$ and $|b - t_2| > \delta$, we let

$$\begin{aligned} T_m(\varphi)(t_1) - T_m(\varphi)(t_2) &= m! \left(\int_{\widehat{at_1}} + \int_{\widehat{t_1t_2}} + \int_{\widehat{t_2b}} \right) w(\tau)[\Psi_{\mathrm{m}}(\tau,t_2) - \Psi_m(\tau,t_1)] d\tau \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{2.27}$$

Similar to the proof of (2.14),

$$|I_2| \le c\|w\| \int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y} dy. \tag{2.28}$$

We rewrite $I_1 + I_3$ as $i_1 + i_2$, where

$$\begin{aligned} i_1 &= m! \int_{\widehat{at_1}} w(\tau) \frac{P_m(\varphi;\tau,t_1) - P_m(\varphi;\tau,t_2)}{(\tau-t_2)^{m+1}} d\tau \\ &\quad + m! \int_{\widehat{t_2b}} w(\tau) \frac{P_m(\varphi;\tau,t_1) - P_m(\varphi;\tau,t_2)}{(\tau-t_1)^{m+1}} d\tau, \\ i_2 &= m! \int_{\widehat{at_1}} w(\tau)[\varphi(\tau) - P_m(\varphi;\tau,t_1)] \left(\frac{1}{(\tau-t_2)^{m+1}} - \frac{1}{(\tau-t_1)^{m+1}} \right) d\tau \\ &\quad + m! \int_{\widehat{t_2b}} w(\tau)[\varphi(\tau) - P_m(\varphi;\tau,t_2)] \left(\frac{1}{(\tau-t_2)^{m+1}} - \frac{1}{(\tau-t_1)^{m+1}} \right) d\tau. \end{aligned} \tag{2.29}$$

Similar to the proof of (2.23),

$$\left|i_2\right| \leq c\|w\|\left(\int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y}dy+\delta\int_\delta^L \frac{\omega(\varphi^{(m)},y)}{y^2}dy\right). \tag{2.30}$$

Using (2.16), we rewrite i_1 as

$$\begin{aligned} &m!\sum_{k=0}^{m-1}\left[\frac{h^{(k)}(t_2)}{k!}\int_{\widehat{at_1}}\frac{w(\tau)d\tau}{(\tau-t_2)^{m-k+1}}+\frac{h^{(k)}(t_1)}{k!}\int_{\widehat{t_2b}}\frac{w(\tau)d\tau}{(\tau-t_1)^{m-k+1}}\right] \\ &\quad+\left(\int_{\widehat{at_1}}\frac{w(\tau)}{\tau-t_2}d\tau+\int_{\widehat{t_2b}}\frac{w(\tau)}{\tau-t_1}d\tau\right)\left[\varphi^{(m)}(t_1)-\varphi^{(m)}(t_2)\right]=i_{11}+i_{12}. \end{aligned} \tag{2.31}$$

Notice that

$$\left|\int_{\widehat{at_1}}\frac{d\tau}{(\tau-t_2)^{m-k+1}}\right| \leq c\delta^{-m+k}, \qquad \left|\int_{\widehat{t_2b}}\frac{d\tau}{(\tau-t_1)^{m-k+1}}\right| \leq c\delta^{-m+k} \tag{2.32}$$

for $k=0,1,\dots,m-1$. Hence, by using the inequality (2.17), we have

$$\left|i_{11}\right| \leq c\|w\|\omega(\varphi^{(m)},\delta). \tag{2.33}$$

Similar to the proof of (2.20),

$$i_{12} \leq c\|w\|\omega(\varphi^{(m)},\delta)\ln\frac{L}{\delta}. \tag{2.34}$$

But if $w\in\Lambda_1$ and $w(a)=w(b)=0$, then

$$\left|\int_{\widehat{at_1}}\frac{w(\tau)}{\tau-t_2}d\tau+\int_{\widehat{t_2b}}\frac{w(\tau)}{\tau-t_1}d\tau\right| \leq c\left(\|w\|+\int_0^1\frac{\omega(w,y)}{y}dy\right) \tag{2.35}$$

(see [15, Section 6]), and thus

$$\left|i_{12}\right| \leq c\left(\|w\|+\int_0^1\frac{\omega(w,y)}{y}dy\right)\omega(\varphi^{(m)},\delta). \tag{2.36}$$

Now from $|i_1| \leq |i_{11}|+|i_{12}|$, we obtain

$$\left|i_1\right| \leq c\|w\|\omega(\varphi^{(m)},\delta)\left(\ln\frac{L}{\delta}+1\right) \tag{2.37}$$

if w is bounded and obtain

$$\left|i_1\right| \leq c\left(\|w\|+\int_0^1\frac{\omega(w,y)}{y}dy\right)\omega(\varphi^{(m)},\delta) \tag{2.38}$$

if $w\in\Lambda_1$ and $w(a)=w(b)=0$, and these, together with (2.28) and (2.30), lead to (2.9)$'$ and (2.10)$'$.

(iii) If $|t_1 - a| \leq \delta$ and $|b - t_2| \leq \delta$, then $|\Gamma| = |\widehat{at_1}| + |\widehat{t_1t_2}| + |\widehat{t_2b}| \leq 3c_0\delta$. From (2.10),

$$
\begin{aligned}
&\left| T_m(\varphi)(t_1) - T_m(\varphi)(t_2) \right| \\
&\quad \leq c\|w\| \int_\Gamma \left[\frac{\omega(\varphi^{(m)}, |\tau - t_2|)}{|\tau - t_2|} + \frac{\omega(\varphi^{(m)}, |\tau - t_1|)}{|\tau - t_1|} \right] |d\tau| \\
&\quad \leq c\|w\| \int_0^\delta \frac{\omega(\varphi^{(m)}, y)}{y} dy
\end{aligned}
\tag{2.39}
$$

and thus (2.9)′ and (2.10)′ are also valid.

Now we have proved that (2.9)′ and (2.10)′ are true in all cases and the constant $c > 0$ depending on m and Γ can be derived from the process of the proof.

The proof is completed. □

Generally, for T_k, $k = 0, 1, \ldots, m$, we have the following results.

Theorem 2.5. *Assume w, φ, and $w\varphi$ are all integrable and $t_0 \in \Gamma$. If w is bounded on some neighborhood of t_0 on Γ and $\varphi \in \Lambda_1^m(t_0, \Gamma)$, then $T(w, \varphi)$ is m-time continuously differentiable at t_0 and*

$$
T_k(w, \varphi)(t_0) = \frac{d^k}{dt^k} T(w, \varphi)(t_0)
\tag{2.40}
$$

for $k = 1, 2, \ldots, m$.

Proof. It is obvious that (2.40) is true for $k = 1, 2, \ldots, m-1$. So, we need only to prove

$$
\frac{d}{dt} T_{m-1}(w, \varphi)(t_0) = T_m(w, \varphi)(t_0)
\tag{2.41}
$$

and $T_m(w, \varphi)$ is continuous at t_0.

According to the given conditions, there is a subarc $\gamma \subset \Gamma$ with $t_0 \in \gamma$ but $t_0 \in \Gamma \setminus \gamma^0$ such that w is bounded on γ and $\varphi \in \Lambda_1^m(\gamma)$, where γ^0 denotes the inner points of γ. Write $T_{m-1}(\varphi)$ as

$$
\left(\int_{\Gamma \setminus \gamma} + \int_\gamma \right) w(\tau) \Psi_{m-1}(\tau, t) d\tau = I_1(t) + I_2(t).
\tag{2.42}
$$

Then $I_1(t)$ is continuously differentiable at t_0 and

$$
I_1'(t_0) = \int_{\Gamma \setminus \gamma} w(\tau) \Psi_m(\tau, t_0) d\tau.
\tag{2.43}
$$

For I_2, we consider

$$
\left(\int_{\gamma \setminus \widehat{t_0 t}} + \int_{\widehat{t_0 t}} \right) w(\tau) \left[\frac{\Psi_{m-1}(\tau, t) - \Psi_{m-1}(\tau, t_0)}{t - t_0} - \Psi_m(\tau, t_0) \right] d\tau = i_1 + i_2,
\tag{2.44}
$$

where $t \in \gamma$ and, without loss of generality, we assume $t_0 \prec t$. Notice that

$$\begin{aligned} i_1 &= \int_{\gamma \setminus \widehat{t_0 t}} w(\tau)\left[\frac{1}{t-t_0}\int_{\widehat{t_0 t}} [\Psi_m(\tau,\zeta) - \Psi_m(\tau,t_0)]d\zeta\right]d\tau \\ &= \frac{1}{t-t_0}\int_{\widehat{t_0 t}}\left[\int_{\gamma \setminus \widehat{t_0 t}} w(\tau)[\Psi_m(\tau,\zeta) - \Psi_m(\tau,t_0)]d\tau\right]d\zeta. \end{aligned} \tag{2.45}$$

By using Theorem 2.4 to the internal integral, we have

$$|i_1| \le c' \|w\|_\gamma \left[\omega(\varphi^{(m)},\delta)\left(1+\ln\frac{1}{\delta}\right) + \int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y}dy\right], \tag{2.46}$$

where $\|w\|_\gamma = \max_{t\in\gamma}|w(t)|$ and $\delta = |t-t_0|$. If $\tau \in \widehat{t_0 t}$, then $|\tau - t| \le c_0|t-t_0|$ and $|\tau - t_0| \le c_0|t-t_0|$, and from (2.10),

$$\begin{aligned} &\left|\frac{\Psi_{m-1}(\tau,t) - \Psi_{m-1}(\tau,t_0)}{t-t_0} - \Psi_m(\tau,t_0)\right| \\ &\quad = \left|\frac{[\Psi_m(\tau,t)(\tau-t)+\varphi^{(m)}(t)] - [\Psi_m(\tau,t_0)(\tau-t_0)+\varphi^{(m)}(t_0)]}{m(t-t_0)} - \Psi_m(\tau,t_0)\right| \\ &\quad \le c\left(\frac{\omega(\varphi^{(m)},|t-t_0|)}{|t-t_0|} + \frac{\omega(\varphi^{(m)},|\tau-t_0|)}{|\tau-t_0|}\right), \end{aligned} \tag{2.47}$$

so that

$$|i_2| \le c'\|w\|_\gamma\left(\omega(\varphi^{(m)},\delta) + \int_0^\delta \frac{\omega(\varphi^{(m)},y)}{y}dy\right). \tag{2.48}$$

Since $\varphi^{(m)} \in \Lambda_1(\gamma)$, (2.46), and (2.48) result in $i_1, i_2 \to 0$ when $t \to t_0$, it follows that

$$I_2'(t_0) = \int_\gamma w(\tau)\Psi_m(\tau,t_0)d\tau. \tag{2.49}$$

On the other hand, according to Theorem 2.4, $\int_\gamma w(\tau)\Psi_m(\tau,t)d\tau$ is continuous on γ because w is bounded on the subarc and $\varphi \in \Lambda_1^m(\gamma)$. Now we have proved that $T_{m-1}(w,\varphi)$ is differentiable at t_0 and there holds (2.41). □

The following corollaries can be verified easily.

Corollary 2.6. *If w is bounded and $\varphi \in \Lambda_1^m$, then $T(w,\varphi) \in C^m$ and*

$$||T(w,\varphi)||_{C^m} \le c||w||\left[\sum_{k=1}^m ||\varphi^{(k)}|| + \int_0^1 \frac{\omega(\varphi^{(m)},y)}{y}dy\right]. \tag{2.50}$$

Corollary 2.7. *If w is bounded and $\varphi \in \Lambda_{n+1}^m$, then $T(w,\varphi) \in \Lambda_n^m$ and*

$$||T(w,\varphi)||_{\Lambda_n^m} \le c||w||\left[\sum_{k=1}^m ||\varphi^{(k)}|| + \int_0^1 \frac{\omega(\varphi^{(m)},y)}{y}\ln^n\frac{1}{y}dy\right], \tag{2.51}$$

where n is a positive integer and $\|\cdot\|_{\Lambda_n^m}$ *is defined by*

$$\|\psi\|_{\Lambda_n^m} = \|\psi\|_{C^m} + \int_0^1 \frac{\omega(\psi^{(m)}, y)}{y} \ln^{n-1}\frac{1}{y}dy, \quad \psi \in \Lambda_n^m. \tag{2.52}$$

Remark 2.8. The space Λ_n^m normed by $\|\cdot\|_{\Lambda_n^m}$ is a Banach space and thus the above corollaries imply that, if the "weight" w is bounded, then $T(w,\cdot) \in \mathcal{L}(\Lambda_1^m, C^m)$ and $T(w,\cdot) \in \mathcal{L}(\Lambda_{n+1}^m, \Lambda_n^m)$ for $n \geq 1$, where $\mathcal{L}(X,Y)$ denotes the space of all bounded linear operators from Banach space X to Banach space Y.

Remark 2.9. Generally, the inequality (2.12) is called Zygmund-type inequality. In this case, if we consider the operator in $C^{m,\lambda}$, then $T(w,\cdot) \in \mathcal{L}(C^{m,\lambda}) \equiv \mathcal{L}(C^{m,\lambda}, C^{m,\lambda})$, or in detail,

$$||T(w,\varphi)||_{C^{m,\lambda}} \leq c\|w\|_{\Lambda_1}\|\varphi\|_{C^{m,\lambda}}, \tag{2.53}$$

where $C^{m,\lambda}$ is the space of functions in C^m whose mth derivative satisfies a Hölder condition with exponent $\lambda \in (0,1)$, and its norm is defined by

$$\|\psi\|_{C^{m,\lambda}} = \|\psi\|_{C^m} + \sup_{0<x\leq 1} \frac{\omega(\psi^{(m)}, x)}{x^\lambda} \tag{2.54}$$

(cf. [15] and [17, Chaptre II, Section 6]). The positive constant c in inequality (2.53) depends only on m, λ, and Γ.

3. Weighted Hadamard-type singular integrals

In this section, we start from the definition of a basic Hadamard-type or finite-part integral, and then give an expression for general ones by means of singularity deletion method (cf. [7, 11]). For convenience, we denote the Hadamard-type integrals of the form (1.3) by $H_m(w,f)(t)$.

Definition 3.1. For $t_0 \in \Gamma^0$, the Hadamard-type singular integral or finite part f.p. $\int_\Gamma (d\tau/(\tau - t_0)^{m+1})$ is defined by

$$\text{f.p.} \int_\Gamma \frac{d\tau}{(\tau - t_0)^{m+1}} = \frac{1}{m}\left[\frac{1}{(a-t_0)^m} - \frac{1}{(b-t_0)^m}\right]. \tag{3.1}$$

If letting $h_0(t) = \text{p.v.} \int_\Gamma (1/(\tau - t))d\tau$, $t \in \Gamma^0$, where p.v. means Cauchy principal value, then we have $h_0(t) = \ln((b-t)/(t-a))$ and

$$\text{f.p.} \int_\Gamma \frac{1}{(\tau - t)^{m+1}}d\tau = \frac{1}{m!}\frac{d^m}{dt^m}h_0(t), \quad t \in \Gamma^0. \tag{3.2}$$

So, the integral is well defined.

Definition 3.2. Let $t_0 \in \Gamma^0$ and $f \in C^m(t_0,\Gamma)$. If $T_m(1,f)(t_0)$ exists, then the m-order Hadamard-type singular integral or finite part integral of f is defined by

$$\text{f.p.} \int_\Gamma \frac{f(\tau)}{(\tau - t_0)^{m+1}} d\tau = \frac{1}{m!} T_m(1,f)(t_0) + \sum_{k=0}^{m} \frac{f^{(k)}(t_0)}{k!} h_{m-k}(t_0), \tag{3.3}$$

where $h_r(t) = \text{f.p.} \int_\Gamma (1/(\tau - t)^{r+1}) d\tau$ with $r = 1,2,\ldots$.

According to Theorem 2.5, if $f \in \Lambda_1^m(t_0,\Gamma)$ at $t_0 \in \Gamma^0$, then $T_m(1,f)$ exists on some neighborhood of t_0 and $T_m(1,f) \in C(t_0,\Gamma)$ as well. Thus, from (3.3), we obtain the following.

Theorem 3.3. *If $f \in \Lambda_1^m(t_0,\Gamma)$, then $H_m(1,f) \in C(t_0,\Gamma)$, where $t_0 \in \Gamma^0$.*

As a special case, we have the following.

Corollary 3.4. *If the function w is integrable on Γ and arbitrary times differentiable on Γ^0, then $H_r(1,w) \in C^\infty(\Gamma^0)$ and*

$$\textit{f.p.} \int_\Gamma \frac{w(\tau)}{(\tau - t)^{r+1}} d\tau = \frac{1}{r!} \frac{d^r}{dt^r} \left[\textit{p.v.} \int_\Gamma \frac{w(\tau)}{\tau - t} d\tau \right] \quad t \in \Gamma^0, \tag{3.4}$$

where $r = 0,1,\ldots$.

Now we consider "weighted" Hadamard-type integrals of the form $H_m(w,f)$.

Definition 3.5. Let w be an integrable function on Γ, $f \in C^m(t_0,\Gamma)$, and $t_0 \in \Gamma^0$. If $T_m(w,f)$, $h_{r,w} = H_r(1,w)$ with $r = 0,1,2,\ldots,m$ are all existent at t_0, then the m-order "weighted" Hadamard-type singular integral or finite part integral of f at t_0 is defined by

$$\text{f.p.} \int_\Gamma \frac{w(\tau)f(\tau)}{(\tau - t_0)^{m+1}} d\tau = \frac{1}{m!} T_m(w,f)(t_0) + \sum_{k=0}^{m} \frac{f^{(k)}(t_0)}{k!} h_{m-k,w}(t_0). \tag{3.5}$$

Theorem 3.6. *Assume the function w is integrable on Γ and arbitrary times differentiable on Γ^0. If $f \in \Lambda_1^m(\Gamma)$, then $H_m(w,f)$ exists on Γ^0. Furthermore, $H_m(w,f) \in C(\Gamma^0)$ and*

$$H_m(w,f)(t) = \frac{1}{m!} \frac{d^m}{dt^m} \left(\textit{p.v.} \int_\Gamma \frac{w(\tau)f(\tau)}{\tau - t} d\tau \right), \quad t \in \Gamma^0. \tag{3.6}$$

Proof. It is clear that $H_m(w,f)(t)$ exists at all $t \in \Gamma^0$ and $H_m(w,f) \in C(\Gamma^0)$ as well. In the following we give the proof of (3.6).

From the given conditions, by Theorem 2.5 and Corollary 3.4, we see that $T(w,\varphi) \in C^m(\Gamma^0)$ and $h_{0,w} \in C^\infty(\Gamma^0)$. Hence, from

$$\text{p.v.} \int_\Gamma \frac{w(\tau)f(\tau)}{\tau - t} d\tau = T(w,f)(t) + f(t)h_{0,w}(t), \tag{3.7}$$

we have

$$\frac{d^m}{dt^m} \left(\text{p.v.} \int_\Gamma \frac{w(\tau)f(\tau)}{\tau - t} d\tau \right) = T_m(w,f)(t) + \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(t) h_{0,w}^{(m-k)}(t), \quad t \in \Gamma^0. \tag{3.8}$$

Corollary 3.4 states that $h_{r,w}(t) = r!h_{0,w}^{(r)}(t)$ for $t \in \Gamma^0$. Thus, from (3.5) we obtain (3.6). □

In (3.5), for each r, $h_{r,w}$ is known. As will be seen in the following examples, these functions are simple in some cases. So, it is the operator T_m that plays a crucial role in Hadamard-type singular integrals.

Example 3.7. Let $w_1(t) = (t-a)^{-\alpha}(b-t)^{-\beta}$ with $0 < \alpha, \beta < 1$ and $\alpha+\beta = 1$. We have

$$\text{p.v.}\ \frac{1}{\pi}\int_\Gamma \frac{w_1(\tau)}{\tau - t}d\tau = \frac{\cos\pi\alpha}{\sin\pi\alpha}w_1(t), \tag{3.9}$$

(cf. [11]), and hence

$$\text{f.p.}\ \int_\Gamma \frac{w_1(\tau)f(\tau)}{(\tau-t)^{m+1}}d\tau = \frac{1}{m!}T_m(w_1,f)(t) + \frac{\pi\cos\pi\alpha}{m!\sin\pi\alpha}\sum_{k=0}^{m}\binom{m}{k}w_1^{(k)}(t)f^{(m-k)}(t) \tag{3.10}$$

for $f \in \Lambda_1^m$.

Now we let $\sigma(t) = (t-a)(b-t)$. From

$$\frac{\sigma(t)}{\tau - t} = \frac{\sigma(\tau)}{\tau - t} + (\tau + t) - (a+b) \tag{3.11}$$

we have

$$\sigma(t)T_m(w_1,f)(t) = T_m(\sigma w_1,f)(t) + m!\int_\Gamma w_1(\tau)p(\tau,t)\frac{f(\tau) - P_m(f;\tau,t)}{(\tau-t)^m}d\tau \tag{3.12}$$

and thus

$$|T_m(w_1,f)(t)| \le \frac{c}{|\sigma(t)|}\left(\|f^{(m)}\| + \int_0^1\frac{\omega(f^{(m)},y)}{y}dy\right), \quad t \in \Gamma^0, \tag{3.13}$$

where $p(\tau,t) = (\tau+t) - (a+b)$ and the constant c depends on w_1, Γ, and m. Therefore,

$$|H_m(w_1,f)(t)| \le c\left(\frac{\|f^{(m)}\|_{\Lambda_1}}{|\sigma(t)|} + \frac{\cos\pi\alpha}{\sin\pi\alpha}\left|\frac{w_1(t)}{\sigma^m(t)}\right|\|f\|_{C^m}\right), \quad t \in \Gamma^0. \tag{3.14}$$

If $\alpha = 1/2$, then $H_m(w_1,f)(t) = (1/m!)T_m(w_1,f)(t)$ and

$$\left|\text{f.p.}\ \int_\Gamma\frac{w_1(\tau)f(\tau)}{(\tau-t)^{m+1}}d\tau\right| \le \frac{c}{|\sigma(t)|}\left(\|f^{(m)}\| + \int_0^1\frac{\omega(f^{(m)},y)}{y}dy\right), \quad t \in \Gamma^0. \tag{3.15}$$

Example 3.8. Let $w_2(t) = (t-a)^{\alpha}(b-t)^{\beta}$ with $0 < \alpha, \beta < 1$ and $\alpha+\beta = 1$. Similarly, we have

$$\begin{aligned}
\text{p.v.}\ \frac{1}{\pi}\int_\Gamma\frac{w_2(\tau)}{\tau-t}d\tau &= -\frac{\cos\pi\alpha}{\sin\pi\alpha}w_2(t) - \frac{t-\alpha a-\beta b}{\sin\pi\alpha},\\
\text{f.p.}\ \int_\Gamma\frac{w_2(\tau)f(\tau)}{(\tau-t)^{m+1}}d\tau &= \frac{1}{m!}T_m(w_2,f)(t) - \frac{\pi\cos\pi\alpha}{m!\sin\pi\alpha}\sum_{k=0}^{m}\binom{m}{k}w_2^{(k)}(t)f^{(m-k)}(t)\\
&\quad - \frac{\pi}{m!\sin\pi\alpha}\left(mf^{(m-1)}(t) + (t-\alpha a-\beta b)f^{(m)}(t)\right)
\end{aligned} \tag{3.16}$$

for $f \in \Lambda_1^m$. In this case

$$\left|\text{f.p.} \int_\Gamma \frac{w_2(\tau)f(\tau)}{(\tau-t)^{m+1}}d\tau\right| \leq c\left[\|w_2\|\int_0^1 \frac{\omega(f^{(m)},y)}{y}dy + \frac{\cos\pi\alpha}{\sin\pi\alpha}\left|\frac{w_2(t)}{\sigma^m(t)}\right|\|f\|_{C^m} + \frac{1}{\sin\pi\alpha}\left(\|f^{(m-1)}\|+\|f^{(m)}\|\right)\right], \quad t\in\Gamma^0. \tag{3.17}$$

By the way, the operator $T_m(w_2,\cdot) \in \mathscr{L}(C^{m,\lambda})$.

Remark 3.9. In fact, we do not redefine Hadamard-type integrals but give an expression of this type of integrals, though Definition 3.5 is more common than the traditional one. In this expression, the main part is the operator T_m. Therefore, it would be easier to identify the characteristics of this type of integrals from T_m.

4. Applications

Consider the strongly singular integral equations of the form

$$\text{f.p.} \frac{1}{\pi}\int_J \frac{\varphi(t)}{(t-x)^2}dt + \int_J k(x,t)\varphi(t)dt = f(x), \quad x \in (-1,1) \tag{4.1}$$

which was discussed in [10], where $k, f \in C^{0,\lambda}$ are given, $J = [-1,1]$, and the unknown function φ is required to be integrable but smooth in the inner of J.

We let $\varphi = wy$ and define the operators H and K by

$$\begin{aligned} Hy(x) &= \text{f.p.} \frac{1}{\pi}\int_J \frac{w(t)y(t)}{(\tau-x)^2}dt, \\ Ky(x) &= \int_J w(t)k(x,t)y(t)dt, \end{aligned} \tag{4.2}$$

respectively, where $w(x) = 1/\sqrt{1-x^2}$. Then the equation becomes

$$(H+K)y = f, \tag*{(4.1)$'$}$$

and we will see that $y \in C^{1,\lambda}$ if the above equation is solvable.

Corresponding to H, we introduce an operator $\hat{H}$ defined by

$$\hat{H}y(x) = -\text{f.p.} \frac{1}{\pi}\int_J \frac{\hat{w}(t)}{t-x}dt \int_0^t y(s)ds, \tag{4.3}$$

where $\hat{w}(x) = \sqrt{1-x^2}$. Let $\sigma = 1-x^2$ be a multiplication operator, that is, $\sigma f(x) = (1-x^2)f(x)$. We have the following.

Theorem 4.1. *$\sigma H \in \mathscr{L}(C^{1,\lambda}, C^{0,\lambda})$, $\hat{H} \in \mathscr{L}(C^{0,\lambda}, C^{1,\lambda})$, and*

$$H\hat{H} = I_0, \qquad \hat{H}H = I_1 - P_1, \tag{4.4}$$

where I_0 and I_1 are identical operators on $C^{0,\lambda}$ and $C^{1,\lambda}$, respectively,

$$P_1 y(x) = p.v.\ \frac{1}{\pi}\int_J \frac{w(t)(x+t)}{t} y(t)dt, \quad y \in C^{1,\lambda} \tag{4.5}$$

and $0 < \lambda < 1$.

Proof. If we let

$$V\psi(x) = \text{p.v.}\ \frac{1}{\pi}\int_J \frac{w(\tau)\psi(t)}{t-x}dt, \qquad \hat{V}\psi(x) = -\text{p.v.}\ \frac{1}{\pi}\int_J \frac{\hat{w}(t)\psi(\tau)}{t-x}dt, \tag{4.6}$$

then $H = DV$ and $\hat{H} = \hat{V}S$, where D is a differential operator and S is an integral operator defined by $Sf(x) = \int_0^x f(t)dt$, $x \in J$. Notice the relations

$$\begin{aligned} V\hat{V}f(x) &= f(x), \\ \hat{V}Vf(x) &= f(x) - \frac{1}{\pi}\int_J w(t)f(t)dt \end{aligned} \tag{4.7}$$

for $f \in C^{0,\lambda}$ (see [11, 14]). Thus we have

$$H\hat{H}f(x) = DV\hat{V}Sf(x) = DSf(x) = f(x) \tag{4.8}$$

for $f \in C^{0,\lambda}$ and

$$\begin{aligned} \hat{H}Hf(x) &= \hat{V}SDVf(x) = \hat{V}(Vf(x) - Vf(0)) \\ &= f(x) - \frac{1}{\pi}\int_J w(t)f(t)dt - xVf(0) = f(x) - P_1 f(x), \end{aligned} \tag{4.9}$$

for $f \in C^{1,\lambda}$. Hence, (4.4) is true.

By noting $Hf = T_1(w,f)$ and (3.13), we have

$$\sigma Hf(x) = T_1(\hat{w},f)(t) + \int_\Gamma w(\tau)(\tau+x)\frac{f(\tau) - P_1(f;\tau,x)}{\tau - x}d\tau \tag{4.10}$$

and it leads to $\sigma H \in \mathcal{L}(C^{1,\lambda}, C^{0,\lambda})$. On the other hand, $\hat{H} = \hat{V}S$, but $S \in \mathcal{L}(C^{0,\lambda}, C^{1,\lambda})$ and $\hat{V} \in \mathcal{L}(C^{1,\lambda})$ (see [15]), and thus $\hat{H} \in \mathcal{L}(C^{0,\lambda}, C^{1,\lambda})$. The proof is completed. □

Let $Z_0^{1,\lambda} = \{\psi \in C^{1,\lambda} : P_1\psi = 0\}$. Then $Z_0^{1,\lambda}$ is a close subspace of $C^{1,\lambda}$ and if restricted in this subspace, H is invertible and its inverse is $\hat{H}$. Therefore, the equation

$$(H+K)y = f, \quad y \in Z_0^{1,\lambda} \tag{4.11}$$

is equivalent to the following Fredholm integral equation:

$$(I+\hat{K})y = f^*, \tag{4.12}$$

where $\hat{K} = \hat{H}K$ and $f^* = \hat{H}f$. Hence, if the Fredholm integral equation is regular, then (4.1)′ has unique solution in $Z_0^{1,\lambda}$.

Let p be a given polynomial with degree of 1 and $Z_p^{1,\lambda} = p + Z_0^{1,\lambda}$. Notice that $P_1\psi = p$ for $\psi \in Z_p^{1,\lambda}$. Thus, the equation

$$(H+K)y = f, \quad y \in Z_p^{1,\lambda} \tag{4.13}$$

is equivalent to

$$(I+\widehat{K})y = f^* + p. \tag{4.14}$$

Since p is arbitrarily given, the solution of $(4.1)'$ has 2 degrees of freedom if the above Fredholm integral equation is regular.

Theorem 4.2. *Under the given assumptions, $(4.1)'$ is equivalent to (4.14), and if the equation is solvable then the solution has at least 2 degrees of freedom and belongs to $C^{1,\lambda}$.*

Usually, we want the solution of (4.1) to be bounded. If we find a solution y from (4.14) satisfying $y(\pm 1) = 0$, then its corresponding solution $\varphi = wy$ of (4.1) is bounded (in fact $\varphi(\pm 1) = 0$, cf. [18]). However, it is possible to choose such a solution from (4.14), because its solution has 2 degrees of freedom. Alternatively, we can define another operator $\widehat{H}_0$ by

$$\widehat{H}_0 y(x) = -\frac{1-x^2}{\pi}\text{p.v.}\int_J \frac{w(t)}{t-x}dt\int_0^t y(s)ds, \tag{4.15}$$

and get the solution from the equation

$$(I+\widehat{K}_0)y = f_0^*, \tag{4.16}$$

where $\widehat{K}_0 = \widehat{H}_0 K$ and $f_0^* = \widehat{H}_0 f$. The reason is stated as follows.

At first, it is easy to verify the following.

Theorem 4.3. $\widehat{H}_0 \in \mathscr{L}(C^{0,\lambda}, C_0^{1,\lambda})$ *and*

$$H\widehat{H}_0 = I_0, \qquad \widehat{H}_0 H = I_1^0, \tag{4.17}$$

where the space $C_0^{1,\lambda} = \{\psi \in C^{1,\lambda} : \psi(\pm 1) = 0\}$ and I_1^0 is an identical operator on it.

Then, from this theorem, we see that (4.16) is equivalent to

$$(H+K)y = f, \quad y \in C_0^{1,\lambda}, \tag{4.18}$$

and hence, the solution of (4.16) is also the solution of $(4.1)'$ but satisfies $y(\pm 1) = 0$.

Example 4.4. In order to solve the equation

$$\text{f.p.}\ \frac{1}{\pi}\int_J \frac{\varphi(t)}{(t-x)^2}dt + \frac{4}{\pi}\int_J (1+xt)\varphi(t)dt = 2, \quad x \in (-1,1) \tag{4.19}$$

we let $\varphi(x) = y(x)/\sqrt{1-x^2}$, that is,

$$\text{f.p.}\ \frac{1}{\pi}\int_J \frac{1}{\sqrt{1-t^2}}\frac{y(t)}{(t-x)^2}dt + \frac{4}{\pi}\int_J \frac{1+xt}{\sqrt{1-t^2}}y(t)dt = 2, \quad x \in (-1,1) \tag{4.15$'$}$$

and use $\widehat{H}$ to act on both sides of it. Then, the equation is converted into

$$y(x) - \frac{1}{\pi}\int_J \frac{2-4x^2+(x-2x^3)t}{\sqrt{1-t^2}}y(t)dt = -1+2x^2+\theta_0+\theta_1 x, \tag{4.20}$$

where θ_0 and θ_1 are arbitrary constants. By solving the equation, we have

$$y(x) = 3\theta_0 - 1 + \frac{7}{2}\theta_1 x + (2-4\theta_0)x^2 - 2\theta_1 x^3. \tag{4.21}$$

If taking $\theta_0 = 1$ and $\theta_1 = 0$, we obtain a special solution $y(x) = 2(1-x^2)$ which satisfies $y(\pm 1) = 0$, and thus $\varphi(x) = 2\sqrt{1-x^2}$ is a bounded as well as unique solution of (4.19). This solution can also be obtained directly by solving the equation

$$y(x) - \frac{1-x^2}{\pi}\int_J \frac{4+2xt}{\sqrt{1-t^2}}y(t)dt = -2(1-x^2), \tag{4.22}$$

which is from (4.15$'$) with $\widehat{H}_0$ acting on it.

Remark 4.5. Obviously, the results on (4.1) obtained in this section are significant for the solution of this kind of equations, and especially, they are helpful for the discussion of error estimation of the approximate solution.

Remark 4.6. Theorem 4.1 means σH is a bounded operator from $C^{1,\lambda}$ to $C^{0,\lambda}$ but H is not. If we consider the operator in the Sobolev spaces $W_w^{m,2}$ and $W_{\hat{w}}^{m,2}$, which are the completions of C^m normed by

$$\begin{aligned} \|f\|_{m,w} &= \left[\sum_{r=0}^{m}\int_J w(t)\left|f^{(r)}(t)\right|^2 dt\right]^{1/2}, \\ \|f\|_{m,\hat{w}} &= \left[\sum_{r=0}^{m}\int_J \hat{w}(t)\left|f^{(r)}(t)\right|^2 dt\right]^{1/2}, \end{aligned} \tag{4.23}$$

respectively, then $H \in \mathcal{L}(W_w^{m,2}, W_{\hat{w}}^{m-1,2})$ and $\widehat{H} \in \mathcal{L}(W_{\hat{w}}^{m-1,2}, W_w^{m,2})$ as well (cf. [17, 19]). In this case, H and $\widehat{H}$ are symmetrical.

References

[1] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover, New York, NY, USA, 1952.

[2] L. Schwartz, *Théorie des distributions. Tome I*, Actualités scientifiques et industrielles, no. 1091 Publ. Inst. Math. Univ. Strasbourg 9, Hermann & Cie., Paris, France, 1950.

[3] L. Schwartz, *Théorie des distributions. Tome II*, Actualités scientifiques et industrielles, no. 1122 Publ. Inst. Math. Univ. Strasbourg 10, Hermann & Cie., Paris, France, 1951.

[4] C. Fox, "A generalization of the Cauchy principal value," *Canadian Journal of Mathematics*, vol. 9, no. 1, pp. 110–117, 1957.

[5] H. R. Kutt, "The numerical evaluation of principal value integrals by finite-part integration," *Numerische Mathematik*, vol. 24, no. 3, pp. 205–210, 1975.

[6] N. I. Ioakimidis, "Application of finite-part integrals to the singular integral equations of crack problems in plane and three-dimensional elasticity," *Acta Mechanica*, vol. 45, no. 1-2, pp. 31–47, 1982.

[7] J. K. Lu, "On singular integrals with singularities of high fractional order and their applications," *Acta Mathematica Scientia*, vol. 2, no. 2, pp. 211–228, 1982.

[8] C. R. Wang, "The Hadamard principal value of the singular integral $\int_L (f(\tau)/(\tau - t)^{n+1})d\tau$," *Chinese Annals of Mathematics*, vol. 3, no. 2, pp. 195–202, 1982.

[9] P. Linz, "On the approximate computation of certain strongly singular integrals," *Computing*, vol. 35, no. 3-4, pp. 345–353, 1985.

[10] A. C. Kaya and F. Erdogan, "On the solution of integral equations with strongly singular kernels," *Quarterly of Applied Mathematics*, vol. 45, no. 1, pp. 105–122, 1987.

[11] J. K. Lu, *Boundary Value Problems for Analytic Functions*, vol. 16 of *Series in Pure Mathematics*, World Scientific, River Edge, NJ, USA, 1993.

[12] D.-H. Yu, *Natural Boundary Integral Method and Its Applications*, vol. 539 of *Mathematics and Its Applications*, Science Press, Beijing, China; Kluwer Academic, Dordrecht, The Netherlands, 2002.

[13] M. Hori and S. Nemat-Nasser, "Asymptotic solution of a class of strongly singular integral equations," *SIAM Journal on Applied Mathematics*, vol. 50, no. 3, pp. 716–725, 1990.

[14] N. I. Muskhelishvili, *Singular Integral Equations. Boundary Problems of Function Theory and Their Application to Mathematical Physics*, P. Noordhoff N. V., Gröningen, The Netherlands, 1953.

[15] Y. J. Xu, "The complex spline approximation of singular integral operators over an open arc," *Journal of Approximation Theory*, vol. 119, no. 1, pp. 86–109, 2002.

[16] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, NY, USA, 1966.

[17] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer, Berlin, Germany, 1986.

[18] Y. J. Xu, "Some properties of singular integral operators over an open curve," to appear in *Science in China. Series A*.

[19] K. Yosida, *Functional Analysis*, vol. 123 of *Fundamental Principles of Mathematical Sciences*, Springer, Berlin, Germany, 6th edition, 1980.

Yong Jia Xu: Department of Statistics, Guangdong University of Business Studies,
Guangzhou 510320, China
Email address: yjxu@gdcc.edu.cn