

NORMING POINTS AND UNIQUE MINIMALITY OF ORTHOGONAL PROJECTIONS

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We study the norming points and norming functionals of symmetric operators on L_p spaces for $p = 2m$ or $p = 2m/(2m - 1)$. We prove some general result relating uniqueness of minimal projection to the set of norming functionals. As a main application, we obtain that the Fourier projection onto $\text{span}[1, \sin x, \cos x]$ is a unique minimal projection in L_p .

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1. Introduction

This paper had been motivated by the question of uniqueness of minimal projections in L_p spaces. As an example, let $L_p(-\pi, \pi)$ be the Banach space of all 2π -periodic functions $f(t)$ such that

$$\|f\|_p^p = \int |f|^p d\mu < \infty, \quad (1.1)$$

where $1 \leq p \leq \infty$ and μ is normalized Lebesgue measure on $(-\pi, \pi)$. Let V be the space of trigonometric polynomials of degree n and let P be the Fourier projection from $L_p(-\pi, \pi)$ onto V . It is well known (see [2, 3, 16]) that P is a minimal projection, that is, P has the least norm among all projections from $L_p(-\pi, \pi)$ onto V . For $p = 1, 2$ and $p = \infty$, it was proved that P is a unique projection that has this property (see [4, 7]). For the other values of p , the uniqueness of the minimal projection is a wide open question.

More generally, let V be a complemented subspace of $L_p(\mu)$. Define

$$\lambda(V, L_p(\mu)) = \inf \{\|P\| : P \text{ is a projection from } L_p(-\pi, \pi) \text{ onto } V\}. \quad (1.2)$$

A projection P from $L_p(-\pi, \pi)$ onto V is called minimal if $\|P\| = \lambda(V, L_p(\mu))$. The study

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of minimal projections have been instigated by Cheney and Price and continued by many authors (see [2, 3, 5, 8, 10–13, 15, 16]).

Yet, as of today the authors of this article do not know a single example of a subspace $V \subset L_p(\mu)$ with $1 < p < \infty$ for which the minimal projection is not unique! There are a few (very few) instances for which the uniqueness of the minimal projection had been verified (see [1, 6, 9, 14, 16–18, 20]). As an application of the general technique developed in this paper we will add a few bits and pieces to the list of examples of uniqueness of minimal projections. In particular we will prove that the minimal projection from $L_p(-\pi, \pi)$ onto $\text{span}\{1, \sin t, \cos t\}$ is unique.

It was clear from the very beginning (see [14, 18, 20]) that the understanding of minimal projections depends strongly on our knowledge of norming points; that is, the functions $f \in L_p(\mu)$ such that $\|Pf\| = \|P\|\|f\|$. By predicting the norming points of a minimal projection, one can find the minimal projections itself and forecast its properties. The spaces $L_1(\mu)$ have few extreme points and thus have few candidates for the norming points. This is one of the reasons for the successful studies of the minimal projections in $L_1(\mu)$. The space $L_p(\mu)$ on the other hand is a very different story. Every point of the unite ball of $L_p(\mu)$ is an extreme point. Even if the operator P is given in advance, the norm of the operator $P : L_p(\mu) \rightarrow L_p(\mu)$ as well as its norming points are hard to come by. Readers, sceptical of our last statement are invited to find the norm of a nontrivial 3×3 matrix as an operator on l_p^3 for $1 < p < \infty$ and $p \neq 2$.

In the next section, we will develop a convenient relationship between the symmetric operators on $L_p(\mu)$ and its norming points. In the last section we will apply these relationships to investigate the uniqueness of minimal projections. We will use the rest of this section to establish some notations.

The symbol $L_p(\mu)$ will always denote the real Banach space with an arbitrary positive Borel measure μ . The index p , throughout this paper, will be of the form

$$p = 2m \quad \text{or} \quad p = \frac{2m}{2m-1} \quad \text{with } m \in \mathbb{N}. \quad (1.3)$$

The dual space to $L_p(\mu)$ is identified with $L_q(\mu)$ with $1/p + 1/q = 1$. For $(f, g) \in L_p(\mu) \times L_q(\mu)$, the pairing $(f, g) \rightarrow \int fg d\mu$ defines the duality.

Definition 1.1. For an operator P from a Banach space X onto its subspace Y , define the set of norming points as

$$\begin{aligned} \mathfrak{N}(P) &:= \{x \in X : \|Px\| = \|P\|\|x\|, x \neq 0\}; \\ \mathfrak{N}_1(P) &= \{x \in \mathfrak{N}(P); \|x\| = 1\} \end{aligned} \quad (1.4)$$

and norming functionals

$$\begin{aligned} \mathfrak{N}\mathfrak{F}(P) &:= \{f \in X^* : \|f \circ P\| = \|P\|\|f\|, f \neq 0\}; \\ \mathfrak{N}\mathfrak{F}_1(P) &= \{f \in \mathfrak{N}\mathfrak{F}(P); \|f\| = 1\}. \end{aligned} \quad (1.5)$$

It is well known (see [16]) that if P is compact, then $\mathfrak{N}\mathfrak{F}(P) \neq \emptyset$ if we additionally know that X is reflexive then also $\mathfrak{N}(P) \neq \emptyset$ (without reflexivity, the last statement is not true—Fourier projection $\mathcal{C}(\pi, \pi) \rightarrow \mathfrak{T}_n$ does not have a norming point). For $x \in X$, we let

$$\begin{aligned} \mathfrak{N}(x) &:= \{f \in X^* : \|fx\| = \|f\| \|x\|, f \neq 0\}; \\ \mathfrak{N}_1(x) &:= \{f \in \mathfrak{N}(P); \|f\| = 1\}. \end{aligned} \tag{1.6}$$

We also define the sets of norming pairs

$$\begin{aligned} \mathfrak{E}(P) &:= \{(f, x) \in X^* \times X : f(Px) = \|P\| \|x\| \|f\|, \|x\| \|f\| \neq 0\}, \\ \mathfrak{E}_1(P) &:= \{(f, x) \in \mathfrak{E}(P) : \|x\| = \|f\| = 1\}. \end{aligned} \tag{1.7}$$

The sets of norming pairs had been also termed *extremal pairs* by Chalmers and Metcalf (see [2, 3]). The Holder inequality establishes an easy relation between norming points and norming pairs. This relationship is especially transparent in cases $p = 2m/(2m - 1)$ or $p = 2m$ for some integer m . From now on, we will only deal with these values for p . To formalize it, we avail ourself of *Holder functionals*.

Definition 1.2. Let f be a nonzero function in L_p . Define

$$H_p(f) = \frac{(f)^{p-1}}{\|(f)^{p-1}\|_q} \in L_q. \tag{1.8}$$

Observe that $H_p : L_p \setminus \{0\} \rightarrow L_q$ is a nonlinear (continuous) mapping. Here are a few simple properties of this functional.

PROPOSITION 1.3. *Let $p = 2m/(2m - 1)$ or $p = 2m$ for some integer m . Then*

- (1) $\|H_p(f)\|_q = 1$,
- (2) $\int f H_p(f) d\mu = \|f\|_p$,
- (3) $H_p(\lambda f) = \text{sign}(\lambda) H_p(f)$,
- (4) $g \in \mathfrak{N}(f)$ if and only if $f \in \mathfrak{N}(g)$,
- (5) if $0 \neq f \in L_p$, then $g \in \mathfrak{N}(f)$ if and only if $g = H_p(f)$.

2. Orthogonal projections onto subspaces of L_p

Let V be a finite-dimensional subspace of $L_p \cap L_2$. By the *orthogonal projection* from L_p onto V we mean an orthogonal projection P from L_2 onto V considered as the mapping on L_p . In the above settings, P^* is an orthogonal projection in L_q with the range V (regarded as a subspace of L_q). The following proposition is obvious.

PROPOSITION 2.1. *Let V be a finite-dimensional subspace of L_p and let P be a projection from L_p onto V . Regard P as an operator from L_p to L_p . Then P^* is a projection regarded as an operator from L_q to L_q . There exists*

$$\|P\|_{L_p \rightarrow L_p} = \|P^*\|_{L_q \rightarrow L_q}. \tag{2.1}$$

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Furthermore, the following are equivalent:

- (1) $0 \neq (g, f) \in \mathfrak{E}_*(P)$,
- (2) $0 \neq (f, g) \in \mathfrak{E}_*(P^*)$,
- (3) $g \in \mathfrak{N}_*(Pf)$ and $f \in \mathfrak{N}_*(P)$,
- (4) $f \in \mathfrak{N}_*(P^*g)$ and $g \in \mathfrak{N}_*(P^*)$,

where \mathfrak{N}_* (resp., \mathfrak{E}_*) stands for either \mathfrak{N} or \mathfrak{N}_1 (resp., \mathfrak{E} or \mathfrak{E}_1).

Keeping in mind the form of the Holder functional H_p , we obtain the following theorem.

THEOREM 2.2. *Let $p = 2m/(2m - 1)$ or $p = 2m$ and let P be a projection from L_p onto V . Suppose that $f \neq 0$. Then*

$$f \in \mathfrak{N}(P) \text{ iff } (\|P\|)^p f^{p-1} = P^*((Pf)^{p-1}). \quad (2.2)$$

In particular if $p = 2m$,

$$f \in \mathfrak{N}(P) \text{ iff } (\|P\|)^{2m/(2m-1)} f = P^*((Pf)^{2m-1})^{1/(2m-1)}, \quad (2.3)$$

and if $p = 2m/(2m - 1)$,

$$f \in \mathfrak{N}(P) \text{ iff } (\|P\|)^{2m} f = P^*((Pf)^{1/(2m-1)})^{2m-1}. \quad (2.4)$$

Proof. Without loss, we can assume that $f \in \mathfrak{N}_1(P)$. Take $g \in L_q$ such that $gPf = \|P\|$. So $g \in \mathfrak{N}_1(Pf)$ and as a result from Proposition 1.3 we have $g = H_p(Pf)$. Therefore $(H_p(Pf), f) \in \mathfrak{E}_1(P)$. Using Proposition 2.1, we have $(f, H_p(Pf)) \in \mathfrak{E}_1(P^*)$. This implies that

$$\begin{aligned} \|P\| &= \|P^*\| = \|P^*(H_p(Pf))\|_q = \left\| P^* \left(\frac{(Pf)^{p-1}}{\|(Pf)^{p-1}\|_q} \right) \right\|_q \\ &= \frac{P^*((Pf)^{p-1})}{\|(Pf)^{p-1}\|_q} = \frac{\|P^*((Pf)^{p-1})\|_q}{(\|Pf\|_p)^{p/q}} = \frac{\|P^*((Pf)^{p-1})\|_q}{\|P\|^{p/q}}, \end{aligned} \quad (2.5)$$

hence

$$\|P^*((Pf)^{p-1})\|_q = (\|P\|)^p, \quad (2.6)$$

$$f \in \mathfrak{N}_1(P^*(H_p(Pf))). \quad (2.7)$$

Using Proposition 1.3, the second equality (2.7) gives us

$$f = H_q(P^*(H_p(Pf))). \quad (2.8)$$

Now applying (2.6) to (2.8), we obtain

$$\begin{aligned}
 f &= H_q(P^*(H_p(Pf))) = H_q\left(P^*\left(\frac{(Pf)^{p-1}}{\|(Pf)^{p-1}\|_q}\right)\right) \\
 &= H_q\left(\frac{P^*((Pf)^{p-1})}{\|(Pf)^{p-1}\|_q}\right) = H_q(P^*((Pf)^{p-1})) \\
 &= \frac{(P^*((Pf)^{p-1}))^{q-1}}{\|(P^*((Pf)^{p-1}))^{q-1}\|_p} = \frac{(P^*((Pf)^{p-1}))^{q-1}}{(\|P^*((Pf)^{p-1})\|_q)^{q/p}} = \frac{(P^*((Pf)^{p-1}))^{q-1}}{(\|P\|)^q}.
 \end{aligned} \tag{2.9}$$

Multiplying and raising both sides to the power $1/(q-1) = p-1$ gives (2.2).

Assume now that f fulfills (2.2). Without loss, we may assume that $\|f\|_p = 1$ and compute

$$1 = \int f^p d\mu = \int f f^{p-1} d\mu = \int \frac{f P^*((Pf)^{p-1})}{(\|P\|)^p} d\mu. \tag{2.10}$$

Hence

$$f(P^*((Pf)^{p-1})) = (\|P\|)^p. \tag{2.11}$$

On the other hand,

$$f(P^*((Pf)^{p-1})) \leq \|f\|_p \cdot \|P^*\| \cdot \|(Pf)^{p-1}\|_q = \|P\| \cdot (\|Pf\|_p)^{p-1}. \tag{2.12}$$

If we combine (2.11) and (2.12), then we get

$$\|P\| \leq \|Pf\|_p, \tag{2.13}$$

hence $f \in N_1(P)$. □

Even though Theorem 2.2 gives us an if and only if condition, it cannot be applied directly to find a norming functionals. In order to check the equality in condition (2.2) one has to know $\|P\|$. We may drop $\|P\|$ and formulate Theorem 2.2 as follows.

COROLLARY 2.3. *Let $p = 2m/(2m-1)$ or $p = 2m$ and let P be a projection from L_p onto V . Suppose that $f \neq 0$ and $f \in \mathfrak{N}(P)$. Then there is a constant M such that*

$$f^{p-1} = MP^*((Pf)^{p-1}). \tag{2.14}$$

In particular if $p = 2m$,

$$f = MP^*((Pf)^{2m-1})^{1/(2m-1)}, \tag{2.15}$$

and if $p = 2m/(2m-1)$,

$$f = MP^*((Pf)^{1/(2m-1)})^{2m-1}. \tag{2.16}$$

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If we now investigate (2.14) then for $M = 0$, we obtain $f = 0$ and for $M = (\|P\|)^{-p} \leq 1$, we obtain all norming points for P . The problem is there are many M for which there is a solution to (2.14). Consider the Fourier projection $P : L_{2m/(2m-1)} \rightarrow \text{span}[1, \cos t, \sin t]$. In Section 3, it is proved that a function $f = (\cos t)^{2m-1}$ is not a norming point for P , yet it satisfies (2.16) for some $M < 1$. On the other hand in general, Corollary 2.3 brings some extremely useful information on norming points.

THEOREM 2.4. *Let P be a projection from L_p onto V . Define*

$$V^\alpha := \{f^\alpha : f \in V\}. \quad (2.17)$$

If $p = 2m/(2m-1)$ for some integer m , then

$$\mathfrak{N}(P) \subset V^{2m-1}. \quad (2.18)$$

If $p = 2m$ for some integer m , then

$$\mathfrak{N}(P) \subset V^{1/(2m-1)}. \quad (2.19)$$

The proof follows immediately from Corollary 2.3.

COROLLARY 2.5. *Assuming in the last theorem that $m = 1$, an obvious result that the orthogonal projection $P : L_2 \rightarrow L_2$ has a norming point in V is obtained and being a projection, it follows that $\|P\| = 1$.*

COROLLARY 2.6. *Let $\mathfrak{T}_n = \text{span}[1, \cos kt, \sin kt]_{k=1}^n$. Let Q be an orthogonal projection from L_p onto \mathfrak{T}_n . If $p = 2m/(2m-1)$ for some integer m , then*

$$\|Q\|_{L_p \rightarrow L_p} = \|Q\|_{\mathfrak{T}_{n(2m-1)} \rightarrow \mathfrak{T}_n}. \quad (2.20)$$

In the language of projectional constants, we can reformulate this result as

$$\lambda(\mathfrak{T}_n, L_p) = \lambda(\mathfrak{T}_n, \mathfrak{T}_{(2m-1)n}). \quad (2.21)$$

In terms of norming points, we conclude that there exists a trigonometric polynomial g of degree $(2m-1)n$ which is a norming point for Q . In stark contrast to the last corollary, we have the following corollary.

COROLLARY 2.7. *Let $p = 2m > 2$ and let P be the orthogonal projection from L_p onto \mathfrak{T}_n . Then for every trigonometric polynomial (of any degree) f ,*

$$\|Pf\| < \|P\| \|f\|. \quad (2.22)$$

Proof. Let us assume that there is a trigonometric polynomial of the exact degree N such that $f \in \mathfrak{N}_1(P)$. Since $\|P\| > 1$ we have $N > n$. According to (2.15) we have $f^{2m-1} \in V = \mathfrak{T}_n$. But f^{2m-1} is a trigonometric polynomial of the exact degree $N(2m-1)$. But $N(2m-1) > n$ which gives a contradiction. \square

From the above, we may conclude that if $p = 2m$, then any norming point for P (an orthogonal projection onto polynomials \mathfrak{T}_n) is not a polynomial.

Remark 2.8. While the norming points of P are not trigonometric polynomials, the previous corollary shows that the norming functionals $g \in \mathfrak{N}_1(Pf)$ with $f \in \mathfrak{N}_1(P)$ are polynomials of exact degree $(2m - 1)n$. Hence, whether $p = 2m$ or $p = 2m/(2m - 1)$, at least one component of the norming pair $(g, f) \in \mathfrak{E}(P)$ is a polynomial of exact degree $(2m - 1)n$.

We finish this section with one more application of this technique to the best approximation problem in L_p .

THEOREM 2.9. *Let $p = 2m/(2m - 1)$ and let $\mathfrak{T}_n^c = \text{Im}(\text{Id} - P)$. That is, \mathfrak{T}_n^c is the closure in L_p of trigonometric polynomials with frequencies larger than n . Then for every $f \in \mathfrak{T}_n$, the element $(\text{Id} - P)(f^{2m-1})$ is the unique best approximation to $P(f^{2m-1})$ from \mathfrak{T}_n^c .*

Proof. Consider

$$H_p(f^{2m-1}) = \frac{(f^{2m-1})^{p-1}}{\|(f)^{p-1}\|_q} = \frac{1}{\|(f)^{p-1}\|_q} f. \tag{2.23}$$

Since $f \in \mathfrak{T}_n$, we conclude that

$$H_p(f^{2m-1}) \perp \mathfrak{T}_n^c. \tag{2.24}$$

Also, by the definition of the Holder functional,

$$\begin{aligned} \|Pf^{2m-1} - (I - P)f^{2m-1}\|_p &= \|f^{2m-1}\|_p = H_p(f^{2m-1})(f^{2m-1}) \\ &= H_p(f^{2m-1})(Pf^{2m-1} - (I - P)f^{2m-1}). \end{aligned} \tag{2.25}$$

Properties (2.24) and (2.25) combined with the well-known criterium (see [19]) for the best approximation imply the conclusion of the theorem. \square

The above theorem is very useful obtaining many examples of best approximation in L_p —what we need is to raise some elements to the appropriate power. Here is a simple illustration of the usefulness of this theorem.

Example 2.10. Let $f = (1 + \cos x + \sin x)$. Then $f^3 = (1 + \cos x + \sin x)^3 = 4 + (9/2) \cos x + (9/2) \sin x + 3 \sin 2x - (1/2) \cos 3x + (1/2) \sin 3x$. Hence

$$\left(3 \sin 2x - \frac{1}{2} \cos 3x + \frac{1}{2} \sin 3x \right) \tag{2.26}$$

is the best approximation to

$$\left(4 + \frac{9}{2} \cos x + \frac{9}{2} \sin x \right) \tag{2.27}$$

from the space \mathfrak{T}_1^c (trigonometric polynomials with frequencies larger 1) in the space $L_{4/3}$.

3. Applications

We start with general theorem which will connect uniqueness of a given projection to its norming points. First we state the following theorem.

THEOREM 3.1 (Rudin) [21, III.B.13]. *Let X be a Banach space and V a complemented subspace. Let G be a compact group which acts as a group of linear operators on X such that*

- (1) $T_g(x)$ is a continuous function of g , for every $x \in X$,
- (2) $T_g(V) \subset V$, for all $g \in G$,
- (3) T_g are isometries, for all $g \in G$.

Furthermore, assume that there exists only one projection $P : X \rightarrow V$ which commutes with G . Then this projection is minimal.

Once we know that there is only one projection P commuting with G , the following can be easily found: fix any projection Q from X onto V , then this projection P equals

$$P(x) = \int_G T_g Q T_{g^{-1}}(x) dg, \quad \text{for } x \in X. \quad (3.1)$$

Now we are ready to prove the following theorem.

THEOREM 3.2. *Let V be a complimented subspace of $L^p(\mu)$ ($1 < p < \infty$) and assume there is a group G acting on $L_p(\mu)$ as in Theorem 3.1. Let $P : L^p(\mu) \rightarrow V$ be a minimal projection given by (3.1). Assume additionally that the set of norming functionals for P is total on V (i.e., if $v \in V$, and $f(v) = 0$ for any $f \in \mathfrak{N}\mathfrak{F}(P)$, then $v = 0$). Then projection P is the unique minimal projection.*

Proof. Assume to the contrary that there is another minimal projection Q . Therefore $\|P\| = \|Q\|$. We will prove now that

$$\mathfrak{E}_1(P) \subset \mathfrak{E}_1(Q). \quad (3.2)$$

To do this assume to contrary that $\|f(Qx)\| < \|Q\| = \|P\|$. As in the proof of Theorem 3.1 observe that the function

$$g \longmapsto T_g(f)QT_{g^{-1}}(x) \quad (3.3)$$

is continuous. Therefore in some open neighborhood U of $g = 1$, we have

$$\|T_g(f)QT_{g^{-1}}(x)\| < \|Q\| = \|P\|, \quad (3.4)$$

and by (3.1)

$$\begin{aligned} \|P\| &= \|f(Px)\| = \left\| \int_G T_g f Q T_{g^{-1}}(x) dg \right\| \leq \int_G \|T_g f Q T_{g^{-1}}(x)\| dg \\ &= \int_U \|T_g f Q T_{g^{-1}}(x)\| dg + \int_{G \setminus U} \|T_g f Q T_{g^{-1}}(x)\| dg \\ &< \int_U \|Q\| dg + \int_{G \setminus U} \|Q\| dg = \|P\|, \end{aligned} \quad (3.5)$$

which is a contradiction. Now (3.2) gives us

$$f(P)(x) = f(Q)(x) = \|P\| = \|Q\|, \quad \text{for any } (f, x) \in \mathfrak{E}_1(P). \quad (3.6)$$

That is

$$\left(\frac{P^* f}{\|P\|}\right)(x) = \left(\frac{Q^* f}{\|Q\|}\right)(x) = 1, \quad \text{for any } (f, x) \in \mathfrak{E}_1(P). \quad (3.7)$$

But L_p is a smooth space—an element x can only have one norming functional, therefore

$$P^* f = Q^* f, \quad \text{for any } f \in \mathfrak{N}_1(P). \quad (3.8)$$

Take any $x \in L_p(\mu)$, we have $Px - Qx \in V$. Now for any $f \in N_1(P)$ using (3.8), we have

$$f(Px - Qx) = (P^* f - Q^* f)x = 0, \quad (3.9)$$

but the set of norming functionals is total on V so $Px = Qx$, hence $P = Q$. \square

We will prove now that for $p = 2m$ or $p = 2m/(2m - 1)$ orthogonal projection onto $\text{span}[1, \cos t, \sin t]$ is unique minimal projection. The main trick is based on Corollary 2.3. For Fourier projection P , we have $P^* = P$. It implies that if f is a norming point for P (for $p = 2m/(2m - 1)$), then for some constant $M \leq 1$ we have

$$f = M(P((Pf)^{1/(2m-1)}))^{2m-1}. \quad (3.10)$$

Hence any norming point has to be of the form

$$\begin{aligned} f &= g^{2m-1}, \quad g \in \text{span}[1, \cos t, \sin t], \\ g &= MP\left((P(g^{2m-1}))^{1/(2m-1)}\right), \end{aligned} \quad (3.11)$$

for some constant M . As we will see later, both (3.10) and (3.11) are only necessary conditions, $g = \cos t$ and $f = (\cos t)^{2m-1}$ fulfill, respectively, (3.11) and (3.10), yet $f = (\cos t)^{2m-1}$ is not a norming point for a Fourier projection. That is actually the main difficulty in showing uniqueness of Fourier projection.

THEOREM 3.3. *Let $\mathfrak{T}_1 = \text{span}[1, \cos t, \sin t]$. Let P be an orthogonal projection from L_p onto \mathfrak{T}_1 . If $p = 2m/(2m - 1)$ for some integer $m > 1$, then there is a norming pair (g, f) for P such that*

$$\begin{aligned} f|_{\text{span}[1, \cos t, \sin t]} &= a_0 + a_1 \cos t + a_2 \sin t, \\ g|_{\text{span}[1, \cos t, \sin t]} &= b_0 + b_1 \cos t + b_2 \sin t, \end{aligned} \quad (3.12)$$

where $a_0, a_1, a_2, b_0, b_1, b_2 \neq 0$.

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Proof. Fourier projection is invariant under the isometries $I_\theta : f(t) \mapsto f(t + \theta)$, (i.e., $PI_\theta = I_\theta P$)—see [2, 3, 16]. This implies

$$f(t) \in \mathfrak{N}(P) \iff f(t + \theta) \in \mathfrak{N}(P), \quad (3.13)$$

for any θ . So assume that f is a norming point for P . According to (3.11),

$$f = (a_0 + a_1 \cos t + a_2 \sin t)^{2m-1}. \quad (3.14)$$

Observe that $a_1^2 + a_2^2 > 0$ (otherwise $\|P\| = 1$), therefore

$$\begin{aligned} f &= (a_0 + a_1 \cos t + a_2 \sin t)^{2m-1} \\ &= (a_1^2 + a_2^2)^{(2m-1)/2} \left(\frac{a_0}{\sqrt{a_1^2 + a_2^2}} + \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \cos t + \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \sin t \right)^{2m-1}. \end{aligned} \quad (3.15)$$

Now there is θ such that $\cos(\theta) = a_1/\sqrt{a_1^2 + a_2^2}$ and $\sin(\theta) = a_2/\sqrt{a_1^2 + a_2^2}$, and now

$$\begin{aligned} f &= (a_0 + a_1 \cos t + a_2 \sin t)^{2m-1} \\ &= (a_1^2 + a_2^2)^{(2m-1)/2} \left(\frac{a_0}{\sqrt{a_1^2 + a_2^2}} + \cos \theta \cos t + \sin \theta \sin t \right)^{2m-1} \\ &= (a_1^2 + a_2^2)^{(2m-1)/2} \left(\frac{a_0}{\sqrt{a_1^2 + a_2^2}} + \cos(t - \theta) \right)^{2m-1}. \end{aligned} \quad (3.16)$$

Using (3.13), it follows that in searching for norming points of P , we can restrict ourselves to the functions of the form

$$f(t) = (x + \cos t)^{2m-1}. \quad (3.17)$$

We will prove that P has a norming point of the form (3.17) for some $x \neq 0$. To do this, we need to prove that the function

$$T(x) = \frac{\|P((x + \cos t)^{2m-1})\|_P^p}{\|(x + \cos t)^{2m-1}\|_P^p} = \frac{\int_0^{2\pi} (P((x + \cos t)^{2m-1}))^{2m/(2m-1)} dt}{\int_0^{2\pi} (x + \cos t)^{2m} dt} \quad (3.18)$$

does not attain its maximum at $x = 0$. Define (whenever it makes sense)

$$F(\nu) = \int_0^{2\pi} (\cos t)^\nu dt. \quad (3.19)$$

Observe that for any $k \in \mathbb{N}$,

$$F(2k) = \int_0^{2\pi} (\cos t)^{2k} dt = 4 \int_0^{\pi/2} (\cos t)^{2k} dt = \frac{2 \cdot \sqrt{\pi} \cdot \Gamma(1/2 + k)}{\Gamma(1 + k)} \quad (3.20)$$

and by the symmetry of the cosine graph,

$$F(2k - 1) = \int_0^{2\pi} (\cos t)^{2k-1} dt = 0, \quad F\left(\frac{1}{2k-1}\right) = \int_0^{2\pi} (\cos t)^{1/2k-1} dt = 0. \quad (3.21)$$

What is more, for any $k \in \mathbb{N}$,

$$\begin{aligned} F\left(\frac{2k}{2k-1}\right) &= \int_0^{2\pi} (\cos t)^{2k/(2k-1)} dt \\ &= 4 \int_0^{\pi/2} (\cos t)^{2k/(2k-1)} dt = \frac{2 \cdot \sqrt{\pi} \cdot \Gamma(1/2 + k/(2k-1))}{\Gamma(1 + k/(2k-1))}, \\ F\left(-\frac{2k-2}{2k-1}\right) &= \int_0^{2\pi} \left(\frac{1}{\cos t}\right)^{(2k-2)/(2k-1)} dt \\ &= 4 \int_0^{\pi/2} \left(\frac{1}{\cos t}\right)^{(2k-2)/(2k-1)} dt = \frac{2 \cdot \sqrt{\pi} \cdot \Gamma(1/(4k-2))}{\Gamma(k/(2k-1))}. \end{aligned} \quad (3.22)$$

First we need to compute $P((x + \cos t)^{2m-1})$. Observe that

$$P((x + \cos t)^{2m-1}) = a_0(x) + a_1(x) \cos t, \quad (3.23)$$

where

$$\begin{aligned} a_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} (x + \cos t)^{2m-1} dt, \\ a_1(x) &= \frac{1}{2\pi} \int_0^{2\pi} (x + \cos t)^{2m-1} \cos t dt. \end{aligned} \quad (3.24)$$

Using Newton formula $(x + \cos t)^{2m-1} = \sum_{i=1}^{2m-1} \binom{2m-1}{i} x^i (\cos t)^{2m-1-i}$ and obtain

$$\begin{aligned} a_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^{2m-1} \binom{2m-1}{i} x^i (\cos t)^{2m-1-i} dt \\ &= \sum_{i=1}^{2m-1} \frac{1}{2\pi} \binom{2m-1}{i} x^i \int_0^{2\pi} (\cos t)^{2m-1-i} dt \\ &= \sum_{i=1}^{2m-1} \frac{1}{2\pi} \binom{2m-1}{i} x^i F(2m-1-i) \\ &= \sum_{i-\text{odd}} \frac{1}{2\pi} \binom{2m-1}{i} x^i F(2m-1-i), \end{aligned} \quad (3.25)$$

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and similarly

$$\begin{aligned}
 a_1(x) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^{2m-1} \binom{2m-1}{i} x^i (\cos t)^{2m-1-i} \cos t \, dt \\
 &= \sum_{i=1}^{2m-1} \frac{1}{2\pi} \binom{2m-1}{i} x^i \int_0^{2\pi} (\cos t)^{2m-1-i} \cos t \, dt \\
 &= \sum_{i=1}^{2m-1} \frac{1}{2\pi} \binom{2m-1}{i} x^i F(2m-i) \\
 &= \sum_{i\text{-even}} \frac{1}{2\pi} \binom{2m-1}{i} x^i F(2m-i).
 \end{aligned} \tag{3.26}$$

We have $T(x) = H(x)/G(x)$, where

$$\begin{aligned}
 H(x) &= \int_0^{2\pi} (P((x + \cos t)^{2m-1}))^{2m/(2m-1)} \, dt = \int_0^{2\pi} (a_0(x) + a_1(x) \cos t)^{2m/(2m-1)} \, dt \\
 G(x) &= \int_0^{2\pi} (x + \cos t)^{2m} \, dt.
 \end{aligned} \tag{3.27}$$

We need to prove that

$$\frac{H(x)}{G(x)} > \frac{H(0)}{G(0)}, \tag{3.28}$$

for some small $x > 0$. Since $H(0), G(0) > 0$, this is equivalent to showing that the function $q(x) = H(x)G(0) - G(x)H(0)$ is increasing for some small $x > 0$. We will prove that by showing that $q'(0) > 0$ (interestingly enough, one may easily show that $q'(0) = 0$ so $q(x)$ as well as $T(x)$, has in fact local minimum at zero).

We will compute $G(0), G'(0), H(0), H'(0)$ next. To do so, observe that

$$\begin{aligned}
 a_0(x) &= \left(\frac{1}{2\pi} \binom{2m-1}{1} F(2m-2) \right) x + \left(\frac{1}{2\pi} \binom{2m-1}{3} F(2m-4) \right) x^3 + \dots \\
 a_1(x) &= \left(\frac{1}{2\pi} F(2m) \right) + \left(\frac{1}{2\pi} \binom{2m-1}{2} F(2m-2) \right) x^2 + \dots,
 \end{aligned} \tag{3.29}$$

therefore

$$\begin{aligned}
 a_0(0) &= 0, & (a_0)'(0) &= \frac{1}{2\pi} (2m-1) F(2m-2), & (a_0)''(0) &= 0, \\
 a_1(0) &= \frac{1}{2\pi} F(2m), & (a_1)'(0) &= 0, & (a_1)''(0) &= \frac{1}{2\pi} (2m-1)(2m-2) F(2m-2).
 \end{aligned} \tag{3.30}$$

Now

$$G(0) = \int_0^{2\pi} (\cos t)^{2m} dt = F(2m),$$

$$G''(x) = \int_0^{2\pi} 2m(2m-1)(x + \cos t)^{2m-2} dt. \quad (3.31)$$

Therefore

$$G''(0) = 2m(2m-1)F(2m-2). \quad (3.32)$$

Also

$$H(0) = \int_0^{2\pi} (a_0(0) + a_1(0) \cos t)^{2m/(2m-1)} dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2\pi} F(2m) \cos t \right)^{2m/(2m-1)} dt \quad (3.33)$$

$$= \left(\frac{1}{2\pi} \right)^{2m/(2m-1)} (F(2m))^{2m/(2m-1)} F\left(\frac{2m}{2m-1} \right),$$

and lastly we will compute $H''(0)$,

$$H''(x) = \left(\int_0^{2\pi} (a_0(x) + a_1(x) \cos t)^{2m/(2m-1)} dt \right)''(x)$$

$$= \frac{2m}{(2m-1)^2} \int_0^{2\pi} (a_0(x) + a_1(x) \cos t)^{-(2m-2)/(2m-1)} (a_0'(x) + a_1'(x) \cos t)^2 dt \quad (3.34)$$

$$+ \frac{2m}{2m-1} \int_0^{2\pi} (a_0(x) + a_1(x) \cos t)^{1/(2m-1)} (a_0''(x) + a_1''(x) \cos t) dt.$$

As a result,

$$H''(0) = \frac{2m}{(2m-1)^2} \int_0^{2\pi} \left(\frac{1}{2\pi} F(2m) \cos t \right)^{-(2m-2)/(2m-1)} \left(\frac{1}{2\pi} (2m-1)F(2m-2) \right)^2 dt$$

$$+ \frac{2m}{2m-1} \int_0^{2\pi} \left(\frac{1}{2\pi} F(2m) \cos t \right)^{1/(2m-1)} \left(\frac{1}{2\pi} (2m-1)(2m-2)F(2m-2) \cos t \right) dt$$

$$= \frac{2m}{(2m-1)^2} \left(\frac{1}{2\pi} F(2m) \right)^{-(2m-2)/(2m-1)} \left(\frac{1}{2\pi} (2m-1)F(2m-2) \right)^2 F\left(-\frac{2m-2}{2m-1} \right)$$

$$+ \frac{2m}{2m-1} \left(\frac{1}{2\pi} F(2m) \right)^{1/(2m-1)} \frac{1}{2\pi} (2m-1)(2m-2)F(2m-2)F\left(\frac{2m}{2m-1} \right)$$

$$= \left(\frac{1}{2\pi} \right)^{2m/(2m-1)} 2m(F(2m))^{-(2m-2)/(2m-1)} (F(2m-2))^2 F\left(-\frac{2m-2}{2m-1} \right)$$

$$+ \left(\frac{1}{2\pi} \right)^{2m/(2m-1)} 2m(2m-2)(F(2m))^{1/(2m-1)} F(2m-2)F\left(\frac{2m}{2m-1} \right). \quad (3.35)$$

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Combining what we have computed, we may state that $H''(0)G(0) - G''(0)H(0) > 0$ is equivalent to

$$\begin{aligned} & \left(\frac{1}{2\pi}\right)^{2m/(2m-1)} 2m(F(2m))^{1/(2m-1)}(F(2m-2))^2 F\left(-\frac{2m-2}{2m-1}\right) \\ & + \left(\frac{1}{2\pi}\right)^{2m/(2m-1)} 2m(2m-2)(F(2m))^{2m/(2m-1)} F(2m-2) F\left(\frac{2m}{2m-1}\right) \\ & > \left(\frac{1}{2\pi}\right)^{2m/(2m-1)} 2m(2m-1)(F(2m))^{2m/(2m-1)} F(2m-2) F\left(\frac{2m}{2m-1}\right), \end{aligned} \quad (3.36)$$

which after simplifications turns out to be

$$F(2m-2)F\left(-\frac{2m-2}{2m-1}\right) > F(2m)F\left(\frac{2m}{2m-1}\right). \quad (3.37)$$

We may observe that $F(2x) > F(2x+2)$ as follows. Since $1 > \cos^2(t)$, we have

$$F(2x) = 4 \int_0^{\pi/4} (\cos(t))^{2x-2} dt > 4 \int_0^{\pi/4} (\cos(t))^{2x-2} (\cos(t))^2 dt = F(2x+2). \quad (3.38)$$

Letting now $2x = 2m - 2$, we get $F(2m - 2) > F(2m)$ and letting $x = -(2m - 2)/(2m - 1)$, we get $F(-(2m - 2)/(2m - 1)) > F(2m/(2m - 1))$. These two inequalities combined together gives (3.37).

Therefore, we have just shown that there is a norming point for P of the form

$$f(t) = (x + \cos t)^{2m-1}, \quad (3.39)$$

for some $x \neq 0$. Now according to (3.23),

$$f|_{\text{span}\{1, \cos t, \sin t\}} = a_0(x) + a_1(x) \cos t, \quad (3.40)$$

where $a_0(x)$, $a_1(x)$ are given by (3.25) and (3.26). Observe that

$$\begin{aligned} a_0(x) &= x \left(\sum_{i=1}^{m-1} p_i x^{2i} \right), \quad p_i > 0 \\ a_1(x) &= \sum_{i=1}^{m-1} q_i x^{2i}, \quad q_i > 0, \end{aligned} \quad (3.41)$$

and since $x \neq 0$ it gives us $a_0(x) \neq 0$ and $a_1(x) \neq 0$. Let g be the norming functional associated with f (i.e., $(f, g) \in \mathfrak{E}_1(P)$). Using (3.40) for some constant K , we have

$$Pf = K(y + \cos t), \quad y \neq 0. \quad (3.42)$$

Moreover $g = \mathfrak{N}(Pf)$ and using Proposition 1.3 for some constant L , we have

$$g = L(y + \cos t)^{1/(2m-1)}. \quad (3.43)$$

We may write

$$g|_{\text{span}[1, \cos t, \sin t]} = b_0(y) + b_1(y) \cos t + b_2(y) \sin t, \quad (3.44)$$

and then we will show that $b_0(y)$. To do so, observe that

$$b_0(y) = L \int_0^2 \pi g(t) dt = 2L \int_0^\pi g(t) dt = 2L \int_0^\pi (y + \cos t)^{1/(2m-1)} dt. \quad (3.45)$$

Without loss of generality, we may assume that $y > 0$. If $y \geq 1$, then the function inside the integral is ≥ 0 and then since $y > 0$, it follows that $b_0(y) > 0$. If $y \in (0, 1)$, then there exists $t_0 \in (0, \pi/2)$ such that $y = \cos(\pi/2 - t_0)$, and we can make some estimations:

$$\begin{aligned} \int_0^\pi (y + \cos t)^{1/(2m-1)} dt &= \int_0^{\pi/2+t_0} (y + \cos t)^{1/(2m-1)} dt + \int_{\pi/2-t_0}^\pi (y + \cos t)^{1/(2m-1)} dt \\ &= \int_0^{\pi/2+t_0} (y + \cos t)^{1/(2m-1)} dt + \int_0^{\pi/2-t_0} (y - \cos t)^{1/(2m-1)} dt \\ &= \int_0^{\pi/2-t_0} \left((y + \cos t)^{1/(2m-1)} - (y - \cos t)^{1/(2m-1)} \right) dt \\ &\quad + \int_{\pi/2-t_0}^{\pi/2+t_0} (y + \cos t)^{1/(2m-1)} dt \geq \int_{\pi/2-t_0}^{\pi/2+t_0} (y + \cos t)^{1/(2m-1)} dt > 0. \end{aligned} \quad (3.46)$$

This proves that $b_0(y) \neq 0$. It is easy to prove that $b_1(y)^2 + b_2(y)^2 > 0$. If not, then $Pg = b$, where b is a constant function. But $f(Pg) = \|P\|$ in L_{2m} and as a result of Proposition 1.3 for some constant C , we would have $f = C(b)^{2m-1}$ which implies that $\|P\| = 1$, a contradiction. Since we know that in (3.40) $a_0(x), a_1(x) \neq 0$ and in (3.44) $b_0(y) \neq 0$ and one of $b_1(y), b_2(y)$ is $\neq 0$, we may apply (3.13) to obtain the result. \square

THEOREM 3.4. *Let $\mathfrak{T}_1 = \text{span}[1, \cos t, \sin t]$. Let P be an orthogonal projection from L_p onto \mathfrak{T}_1 . If $p = 2m$ for some integer $m > 1$, then there is a norming pair (g, f) for P such that*

$$\begin{aligned} f|_{\text{span}[1, \cos t, \sin t]} &= a_0 + a_1 \cos t + a_2 \sin t, \\ g|_{\text{span}[1, \cos t, \sin t]} &= b_0 + b_1 \cos t + b_2 \sin t, \end{aligned} \quad (3.47)$$

where $a_0, a_1, a_2, b_0, b_1, b_2 \neq 0$.

The proof is a straightforward consequence of Theorem 3.3 and Proposition 2.1.

THEOREM 3.5. *Let $\mathfrak{T}_1 = \text{span}[1, \cos t, \sin t]$. Let P be the orthogonal projection from L_p onto \mathfrak{T}_1 . If $p = 2m/(2m - 1)$ or $p = 2m$ for some integer $m > 1$, then P is a unique minimal projection.*

Proof. Having Theorems 3.3 and 3.4, we may apply Theorem 3.2 to obtain uniqueness of the orthogonal projection P . \square

Observe that, due to (3.13), Fourier projection always has infinite many norming points and norming functionals. Therefore, Corollary 2.6 produces us an extremely interesting example.

Remark 3.6. There is a finite dimensional smooth space X and its subspace V such that the minimal projection from X onto V has norm greater than one and it has infinitely many norming points and norming functionals.

The above example shows us the crucial difference between the study of minimal projections in ℓ_p and ℓ_1 spaces.

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