ON THE EXTREMAL SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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We investigate here the properties of extremal solutions for semilinear elliptic equation $-\Delta u = \lambda f(u)$ posed on a bounded smooth domain of $\mathbb{R}^n$ with Dirichlet boundary condition and with f exploding at a finite positive value a.

1. Introduction

We consider the following semilinear elliptic problem:

(P_λ)

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and f satisfies the following condition:

(H) f is a C^2 positive nondecreasing convex function on $[0,\infty)$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

It is well known that under this condition (H), there exists a critical positive value $\lambda^* \in (0,\infty)$ for the parameter λ such that the following holds.

(C_1) For any $\lambda \in (0,\lambda^*)$, there exists a positive, minimal, classical solution $u_\lambda \in C^2(\bar{\Omega})$. The function u_λ is minimal in the following sense: for every solution u of (P_λ), we have $u_\lambda \le u$ on Ω. In addition, the function $\lambda \mapsto u_\lambda$ is increasing and $\lambda_1(-\Delta - \lambda f'(u_\lambda)) > 0$, for example, for any $\varphi \in H_0^1(\Omega)\backslash\{0\}$,

$$\lambda \int_\Omega f'(u_\lambda)\varphi^2\,dx < \int_\Omega |\nabla\varphi|^2 dx. \tag{1.3}$$

(C_2) For any $\lambda > \lambda^*$, there exists no classical solution for (P_λ).

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When λ tends to λ^*,

$$u^* = \lim_{\lambda\to\lambda^*} u_\lambda \tag{1.4}$$

always exists by the monotonicity of u_λ. In [3], Brezis et al. have introduced a notion of weak solution as follows: we say u is a weak solution for (P_λ) if $u \in L^1(\Omega)$, $u \geq 0$, $f(u)\delta \in L^1(\Omega)$ with $\delta(x) = \text{dist}(x,\partial\Omega)$, and

$$\int_\Omega u(-\Delta\xi)dx = \lambda \int_\Omega f(u)\xi\, dx, \tag{1.5}$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi|_{\partial\Omega} = 0$. They then proved the following.

(C_3) u^* is always a weak solution of the problem (P_{λ^*}), and for $\lambda > \lambda^*$ no solution exists even in the weak sense.

Later, Martel proved in [6] that u^* is the unique weak solution of (P_{λ^*}), the so called extremal solution.

The typical examples are when the nonlinearity of f is either exponential $f(u) = e^u$ or power-like $f(u) = (1+u)^p$, $p > 1$ (see [4, 5, 7]). For $f(u) = e^u$, u^* is smooth when $n \leq 9$, if $n \geq 10$, $u^* = -2\ln|x|$ is the extremal solution on $B_1(0)$. When $f(u) = (1+u)^p$, if $n < n_p = 6+4(1+\sqrt{p(p-1)})/(p-1)$, u^* is regular, and for $n \geq n_p$, $u^* = |x|^{-2/(p-1)} - 1$ is the extremal solution on $B_1(0)$. An immediate consequence is that with any $p > 1$ and $n \leq 10$, u^* is a smooth solution. It is natural to ask the following question: for small dimension n, is u^* always a classical solution for any function f satisfying (H) and any domain $\Omega \subset \mathbb{R}^n$? Nedev in [9] and Ye and Zhou in [10] had given some partial answers to this question.

Theorem 1.1 [9]. *Suppose that f satisfies (H), then for $n = 2$ or 3, u^* is always a classical solution. Moreover, when $n \geq 4$, $u^* \in L^q(\Omega)$, for any $q < n/(n-4)$ and $f(u^*) \in L^q(\Omega)$, for any $q < n/(n-2)$.*

Theorem 1.2 [10]. *Let f verify (H), rewrite $f(t) = f(0) + te^{g(t)}$. Assume that there exists t_0 positive such that $t^2g'(t)$ is nondecreasing in $[t_0,\infty)$, then for any $\Omega \subset \mathbb{R}^n$ with $n \leq 9$, u^* is a classical solution.*

On the other hand, Brezis and Vazquez have given a characterization of unbounded extremal solutions in $H_0^1(\Omega)$ as follows: if $v \in H_0^1(\Omega)$ is an unbounded weak solution of (P_λ) with $\lambda > 0$ and satisfying the stability condition

$$\lambda \int_\Omega f'(v)\varphi^2\, dx \leq \int_\Omega |\nabla\varphi|^2 dx, \quad \forall \varphi \in C_1(\bar{\Omega}),\ \varphi|_{\partial\Omega} = 0; \tag{1.6}$$

then $\lambda = \lambda^*$ and $v = u^*$. They remarked also that there exist unbounded weak solutions which satisfy (1.6), but do not belong to $H_0^1(\Omega)$, and which are not extremal solutions.

In this paper, we investigate some similar problems with f exploding at a finite positive value a. More precisely, let f satisfy the following condition:

(H') f is a C^1 positive, nondecreasing, convex function on $[0,a)$ with $a \in (0,\infty)$ and

$$\lim_{t\to a^-} f(t) = +\infty. \tag{1.7}$$

We consider the following problem:

(E_λ)

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega,$$
$$u \in (0,a] \quad \text{in } \Omega, \tag{1.8}$$
$$u = 0 \quad \text{on } \partial\Omega.$$

By the work of Mignot and Puel (see [7]), we have always a critical value $\lambda^* \in (0,\infty)$ such that for any $\lambda \in (0,\lambda^*)$, there exists a positive, minimal, classical solution $u_\lambda \in C^2(\bar{\Omega})$, that is, $u_\lambda < a$ in $(\bar{\Omega})$ and for $\lambda > \lambda^*$, no classical solution exists. The aim of this work is to study the propriety of the solution of (E_λ) at the extremal value $\lambda = \lambda^*$ and to prove the nonexistence of weak solution when $\lambda > \lambda^*$. We define that ω is a weak solution of (E_λ), if $\omega \in L^1(\Omega,[0,a])$ such that $f(\omega)\delta \in L^1(\Omega)$, and for all $\zeta \in C^2(\bar{\Omega})$, with $\zeta = 0$ on $\partial\Omega$,

$$-\int_\Omega \omega\Delta\zeta = \lambda\int_\Omega f(\omega)\zeta. \tag{1.9}$$

Similarly, we say that ω is a weak supersolution of (E_λ), if $\omega \in L^1(\Omega,[0,a])$, such that $(\Delta\omega)\delta \in L^1(\Omega)$, and for all $\zeta \in C^2(\bar{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$,

$$-\int_\Omega \omega\Delta\zeta \geq \lambda\int_\Omega f(\omega)\zeta. \tag{1.10}$$

Our main results are the following.

Theorem 1.3. *Given f satisfying (H'), if $\lambda > \lambda^*$, then there is no weak solution of (E_λ).*

Theorem 1.4. *The function $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is the unique weak solution of (E_{λ^*}). Moreover, for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$,*

$$\lambda^*\int_\Omega f'(u^*)\varphi^2\,dx \leq \int_\Omega |\nabla\varphi|^2 dx. \tag{1.11}$$

Theorem 1.5. *Assume that $v \in H_0^1(\Omega)$ is a weak solution of (E_λ) for some $\lambda > 0$, assume also that* $\sup_\Omega(v) = a$ *and*

$$\lambda\int_\Omega f'(v)\varphi^2\,dx \leq \int_\Omega |\nabla\varphi|^2 dx, \tag{1.12}$$

for all $\varphi \in C^1(\bar{\Omega})$, $\varphi = 0$ on $\partial\Omega$, then $\lambda = \lambda^*$ and $v = u^*$.

2. Proof of Theorem 1.3

In fact, Theorem 1.3 is deduced from a general result, which is the following proposition.

Proposition 2.1. *Given g satisfying (H'), if there exists a weak solution ω of*

$$-\Delta\omega = g(\omega) \quad \text{in } \Omega,$$
$$\omega = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

then, for any $\varepsilon \in (0,1)$, *there exists a classical solution* ω_ε *of*

$$\begin{aligned} -\Delta\omega_\varepsilon &= (1-\varepsilon)g(\omega_\varepsilon) \quad \text{in } \Omega, \\ \omega_\varepsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

For the proof of this result, we need the following lemmas which are proved in [3].

Lemma 2.2. *Given* $g \in L^1(\Omega, \delta(x)dx)$, *there exists a unique* $v \in L^1(\Omega)$ *which is a weak solution of*

$$\begin{aligned} -\Delta v &= g \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

where $\|v\|_{L^1} \le C\|g\|_{L^1(\Omega,\delta(x)dx)}$, *for some* C *constant independent of* g. *In addition, if* $g \ge 0$ *a.e. in* Ω, *then* $v \ge 0$ *a.e. in* Ω.

Lemma 2.3. *Assume* $g(0) > 0$ *and set*

$$h(u) = \int_0^u \frac{ds}{g(s)}, \tag{2.4}$$

for all $0 \le u \le a$. *Let* $\tilde{g}$ *be a* C^1 *positive function on* $[0,a)$ *such that* $\tilde{g} \le g$ *and* $\tilde{g}' \le g'$. *Set*

$$\tilde{h}(u) = \int_0^u \frac{ds}{\tilde{g}(s)}, \qquad \Phi(u) = \tilde{h}^{-1}(h(u)), \tag{2.5}$$

for all $u \in [0,a]$. *Then,*

(i) $\Phi(0) = 0$ *and* $0 \le \Phi(u) \le u$ *for all* $0 \le u \le a$,
(ii) Φ *is increasing, concave, and* $\Phi'(u) \le 1$ *for all* $0 \le u \le a$,
(iii) $h(a) < \infty$ *and* $\Phi(a) < a$, *if* $\tilde{g} \not\equiv g$ *in* $[0,a]$.

Proof. It is easy to see that (i) and (iii) hold. We prove (ii), in fact $\Phi'(u) = \tilde{g}(\Phi(u))/g(u) > 0$, and

$$\Phi''(u) = \frac{g(u)\tilde{g}'(\Phi(u))\Phi'(u) - \tilde{g}(\Phi(u))g'(u)}{g(u)^2} = \frac{\tilde{g}(\Phi(u))(\tilde{g}'(\Phi(u)) - g'(u))}{g(u)^2}. \tag{2.6}$$

Since $\tilde{g}'(\Phi(u)) \le g'(\Phi(u)) \le g'(u)$, it follows that Φ is concave, which completes the proof. □

Proof of Proposition 2.1 and Theorem 1.3. Choosing $\tilde{g} = (1-\varepsilon)g$ in Lemma 2.3 and denote by $v = \Phi(\omega)$, where ω is the weak solution of (2.1) and using an approximating argument for ω, we get

$$\begin{aligned} -\int_{\Omega} v\Delta\zeta &= -\int_{\Omega}\Phi(\omega)\Delta\zeta = -\int_{\Omega}\Delta\Phi(\omega)\zeta = -\int_{\Omega}[\Phi'(\omega)\Delta\omega + \Phi''(\omega)|\nabla\omega|^2]\zeta \\ &\geq \int_{\Omega}\Phi'(\omega)g(\omega)\zeta = \int_{\Omega}\tilde{g}(\Phi(\omega))\zeta = \int_{\Omega}(1-\varepsilon)g(v)\zeta \end{aligned} \tag{2.7}$$

for any $\zeta \in C^1(\bar{\Omega})$, $\zeta \geq 0$ with $\zeta = 0$ on $\partial\Omega$. Hence, v is a weak supersolution of (2.2). The result of Proposition 2.1 follows by standard barrier method as follows. We define a sequence $(\omega_k)_{k\geq 0}$ by

$$\begin{aligned} -\Delta\omega_{k+1} &= (1-\varepsilon)g(\omega_k) \quad \text{in } \Omega, \\ \omega_{k+1} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

for $k \in \mathbb{N}$, with $\omega_0 = v$. Using Lemma 2.2, it is easy to check that $\omega_k \geq \omega_{k+1} \geq 0$, for all $k \in \mathbb{N}$, so the sequence ω_k is nonincreasing and converges in $L^1(\Omega)$ to a weak solution u of (2.2). Since $\sup_{\Omega}(u) \leq \sup_{\Omega}(v) < a$, u is a classical solution, Proposition 2.1 is proved. Theorem 1.3 is deduced by taking $g = \lambda f$ in Proposition 2.1. For any $\lambda > \lambda^*$, let $\varepsilon \in (0,1)$ such that $\lambda^* < (1-\varepsilon)\lambda < \lambda$, since there is no classical solution of

$$\begin{aligned} -\Delta\omega_{\varepsilon} &= (1-\varepsilon)\lambda f(\omega_{\varepsilon}) \quad \text{in } \Omega, \\ \omega_{\varepsilon} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.9}$$

it follows by Proposition 1.3 that there is no weak solution of (E_λ). □

3. Proof of Theorem 1.4

We know that u^* is the increasing limit of classical solution u_λ with positive first eigenvalue, that is, for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$,

$$\lambda\int_{\Omega} f'(u_\lambda)\varphi^2\,dx \leq \int_{\Omega}|\nabla\varphi|^2 dx. \tag{3.1}$$

Passing to the limit, the inequality (1.11) holds. To prove the uniqueness, we will in fact also prove a slightly stronger result.

Proposition 3.1. *Let $v \in L^1(\Omega,[0,a])$ be a weak supersolution of (E_{λ^*}), then $v = u^*$.*

Proof. We proceed in two steps. First, we show that v is a weak solution of (E_{λ^*}). Next, we prove that if $v \not\equiv u^*$, then we obtain a contradiction.

Step 1. Suppose that v is not a weak solution of (E_{λ^*}), then we can assume that there exists $\beta > 0$ and $\xi_0 \in C^2(\bar{\Omega})$, $\xi_0 \geq 0$, with $\xi_0|_{\partial\Omega} = 0$ such that

$$-\int_{\Omega} v\Delta\xi_0 = \lambda^*\int_{\Omega} f(v)\xi_0 + \beta, \tag{3.2}$$

it follows that there exists a nonnegative measure $\mu \not\equiv 0$, with $\mu\delta$ bounded on Ω, such that

$$-\int_{\Omega} v\Delta\xi = \int_{\Omega}(\lambda^* f(v) + \mu)\xi, \tag{3.3}$$

for all $\xi \in C^2(\bar{\Omega})$ with $\xi|_{\partial\Omega} = 0$. Consider φ and χ, the solutions of

$$-\Delta\varphi = \mu \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega,$$
$$-\Delta\chi = 1 \quad \text{in } \Omega, \qquad \chi = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

By $\mu \not\equiv 0$, it follows from the properties of the Laplacian that there exists $\varepsilon > 0$ such that $\varepsilon\chi \leq \varphi$. Set $z = v + \varepsilon\chi - \varphi \leq v$. Then, since f is nondecreasing,

$$-\int_\Omega z\Delta\xi = \int_\Omega (\lambda^* f(v) + \varepsilon)\xi \geq \int_\Omega (\lambda^* f(z) + \varepsilon)\xi, \tag{3.5}$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi \geq 0$, with $\xi|_{\partial\Omega} = 0$. This means that z is a weak supersolution for $-\Delta\omega = g(\omega)$, where $g(v) = \lambda^* f(v) + \varepsilon$. Using the proof of Proposition 2.1 and Lemma 2.3 with $\tilde{g}(v) = \lambda^* f(v) + \varepsilon/2$, we can get a classical solution v_1 of

$$-\Delta v_1 = \lambda^* f(v_1) + \left(\frac{\varepsilon}{2}\right) \quad \text{in } \Omega,$$
$$v_1 = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

Moreover, there exists $\alpha > 0$, such that $2\alpha v_1 \leq \varepsilon\chi$. Set $z = v_1 + \alpha v_1 - (\varepsilon/2)\chi$. It is clear that $0 < z \leq v_1$ and z satisfies $-\Delta z \geq (1+\alpha)\lambda^* f(v_1) \geq (1+\alpha)\lambda^* f(z)$ in Ω. Thus, the classical barrier method gives a solution of $(E_{(1+\alpha)\lambda^*})$, which contradicts then the definition of λ^*, so v is a solution of (E_{λ^*}).

Step 2. Clearly, $v \geq u_\lambda$ for any $\lambda < \lambda^*$, hence $v \geq u^*$. Suppose that $v \not\equiv u^*$, take $\Psi = f(v) - f(u^*) \geq 0$, it is clear that $\Psi\delta \in L^1(\Omega)$. We have then $\Psi \not\equiv 0$, because otherwise $f(v) = f(u^*)$ a.e. on Ω, and Lemma 2.2 will give $v = u^*$ a.e. on Ω. Let g be the weak solution of

$$-\Delta g = \Psi \quad \text{in } \Omega,$$
$$g = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

By the maximum principle, we have $g \geq c\delta$ on Ω for some $c > 0$. Hence,

$$-\int_\Omega (v - u^* - \lambda^* g)\Delta\xi = 0, \tag{3.8}$$

for all $\xi \in C^2(\bar{\Omega})$, with $\xi|_{\partial\Omega} = 0$. We obtain by Lemma 2.2 that $v - u^* = \lambda^* g \geq \lambda^* c\delta$ a.e. on Ω, set $Z = (v + u^*)/2$, then

$$-\int_\Omega Z\Delta\xi = \frac{\lambda^*}{2}\int_\Omega (f(v) + f(u^*))\xi = \lambda^* \int_\Omega (f(Z) + h)\xi > \lambda^* \int_\Omega f(Z)\xi \tag{3.9}$$

for all $\xi \in C^2(\bar{\Omega})$, $\xi \geq 0$, with $\xi|_{\partial\Omega} = 0$, where h is given by

$$h = \frac{1}{2}(f(v) + f(u^*)) - f\left(\frac{v + u^*}{2}\right) = \frac{1}{2}\int_{u^*}^{v} ds \int_{(s+u^*)/2}^{s} f''(\sigma)d\sigma. \tag{3.10}$$

Clearly, $h\delta \in L^1(\Omega)$. Suppose first that $h \equiv 0$, then $f''(\sigma) = 0$ if $\sigma \in [u^*, v]$, hence $f(\sigma) = f(0) + f'(0)\sigma$ on $\cup_{x\in\Omega}[u^*(x), v(x)] = [0, \sup_\Omega v]$, since $v > u^*$ in Ω. Then, if $\sup_\Omega v = a$, we obtain a contradiction by (1.7), and if $\sup_\Omega v < a$, both u^* and v are classical solutions of a linear problem with $f(t) = A + Bt$ for which the uniqueness is known (see, for instance, [8]). If $h \not\equiv 0$, it follows that Z is a strict supersolution of (E_{λ^*}) and we obtain also a contradiction by Step 1. □

4. Proof of Theorem 1.5

Suppose that $\lambda < \lambda^*$. We observe that by a density argument, the inequality (1.12) holds for every $\Phi \in H_0^1(\Omega)$. Taking $\Phi = v - u_\lambda$ in (1.12), we get

$$\lambda \int_\Omega f'(v)(v-u_\lambda)^2 dx \leq \int_\Omega |\nabla(v-u_\lambda)|^2 dx = \lambda \int_\Omega [f(v) - f(u_\lambda)](v-u_\lambda)dx, \quad (4.1)$$

that is,

$$\lambda \int_\Omega [f(v) - f(u_\lambda) - f'(v)(v-u_\lambda)](v-u_\lambda)dx \geq 0. \quad (4.2)$$

Since f is convex and $v \geq u_\lambda$, we get $f(v) = f(u_\lambda) + f'(v)(v-u_\lambda)$ a.e. on Ω. Hence, f must be linear in the interval $[u_\lambda(x), v(x)]$ for a.e. $x \in \Omega$. If $v > u_\lambda$, we get that f is linear in $\cup_x[u(x), v(x)] = [0, \sup_\Omega v) = [0, a)$, which contradicts (1.7). So, $v = u_\lambda$, as v is not a classical solution, we get a contradiction, so $\lambda = \lambda^*$. The similar argument with (1.11) shows that $v = u^*$.

5. Application

Now, we consider a special case $f(u) = 1/(1-u)^p$ with $p > 0$ and $\Omega = B_1(0)$, this problem was studied by Brauner and Nicolaenko in [1, 2]. When $p = 1$, this equation appears as a limit of some problem of disruption in biochemistry; it allows then to justify some phenomenon in kinetic enzymatic and the kinetic of reactors associated to some limit coat. For $n \geq 2$, we know an explicit weak solution

$$U(x) = 1 - |x|^{2/(p+1)}, \quad (5.1)$$

which is obviously in $H_0^1(\Omega)$, it corresponds to the parameter value

$$\lambda^\sharp(n,p) = \frac{2}{p+1}\left(n - \frac{2p}{p+1}\right) > 0. \quad (5.2)$$

The linearized operator is

$$L_\sharp \Phi = -\Delta\Phi - \frac{2p}{p+1}\left(n - \frac{2p}{p+1}\right)\frac{\Phi}{r^2}, \quad (5.3)$$

where $r = |x|$. By Theorem 1.5, U is the extremal solution if and only if for any $\Phi \in H_0^1(\Omega)$,

$$\frac{2p}{p+1}\left(n - \frac{2p}{p+1}\right)\int_B \frac{\Phi^2}{r^2} \leq \int_B |\nabla\Phi|^2 dx. \tag{5.4}$$

Thanks to Hardy's inequality, this holds if and only if (see [4])

$$\frac{2p}{p+1}\left(n - \frac{2p}{p+1}\right) \leq H = \frac{(n-2)^2}{4}. \tag{5.5}$$

Thus, we have the following proposition.

Proposition 5.1. *For any $p > 0$, let*

$$n_0(p) = \frac{2}{p+1}\left[(3p+1) + 2\sqrt{p(p+1)}\right]. \tag{5.6}$$

Then,

(i) *if $n \geq n_0(p)$, $u^*(x) = 1 - |x|^{2/(p+1)}$, and $\lambda^* = \lambda^\sharp$;*
(ii) *if $n < n_0(p)$, $\lambda^* > \lambda^\sharp$ and u^* is smooth.*

Proof. By an easy computation, we have that $n \geq n_0(p)$ is equivalent to (5.5), so (i) is proved by Theorem 1.5. The proof of (ii) is given in [7]. □

We remark that when p tends to 0, $n_0(p)$ tends to 2. So, for any $n \geq 3$, we can meet some nonlinearities f (by choosing appropriate p) such that the extremal solution is no longer classical, this fact is different from the situation for $a = \infty$, if we compare with the results in [9, 10]. Thus, a natural question is raised, for f satisfying (H') and Ω bounded smooth domain in $\mathbb{R}^2$, do we have always that u^* is a classical solution?

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