EXISTENCE OF TRIPLE POSITIVE PERIODIC SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION DEPENDING ON A PARAMETER

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We establish the existence of three positive periodic solutions for a class of delay functional differential equations depending on a parameter by the Leggett-Williams fixed point theorem.

1. Introduction

In this paper, we will discuss the existence of positive periodic solutions for the first-order functional differential equations

$$y'(t) = -a(t)y(t) + \lambda h(t) f\big(y(t-\tau(t))\big), \quad t \in \mathbb{R}, \tag{1.1}$$

$$x'(t) = a(t)x(t) - \lambda h(t) f\big(x(t-\tau(t))\big), \quad t \in \mathbb{R}, \tag{1.2}$$

where $a = a(t)$, $h = h(t)$, and $\tau = \tau(t)$ are continuous T-periodic functions, and $f = f(u)$ is a nonnegative continuous function. We assume that T is a fixed positive number and that $a = a(t)$ satisfies the condition $\int_0^T a(u)du > 0$. The number λ will be treated as a parameter in both equations.

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equations can be interpreted as the standard Malthusian population model $y' = -a(t)y$ subject to perturbations with periodic delays. One important question is whether these equations can support positive periodic solutions. The existence of one or two positive periodic solutions for these functional differential equations has been studied, see for examples [1, 2, 3, 5, 6, 7, 8, 9]. In this paper, we will obtain some existence criteria for three positive periodic solutions when the parameter λ varies.

$E = (E, \|\cdot\|)$ in the sequel is a Banach space, and $C \subset E$ is a cone. By a concave nonnegative continuous functional ψ on C, we mean a continuous mapping $\psi : C \to [0, +\infty)$ with

$$\psi\big(\mu x + (1-\mu)y\big) \geq \mu\psi(x) + (1-\mu)\psi(y), \quad x, y \in C,\ \mu \in [0,1]. \tag{1.3}$$

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Let ξ, α, β be positive constants, we will employ the following notations:

$$\begin{aligned} C_\xi &= \{y \in C : \|y\| < \xi\}, \\ \overline{C}_\xi &= \{y \in C : \|y\| \leq \xi\}, \\ C(\psi,\alpha,\beta) &= \{y \in \overline{C}_\beta : \psi(y) \geq \alpha\}. \end{aligned} \tag{1.4}$$

Our existence criteria will be based on the Leggett-Williams fixed point theorem (see [4]).

Theorem 1.1. *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subset E$ a cone of E and $R > 0$ a constant. Suppose there exists a concave nonnegative continuous functional ψ on C with $\psi(y) \leq \|y\|$ for $y \in \overline{C}_R$. Let $A : \overline{C}_R \to \overline{C}_R$ be a completely continuous operator. Assume there are numbers r, L, and K with $0 < r < L < K \leq R$ such that*

(H1) *the set $\{y \in C(\psi,L,K) : \psi(y) > L\}$ is nonempty and $\psi(Ay) > L$ for all $y \in C(\psi, L,K)$;*
(H2) *$\|Ay\| < r$ for $y \in \overline{C}_r$;*
(H3) *$\psi(Ay) > L$ for all $y \in C(\psi,L,R)$ with $\|Ay\| > K$.*

Then A has at least three fixed points y_1, y_2, and $y_3 \in \overline{C}_R$. Furthermore, $y_1 \in C_r$, $y_2 \in \{y \in C(\psi,L,R) : \psi(y) > L\}$, and $y_3 \in \overline{C}_R \backslash (C(\psi,L,R) \cup \overline{C}_r)$.

2. Existence of triple solutions for (1.1)

A continuously differentiable and T-periodic function $y : \mathbb{R} \to \mathbb{R}$ is called a T-periodic solution of (1.1) associated with ω if it satisfies (1.1) when $\lambda = \omega$ in (1.1).

It is not difficult to check that any T-periodic continuous function $y(t)$ that satisfies

$$y(t) = \lambda \int_t^{t+T} G(t,s)h(s)f\left(y(s-\tau(s))\right)ds, \quad t \in \mathbb{R}, \tag{2.1}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_0^T a(u)du\right) - 1}, \quad t,s \in \mathbb{R}, \tag{2.2}$$

is also a T-periodic solution of (1.1) associated with λ. Note further that

$$\begin{gathered} 0 < N \equiv \min_{t,s\in[0,T]} G(t,s) \leq G(t,s) \leq \max_{t,s\in[0,T]} G(t,s) \equiv M, \quad t \leq s \leq t+T; \\ 1 \geq \frac{G(t,s)}{\max_{t,s\in[0,T]} G(t,s)} \geq \frac{\min_{t,s\in[0,T]} G(t,s)}{\max_{t,s\in[0,T]} G(t,s)} = \frac{N}{M} > 0. \end{gathered} \tag{2.3}$$

For the sake of convenience, we set

$$A_0 = \max_{t\in[0,T]} \int_t^{t+T} G(t,s)h(s)ds,$$
$$B_0 = \min_{t\in[0,T]} \int_t^{t+T} G(t,s)h(s)ds. \tag{2.4}$$

We use Theorem 1.1 to establish the existence of three positive periodic solutions to (2.1). To this end, one or several of the following conditions will be needed:

(S1) $f : [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function,
(S2) $h(t) > 0$ for $t \in \mathbb{R}$,
(S3) $\lim_{x\to 0} f(x)/x = l_1$,
(S4) $\lim_{x\to +\infty} f(x)/x = l_2$.

Let E be the set of all real T-periodic continuous functions endowed with the usual operations and the norm $\|y\| = \max_{t\in[0,T]} |y(t)|$. Then E is a Banach space with cone

$$C = \{y \in E : y(t) \geq 0,\ t \in (-\infty,+\infty)\}. \tag{2.5}$$

Theorem 2.1. *Suppose (S1)–(S4) hold such that $l_1 = l_2 = 0$. Suppose further that there is a number $L > 0$ such that $f(L) > 0$. Let R, K, L, and r be four numbers such that*

$$R \geq K > \frac{LM}{N} \geq L > r > 0, \tag{2.6}$$
$$\frac{f(r)}{r} < \frac{f(R)}{R} < \frac{B_0 f(L)}{A_0 L}. \tag{2.7}$$

Then for each $\lambda \in (L/(B_0 f(L)), R/(A_0 f(R))]$, there exist three nonnegative periodic solutions y_1, y_2, and y_3 of (1.1) associated with λ such that $y_1(t) < r < y_2(t) < L < y_3(t) \leq R$ for $t \in \mathbb{R}$.

Proof. First of all, in view of (S2), $A_0, B_0 > 0$. Note further that if $f(L) > 0$, then by (S1), $f(R) > 0$ for any R greater than L. In view of (S4), we may choose $R \geq K > L$ such that the second inequality in (2.7) holds, and in view of (S3), we may choose $r \in (0,L)$ such that the first inequality in (2.7) holds. We set $\lambda_1 = L/(B_0 f(L))$ and $\lambda_2 = R/(A_0 f(R))$. Then $\lambda_1, \lambda_2 > 0$. Furthermore, $\lambda_1 < \lambda_2$ in view of (2.7). We now define for each $\lambda \in (\lambda_1, \lambda_2]$ a continuous mapping $A : C \to C$ by

$$(Ay)(t) = \lambda \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds, \quad t \in \mathbb{R} \tag{2.8}$$

and a functional $\psi : C \to [0,\infty)$ by

$$\psi(y) = \min_{t\in[0,T]} y(t). \tag{2.9}$$

In view of (S1), (S2), and (2.7), we have

$$\begin{aligned}(Ay)(t) &= \lambda\int_t^{t+T} G(t,s)h(s)f\big(y(s-\tau(s))\big)ds\\ &\le \lambda f(\|y\|)\int_t^{t+T} G(t,s)h(s)ds\\ &\le \lambda f(R)\int_t^{t+T} G(t,s)h(s)ds\\ &\le \lambda A_0 f(R) \le \lambda_2 A_0 f(R) = R,\end{aligned} \tag{2.10}$$

for $t \in [0,T]$ and all $y \in \overline{C}_R$. Therefore, $A(\overline{C}_R) \subset \overline{C}_R$. We assert that A is completely continuous on $\overline{C}_R$. Indeed, in view of the theorem of Arzela-Ascoli, it suffices to show that $A(\overline{C}_R)$ is equicontinuous. To see this, note that for $t_1 < t_2$,

$$\begin{aligned}&\int_{t_2}^{t_2+T} G(t_2,s)h(s)f\big(y(s-\tau(s))\big)ds - \int_{t_1}^{t_1+T} G(t_1,s)h(s)f\big(y(s-\tau(s))\big)ds\\ &\quad= \int_{t_1+T}^{t_2+T} G(t_2,s)h(s)f\big(y(s-\tau(s))\big)ds\\ &\qquad+ \int_{t_2}^{t_1+T}\{G(t_2,s)-G(t_1,s)\}h(s)f\big(y(s-\tau(s))\big)ds\\ &\qquad- \int_{t_1}^{t_2} G(t_1,s)h(s)f\big(y(s-\tau(s))\big)ds.\end{aligned} \tag{2.11}$$

Furthermore,

$$\begin{aligned}&\left|\int_{t_1}^{t_2} G(t_1,s)h(s)f\big(y(s-\tau(s))\big)ds\right|\\ &\qquad\le \left\{f(\|y\|)\int_{t_1}^{t_1+T} G(t,s)h(s)ds\right\}|t_2-t_1|\\ &\qquad\le A_0 f(\|y\|)\,|t_2-t_1|,\\ &\left|\int_{t_1+T}^{t_2+T} G(t,s)h(s)f\big(y(s-\tau(s))\big)ds\right|\\ &\qquad\le \left\{f(\|y\|)\int_{t_1+T}^{t_1+2T} G(t,s)h(s)ds\right\}|t_2-t_1|\\ &\qquad\le A_0 f(\|y\|)\,|t_2-t_1|,\\ &\left|\int_{t_2}^{t_1+T}\{G(t_2,s)-G(t_1,s)\}h(s)f\big(y(s-\tau(s))\big)ds\right|\\ &\qquad\le f(\|y\|)\max_{x\in[0,T]} h(x)\int_{t_2}^{t_1+T}|G(t_2,s)-G(t_1,s)|\,ds\\ &\qquad\le f(\|y\|)\max_{x\in[0,T]} h(x)\int_0^{2T}|G(t_2,s)-G(t_1,s)|\,ds.\end{aligned} \tag{2.12}$$

In view of the uniform continuity of G in $\{(t,s) \mid 0 \le t, s \le 2T\}$, for any $\varepsilon > 0$, there is δ which satisfies

$$0 < \delta < \min\left\{T, \frac{\varepsilon}{3\lambda_2 A_0 f(R)}, \frac{\varepsilon}{3\lambda_2 f(R)[\max_{0\le t,\,s\le 2T} G(t,s)][\max_{x\in[0,T]} h(x)]}\right\}, \quad (2.13)$$

and for $0 < t_2 - t_1 < \delta$, we have

$$|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{6\lambda_2 T f(R)\max_{0\le t\le T} h(t)}, \quad s \in [0,2T]. \quad (2.14)$$

Thus

$$\begin{aligned}
&|(Ay)(t_1) - (Ay)(t_2)| \\
&\quad = \lambda\left|\int_{t_2}^{t_2+T} G(t,s)h(s)f(y(s-\tau(s)))ds - \int_{t_1}^{t_1+T} G(t,s)h(s)f(y(s-\tau(s)))ds\right| \\
&\quad \le \lambda_2\left|\int_{t_1+T}^{t_2+T} G(t_2,s)h(s)f(y(s-\tau(s)))ds\right| \\
&\qquad + \lambda_2\left|\int_{t_2}^{t_1+T} \{G(t_2,s) - G(t_1,s)\}h(s)f(y(s-\tau(s)))ds\right| \\
&\qquad + \lambda_2\left|\int_{t_1}^{t_2} G(t_1,s)h(s)f(y(s-\tau(s)))ds\right| \\
&\quad \le 2\lambda_2 A_0 f(R)|t_2 - t_1| + \lambda_2 f(R)2T\max_{t\in[0,T]} h(t)\frac{\varepsilon}{6\lambda_2 T f(R)\max_{0\le t\le T} h(t)} < \varepsilon
\end{aligned} \quad (2.15)$$

for any $y(t) \in \overline{C}_R$. This means that $A(\overline{C}_R)$ is equicontinuous.

We now assert that Theorem 1.1(H2) holds. Indeed,

$$\begin{aligned}
(Ay)(t) &= \lambda\int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds \\
&\le \lambda f(\|y\|)\int_t^{t+T} G(t,s)h(s)ds \\
&\le \lambda f(r)\int_t^{t+T} G(t,s)h(s)ds \le \lambda_2 A_0 f(r) < r
\end{aligned} \quad (2.16)$$

for all $y \in \overline{C}_r$, where the last inequality follows from (2.7).

In addition, we can show that the condition (H1) of Theorem 1.1 holds. Obviously, $\psi(y)$ is a concave continuous function on C with $\psi(y) \le \|y\|$ for $y \in \overline{C}_R$. We notice that if $u(t) = (1/2)(L+K)$ for $t \in (-\infty,+\infty)$, then $u \in \{y \in C(\psi,L,K) : \psi(y) > L\}$ which implies that $\{y \in C(\psi,L,K) : \psi(y) > L\}$ is nonempty. For $y \in C(\psi,L,K)$, we have $\psi(y) = \min_{t\in[0,T]} y(t) \ge L$ and $\|y\| \le K$. In view of (S1)–(S4), we have

$$\psi(Ay) = \lambda\min_{t\in[0,T]}\int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds \ge \lambda B_0 f(L) > \lambda_1 B_0 f(L) = L \quad (2.17)$$

for all $y \in C(\psi,L,K)$.

Finally, we prove condition (H3) in Theorem 1.1. Let $y \in C(\psi, L, R)$ with $\|Ay\| > K$. We notice that (2.8) implies

$$\|Ay\| \leq \lambda M \int_0^T h(s) f(y(s-\tau(s)))\,ds. \tag{2.18}$$

Thus

$$\begin{aligned}\psi(Ay) &= \lambda \min_{t\in[0,T]} \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))\,ds \\ &\geq \lambda N \int_0^T h(s)f(y(s-\tau(s)))\,ds \\ &\geq \frac{N}{M}\|Ay\| > \frac{N}{M}K > L.\end{aligned} \tag{2.19}$$

An application of Theorem 1.1 stated above now yields our proof. □

We remark that the assumptions of Theorem 2.1 are not vacuous as can be seen by letting $T = 3$, $a(t) \equiv 1$, $h(t) \equiv 1$, and

$$f(x) = \begin{cases} x\ln(1+x), & 0 \leq x < 1, \\ e^x - e + \ln 2, & 1 \leq x < 6, \\ \sqrt{x} - \sqrt{6} + e^6 - e + \ln 2, & x \geq 6. \end{cases} \tag{2.20}$$

Then by taking $r = 3/2$, $L = 5$, $K = 101$, and $R = 5.0135 \times 10^{41}$, we easily check that $A_0 = 1.0524$, $B_0 = 5.239 \times 10^{-2}$ and all the conditions of Theorem 2.1 hold.

Theorem 2.2. *Suppose (S1)–(S4) hold such that $0 < l_1 < l_2$. Suppose there is a number $L > 0$ such that $f(L) > 0$ and $0 < l_1 < l_2 < B_0 f(L)/(A_0 L)$. Let R, K, L, and r be four numbers such that*

$$R \geq K > \frac{LM}{N} \geq L > r > 0, \tag{2.21}$$

$$\frac{f(R)}{R} < l_2 + \varepsilon, \tag{2.22}$$

$$\frac{f(r)}{r} < l_1 + \varepsilon, \tag{2.23}$$

where ε is a positive number such that

$$l_2 + \varepsilon < \frac{B_0 f(L)}{A_0 L}. \tag{2.24}$$

Then for each $\lambda \in (L/(B_0 f(L)), 1/(A_0 l_2))$, (1.1) has at least three nonnegative periodic solutions y_1, y_2, and y_3 associated with λ such that $y_1(t) < r < y_2(t) < L < y_3(t) \leq R$ for $t \in \mathbb{R}$.

Proof. First of all, $A_0, B_0 > 0$ by (S2). Note further that if $f(L) > 0$, then by (S1), $f(R) > 0$ for any R greater than L. Let $\lambda_1 = L/(B_0 f(L))$ and $\lambda_2 = 1/(A_0 l_2)$. Then $\lambda_1, \lambda_2 > 0$. Furthermore, $0 < \lambda_1 < \lambda_2$ in view of the condition $0 < l_1 < l_2 < B_0 f(L)/(A_0 L)$. For positive ε that satisfies (2.24) and any $\lambda \in (\lambda_1, \lambda_2)$, in view of (S4) (and the fact that $\lambda \le 1/(A_0(l_2+\varepsilon))$), there is $R \ge K > L$ such that (2.22) holds, and in view of (S3), there is $r \in (0, L)$ such that (2.23) holds.

We now define for each $\lambda \in (\lambda_1, \lambda_2)$ a continuous mapping $A : C \to C$ by (2.8) and a functional $\psi : C \to [0, +\infty)$ by (2.9). As in the proof of Theorem 2.1, it is easy to see that A is completely continuous on $\overline{C}_R$ and maps $\overline{C}_R$ into $\overline{C}_R$. For all $y \in \overline{C}_R$, we have

$$\begin{aligned}(Ay)(t) &= \lambda \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds \\ &\le \lambda f(\|y\|)\int_t^{t+T} G(t,s)h(s)ds \\ &\le \lambda f(R)\int_t^{t+T} G(t,s)h(s)ds \\ &\le \lambda A_0 f(R) < \lambda A_0(l_2+\varepsilon)R \le R.\end{aligned} \tag{2.25}$$

Furthermore, condition (H2) of Theorem 1.1 holds. Indeed, for $y \in \overline{C}_r$, we have

$$\begin{aligned}(Ay)(t) &= \lambda \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))ds \\ &\le \lambda f(\|y\|)\int_t^{t+T} G(t,s)h(s)ds \\ &\le \lambda f(r)\int_t^{t+T} G(t,s)h(s)ds \\ &\le \lambda A_0 f(r) \le \lambda A_0(l_1+\varepsilon)r < r.\end{aligned} \tag{2.26}$$

Similarly, we can prove that the conditions (H1) and (H3) of Theorem 1.1 hold. An application of Theorem 1.1 now yields our proof. □

Theorem 2.3. *Suppose (S1) and (S2) hold and* $f(0) > 0$. *Suppose there exist four numbers* L, R, K, *and* r *such that (2.6) and (2.7) hold. Then for each* $\lambda \in (L/(B_0 f(L)), R/(A_0 f(R))]$, *(1.1) has at least three positive periodic solutions* y_1, y_2, *and* y_3 *associated with* λ *such that* $0 < y_1(t) < r < y_2(t) < L < y_3(t) \le R$ *for* $t \in \mathbb{R}$.

The proof is similar to Theorem 2.1 and is hence omitted.

3. Existence of triple solutions for (1.2)

Equation (1.2) can be regarded as a dual of (1.1). Therefore, dual existence theorems can be found. Their proofs are obtained by arguments parallel to those for our previous theorems. Therefore, only a short summary will be given. First, (1.2) is transformed into

$$x(t) = \lambda \int_t^{t+T} H(t,s)h(s)f(x(s-\tau(s)))ds, \tag{3.1}$$

where

$$H(t,s)=\frac{\exp\left(-\int_t^s a(u)du\right)}{1-\exp\left(-\int_0^T a(u)du\right)}=\frac{\exp\left(\int_s^{t+T} a(u)du\right)}{\exp\left(\int_0^T a(u)du\right)-1},\quad t\le s\le t+T, \tag{3.2}$$

which satisfies

$$M'\equiv\max_{t,s\in[0,T]}H(t,s)\ge H(t,s)\ge\min_{t,s\in[0,T]}H(t,s)\equiv N',\quad t\le s\le t+T. \tag{3.3}$$

Let

$$\begin{aligned} A' &= \max_{t\in[0,T]}\int_t^{t+T}H(t,s)h(s)ds,\\ B' &= \min_{t\in[0,T]}\int_t^{t+T}H(t,s)h(s)ds. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Suppose (S1)–(S4) hold such that $l_1=l_2=0$. Suppose further that there is a number $L>0$ such that $f(L)>0$. Let R, K, L, and r be four numbers such that*

$$\begin{gathered} R\ge K>\frac{LM'}{N'}\ge L>r>0,\\ \frac{f(r)}{r}<\frac{f(R)}{R}<\frac{B'f(L)}{A'L}. \end{gathered} \tag{3.5}$$

Then for each $\lambda\in(L/(B'f(L)),R/(A'f(R))]$, there exist three nonnegative periodic solutions x_1, x_2, and x_3 of (1.2) associated with λ such that $x_1(t)<r<x_2(t)<L<x_3(t)\le R$ for $t\in\mathbb{R}$.

Theorem 3.2. *Suppose (S1)–(S4) hold such that $0<l_1<l_2$. Suppose there is a number $L>0$ such that $f(L)>0$ and $0<l_1<l_2<B'f(L)/(A'L)$. Let R, K, L, and r be four numbers such that*

$$\begin{gathered} R\ge K>\frac{LM'}{N'}\ge L>r>0,\\ \frac{f(R)}{R}<l_2+\varepsilon,\\ \frac{f(r)}{r}<l_1+\varepsilon, \end{gathered} \tag{3.6}$$

where ε is a positive number such that

$$l_2+\varepsilon<\frac{B'f(L)}{A'L}. \tag{3.7}$$

Then for each $\lambda\in(L/(B'f(L)),1/(A'l_2))$, (1.2) has at least three nonnegative periodic solutions x_1, x_2, and x_3 associated with λ such that $x_1(t)<r<x_2(t)<L<x_3(t)\le R$ for $t\in\mathbb{R}$.

Theorem 3.3. *Suppose (S1) and (S2) hold and $f(0)>0$. Suppose there exist four numbers L, R, K, and r such that (3.5) hold. Then for each $\lambda\in(L/(B'f(L)),R/(A'f(R))]$, (1.2) has at least three positive periodic solutions x_1, x_2, and x_3 associated with λ such that $0<x_1(t)<r<x_2(t)<L<x_3(t)\le R$ for $t\in\mathbb{R}$.*

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