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Variational Iteration Method for Solving Volterra and Fredholm Integral Equations of the Second Kind

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Abstract

In this paper, variational iteration method (VIM) is used to give the approximate solution of Volterra and Fredholm integral equations of the second kind. The method constructs a convergent sequence of functions, which approximates the exact solution with few iterations. To illustrate the ability and reliability of the method, some examples are given, revealing its effectiveness and simplicity.

Keywords: *Variational iteration method; Volterra and Fredholm integral equations*

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1 Introduction

Let $u(x)$ is an unknown functions, $f(x)$ is a given know function, and $k(x,t)$ a know integral kernel.

The Volterra integral equation of the second kind is an integral equation of the form

$$u(x) = f(x) + \int_a^x k(x,t)u(t)\,dt, \qquad (1)$$

And the Fredholm integral equation of the second kind is an integral equation of the form

$$u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt, \qquad (2)$$

The variational iteration method is a new method for solving linear and nonlinear problems and was introduced by a Chinese mathematician, He [1-3]. In [4] He modified the general Lagrange multiplier method [5] and constructed an iterative sequence of functions which converges to the exact solution. In most linear problem the Lagrange multiplier, the approximate solution turns into the exact solution and is available with just one iteration.

To illustrate the method, consider the following general functional equation

$$Lu(x) + N(x) = g(x), \qquad (3)$$

Where L is a linear operator, N is a non-linear operator and $g(t)$ is a known analytical function. According to the variational iteration method, we can construct the following correction functional

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right\}d\xi, \qquad (4)$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory, u_0 is an initial approximation with possible unknowns, and $\tilde{u}_n$ is considered as restricted variation, i.e., $\delta\tilde{u}_n = 0$.

2 Solution of the Volterra and Fredholm Integral Equation of the Second Kind

Consider the Volterra and Fredholm integral equation of the second kind given in Eqs. (1) and (2):

For Eqs. (1) and (2) first we take the partial derivative with respect to x.

For the Volterra integral equation of the second kind we have

$$u'(x) = f'(x) + \frac{d}{dx}\int_a^x k(x,t)u(t)\,dt, \tag{5}$$

and for the Fredholm integral equation of the second kind we have

$$u'(x) = f'(x) + \int_a^b k'(x,t)u(t)\,dt, \tag{6}$$

Consider

$\frac{d}{dx}\int_a^x k(x,t)u(t)\,dt$, and $\int_a^b k'(x,t)u(t)\,dt$, as a restricted variation; we use the variational iteration method in direction x. Then we have the following iteration sequence:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left\{u_n'(\xi) - f'(\xi) - \frac{d}{d\xi}\int_a^\xi k(\xi,t)u_n(t)\,dt\right\}d\xi, \tag{7}$$

Taking the with respect to the independent variable u_n and noticing that $\delta u_n(0) = 0$, we get

$$\delta u_{n+1} = \delta u_n + \lambda \delta u_n\Big|_{\xi=x} - \int_0^x \lambda' \delta u_n\, d\xi = 0\,. \tag{8}$$

Then we apply the following stationary conditions:

$$1 + \lambda(\xi)\Big|_{\xi=x} = 0, \qquad \lambda'(\xi)\Big|_{\xi=x} = 0$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda = -1\,, \tag{9}$$

And as result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{u_n'(\xi) - f'(\xi) - \frac{d}{d\xi}\int_a^\xi k(\xi,t)u_n(t)\,dt\right\}d\xi, \tag{10}$$

3 Numerical Examples

In this section, we applied the method presented in this paper to two examples to show the efficiency of the approach.

Example1. Consider the linear Volterra integral equation

$$u(x) = \cos x - \sin x + 2\int_0^x \sin(x-t)u(t)\,dt \tag{11}$$

The analytical solution of the above problem is given by,

$$u(x) = \exp(-x). \tag{12}$$

In the view of the variational iteration method, we construct a correction functional in the following form:

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{ u_n'(\xi) + \sin\xi + \cos\xi - 2\frac{d}{d\xi}\int_0^{\xi} \{\sin(\xi-t)u_n(t)\}\,dt \right\} d\xi, \tag{13}$$

Starting with the initial approximation $y_0 = \cos x - \sin x$ in Eq. (13) successive approximations $u_i(x)$'s will be achieved. The plot of exact solution Eq.(11), the 5th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 1.

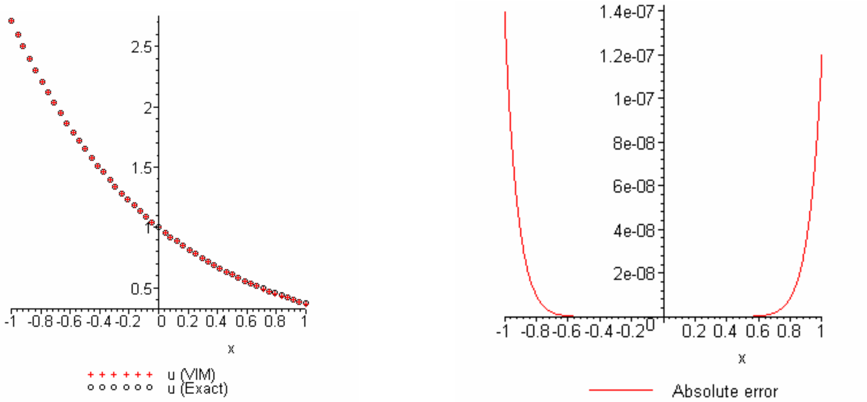

Fig. 1. The plots of approximate solution, exact solution and absolute error for Example 1.

Example 2. Consider the linear Fredholm integral equation

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 x\,t\,u^2(t)\,dt \tag{14}$$

The analytical solution of the above problem is given by,

$$u(x) = x. \tag{15}$$

In the view of the variational iteration method, we construct a correction functional in the following form:

$$u_{n+1}(x)=u_n(x)-\int_0^x\left\{u_n'(\xi)-\frac{7}{8}-\frac{1}{2}\int_0^1 t\,u^2{}_n(t)\,dt\right\}d\xi, \tag{16}$$

Starting with the initial approximation $y_0=\frac{7}{8}x$ in Eq. (16) successive approximations $u_i(x)$'s will be achieved. The plot of exact solution Eq. (14), the 5th order of approximate solution obtained using the VIM and absolute error between the exact and numerical solutions of this example are shown in Fig. 2.

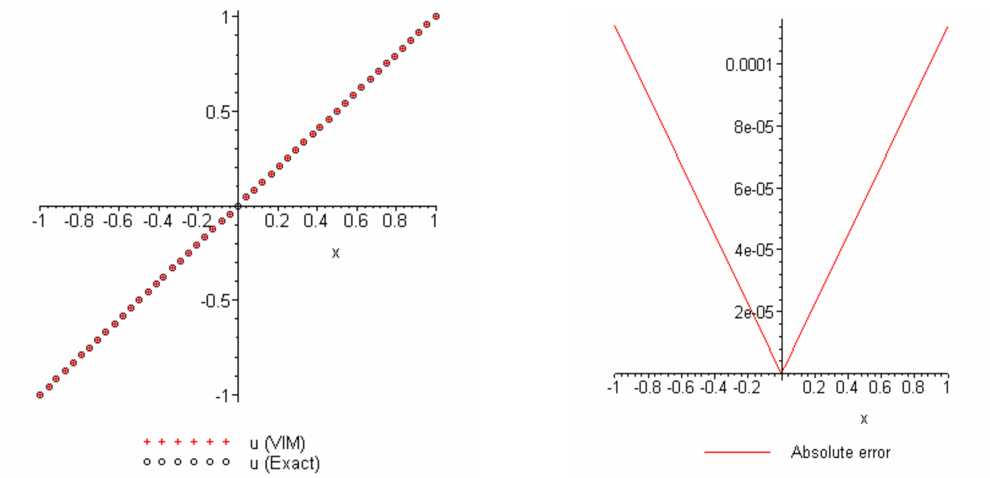

Fig. 2. The plots of approximate solution, exact solution and absolute error for Example 2.

4 Conclusion

In this paper the variational iteration method is used to solve the Volterra and Fredholm integral equations. The results showed that the convergence and accuracy of the variational iteration method for numerically analyzed the Volterra and Fredholm integral equations were in a good agreement with the analytical solutions. The computations associated with the examples in this paper were performed using maple 13.

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