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A Comparison between the Variational Iteration Method and Adomian Decomposition Method

K.R. Raslan[1] and H.A. Baghdady[2]

[1]Department of Mathematics, Faculty of Science
AL-Azhar University, Nasr City, Egypt
E-mail: Kamal_Raslan@yahoo.com
[2]Department of Mathematics, Faculty of Science
AL-Azhar University, (Girls Branch), Nasr City, Egypt
E-mail: hbaghdady@yahoo.com

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Abstract

In this paper, we present a comparative study between the variational iteration method and Adomian decomposition method. The study outlines the significant features of the two methods. The analysis will be illustrated by investigating the "Improved" Modified Kortweg-de Varies equation.

Keywords: *Variational iteration method, Adomian decomposition method, Improved Modified Kortweg-de Varies equation.*

1 Introduction

This paper outlines a reliable comparison between two powerful methods that were recently developed. The first is the variational iteration method (VIM) developed by He in [15–22] and used in [23, 12, 24, 25] among many others. The second is Adomian decomposition method (ADM) developed by Adomian in [13,14], and used heavily in the literature in [2-9] and the references therein. The two methods give rapidly convergent series with specific significant features for each scheme. In this paper, our work stems mainly on two of the most recently developed methods, the VIM and ADM. The two methods, which accurately compute the solutions in a series form or in an exact form, are of great interest to applied sciences. The main advantage of the two methods is that it can be applied directly for all types of differential and integral equations, homogeneous or inhomogeneous. Another important advantage is that the methods are capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. The effectiveness and the usefulness of both methods are demonstrated by finding exact solutions to the models that will be investigated. However, each method has its own characteristic and significance that will be examined.

2 The Governing Equation

In this section, we consider the IMKdV equation which can be written in the form

$$u_t + \varepsilon u^p u_x + \mu u_{xxx} - \nu u_{xxt} = 0, \quad a \le x \le b \tag{1}$$

Where p=1, 2,… is positive integer, ε, μ and ν are positive parameters and the subscripts x and t denote differentiation w.r.t. x and t respectively, with the physical boundary conditions

$$u(a,t) = u(b,t) = 0, u_x(a,t) = u_x(b,t) = 0, u_{xx}(a,t) = u_{xx}(b,t) = 0, \tag{2}$$

in addition to the initial condition $u(x,0) = f(x)$.

It will known that when $p = 1, 2, \ldots$ the "improved" MKdV equation has the single soliton analytic solution [1]:

$$U^p(x,t) = \frac{c(p+1)(p+2)}{2\varepsilon} \operatorname{sech}^2\{k(x - x_0 - ct)\} \tag{3}$$

Where

$$k = \frac{p}{2}\sqrt{\frac{c}{\mu + \nu c}}$$

3 The Conservation Laws

It is of great importance to discuss the conservation laws for our problems, the IMKdV equation possesses three polynomial invariants, these invariants can be derived, easily to be shown in that case as follows[1]:

$$I_1 = \int_a^b U dx$$
$$I_2 = \int_a^b (U^2 + \nu U_x^2) dx \qquad (4)$$
$$I_3 = \int_a^b (U^{p+2} - \frac{(p+1)(p+2)\mu u_x^2}{2\varepsilon}) dx$$

4 Solution by Variational Iteration Method

To illustrate the basic idea of variational iteration method, we consider the following nonlinear functional equation [9, 10, 11, 18]

$$\text{Lu(x) + Nu(x) = g(x)} \qquad (4)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ an inhomogeneous term. According to the VIM, we can express the following correction functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ L_s u_n + \left(L_x + N \right) \tilde{u}_n - g \right\} ds, \qquad (5)$$

where λ are general Lagrange multiplier which can be identified optimally via the variational theory [3], and $\tilde{u}_n$ is a restricted variation which means $\delta \tilde{u}_n = 0$. By this method, we determine first the Lagrange multipliers λ_i (i = 1, 2, 3, 4) which will be identified optimally. The successive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained by suitable choice of trail function u_0. Consequently, the solution is given as

$$\text{u(x, y, z, t)} = \lim_{n \to \infty} u_n(x, y, z, t). \qquad (3)$$

for Eq. (1) in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left\{ \left(u_n(x,s) \right)_s + \mu \left(u_n(x,s) \right)_{xxx} - \nu \left(u_n(x,s) \right)_{xxs} + \varepsilon \tilde{u}_n^p(x,s) \left(\tilde{u}_n(x,s) \right)_x \right\} ds, \qquad (7)$$

After some calculations, we obtain the following stationary conditions

$$1 + \lambda(s)\big|_{s=t} = 0, \qquad \lambda(s)\big|_{s=t} = 0, \qquad \lambda'(s)\big|_{s=t} = 0. \qquad (8)$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda = -1 \qquad (9)$$

Substituting this value of the Lagrange multiplier into the functional (7) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \left(u_n(x,s)\right)_s + \mu\left(u_n(x,s)\right)_{xxx} - \nu\left(u_n(x,s)\right)_{xxs} + \varepsilon \tilde{u}_n^{\,p}(x,s)\left(\tilde{u}_n(x,s)\right)_x \right\} ds, \tag{10}$$

This is turn the first few components if p=2

$$U(x,0) = u_0(x,t) = \frac{c(p+1)(p+2)}{2\varepsilon}\operatorname{sec}h\{k(x-x_0)\}$$

$$u_1(x,t) = \frac{3c}{2\varepsilon^2}\operatorname{sec}h^4\{k(x-x_0)\} + (3\varepsilon\cosh\{k(x-x_0) + \varepsilon\cosh\{3k(x-x_0)$$
$$+ 2kt(72c^2 - 11k^2\varepsilon\mu + k^2\varepsilon\mu\cosh\{2k(x-x_0)\})\sinh\{k(x-x_0)\})$$

$$u_2(x,t) = -\frac{6c}{\varepsilon^5}\operatorname{sec}h\{k(x-x_0)\}((18k^3t^3(36c^3 - 5ck^2\varepsilon\mu)^2\operatorname{sec}h^{11}\{k(x-x_0)\}(3\varepsilon\cosh\{k(x-x_0)\}$$
$$+\varepsilon\cosh\{3k(x-x_0)\} + 8kt(-27c^2 + 5k^2\varepsilon\mu)\operatorname{sec}h^3\{k(x-x_0)\})\tanh\{k(x-x_0)\}$$
$$36c^2k^4t^4(36c^2 - 5k^2\varepsilon\mu)^3\operatorname{sec}h^{10}\{k(x-x_0)\}\tanh^2\{k\{x-x_0)\} - \varepsilon^4(1$$
$$+k^3t\mu\tanh^3\{k(x-x_0)\} + k^5t\mu\nu\tanh^5\{k(x-x_0)\} + k^6t^2\mu^2\tanh^6\{k(x-x_0)\})$$
$$+k^2t^2\varepsilon\operatorname{sec}h^6\{k(x-x_0)\}(\varepsilon(1296c^4 - 576c^2k^2\varepsilon\mu + 61k^4\varepsilon^2\mu^2) - 36c^2kt(9072c^4$$
$$-2808c^2k^2\varepsilon\mu + 1512c^2k^4t^2\mu(216c^4 - 66c^2k^2\varepsilon\mu + 5k^4\varepsilon^2\mu^2)\tanh^6\{k(x-x_0)\})$$
$$+k^2t\varepsilon^2\operatorname{sec}h^4\{k(x-x_0)\}\tanh\{k(x-x_0)\}(k\varepsilon(369c^2 - 61k^2\varepsilon\mu)\nu - t(6480c^4$$
$$-3960c^2k^2\varepsilon\mu + 479k^4\varepsilon^2\mu^2)\tanh\{k(x-x_0)\} - 288c^2k^3t^2\mu(45c^2 - 7k^2\varepsilon\mu)\tanh^4\{k(x-x_0)\}$$
$$-144c^2k^6t^3\mu^2(45c^2 - 7k^2\varepsilon\mu)\tanh^7\{k(x-x_0)\}) - kt\varepsilon^3\operatorname{sec}h^2\{k(x-x_0)\}\tanh\{k(x-x_0)\}$$
$$(36c^2 - 5k^2\varepsilon\mu + 2k^2(162c^2 - 29k^2\varepsilon\mu)\nu\tanh^2\{k(x-x_0)\} + k^3t\mu(1080c^2 - 179k^2\varepsilon\mu)$$
$$\tanh^3\{k(x-x_0)\} + 108c^2k^6t^2\mu^2\tanh^6\{k(x-x_0)\} + 36c^2k^9t^3\mu^3\tanh^9\{k(x-x_0)\}))$$

5 Solution by the Adomian Decomposition Method

Let us rewrite equation (1) in the form

$$L_t(u) = -\varepsilon N(u) - \mu L_x(u_x) + \nu L_x(u_t), \tag{11}$$

with the initial condition

$$U^{p}(x,t)=\frac{c(p+1)(p+2)}{2\varepsilon}\operatorname{sech}^{2}\{k(x-x_{0})\} \tag{12}$$

where $L_t(\bullet)=\frac{\partial}{\partial t}, L_x=\frac{\partial^2}{\partial x^2}(\bullet)$

and N(u) represents the nonlinear term $u^p u_x$.

Assuming that the inverse operator is $L_t^{-1}(\bullet)=\int_0^t(\bullet)dt$ and then operating on both sides of Eq. (11) with L_t^{-1}

$$u(x,t)=u(x,0)-\varepsilon L_t^{-1}(N(u))-\mu L_t^{-1}(L_x(u_x))+\nu L_t^{-1}(L_x(u_t)) \tag{13}$$

The ADM assumes that the unknown function u(x, t) can be expressed as a sum of components defined in a series of the form:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{14}$$

And the nonlinear operator N(u) can be written as

$$N(u)=\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n) \tag{15}$$

where A_n are called Adomian polynomials. The Adomian polynomials can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [13, 14]. The polynomials A_n are given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)^p\left(\sum_{i=0}^{\infty}\lambda^i(u_i)_x\right)\right)\Bigg|_{\lambda=0}, n\geq 0 \tag{16}$$

Substituting (14), (16) into the equation (13) gives

$$u(x,t)=u(x,0)-\varepsilon L_t^{-1}\left(\sum_{n=0}^{\infty}A_n\right)-\mu L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}(u_n)_x\right)\right)+\nu L_t^{-1}\left(L_x\left(\sum_{n=0}^{\infty}(u_n)_t\right)\right) \tag{17}$$

And we identify the zeros component $u_0 = u(x,0)$ by terms arising from initial conditions, and we obtain the subsequent components using the following recursive relation:

$$\left.\begin{aligned}&u_0=u(x,0)\\&u_{n+1}=-\varepsilon L_t^{-1}(A_n)-\mu L_t^{-1}(L_x(u_n)_x)+\nu L_t^{-1}(L_x(u_n)_t), \qquad n\geq 0,\end{aligned}\right\} \tag{18}$$

Where A_n are Adomian polynomials that represent the nonlinear term $u^2 u_x$ and given by:

$$
\begin{aligned}
&A_0 = u_0^p (u_0)_x, \quad A_1 = p u_0^{p-1} u_1 (u_0)_x + u_0^p (u_1)_x \\
&A_2 = \frac{p(p-1)}{2} u_0^{p-2} u_1^2 (u_1)_x + p u_0^{p-1} u_2 (u_0)_x + p u_0^{p-1} u_1 (u_1)_x + u_0^p (u_2)_x \\
&A_3 = p u_0^{p-1} u_3 (u_0)_x + p(p-1) u_0^{p-2} u_1 u_2 (u_0)_x + \frac{p(p-1)}{2} u_0^{p-2} u_1^3 (u_0)_x + \\
&\quad p u_0^{p-1} u_2 (u_1)_x + p u_0^{p-1} u_1 (u_2)_x + u_0^p (u_3)_x
\end{aligned}
\tag{19}
$$

And so on. The rest of polynomials can be constructed in a similar manner.

The first few components of $u_n(x,t)$ follows immediately upon setting

$$
\begin{aligned}
&u_0(x) = f(x) \\
&u_1(x) = L^{-1}\left(-\varepsilon A_0 - \mu L_x (u_0)_x + \nu L_x (u_0)_t\right) \\
&u_2(x) = L^{-1}\left(-\varepsilon A_1 - \mu L_x (u_1)_x + \nu L_x (u_1)_t\right) \\
&u_3(x) = L^{-1}\left(-\varepsilon A_2 - \mu L_x (u_2)_x + \nu L_x (u_2)_t\right) \\
&u_4(x) = L^{-1}\left(-\varepsilon A_3 - \mu L_x (u_3)_x + \nu L_x (u_3)_t\right)
\end{aligned}
\tag{20}
$$

The scheme in (20) can easily determine the components $u_n(x,t), n \geq 0$.

It is possible to calculate more components in the decomposition series to enhance the approximation. Consequently, one can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n(x,t)$, and hence the solution u(x,t) is readily obtained in a series form. The obtained series may lead to the exact solution.

Adomian decomposition method gives the recurrence relation:

$$
\left.
\begin{aligned}
&u_0(x,t) = \frac{c(p+1)(p+2)}{2\varepsilon} \operatorname{sech}^2\{k(x-x_0)\} \\
&u_{n+1} = L_t^{-1}(-\varepsilon(A_n) - \mu(L_x(u_n)_x) + \nu(L_x(u_n)_t)), \qquad n \geq 0,
\end{aligned}
\right\}
\tag{21}
$$

In our test problems we pay attention to these three invariants, and make sure that these laws are always satisfied.

6 Applications

Following we apply our numerical scheme on the type of nonlinear equation we are handling which is the IMKdV equation.

6.1 Single Soliton Solution

Consider the initial value problem associated with the IMKdV equation (1) with initial condition Eq. (12)

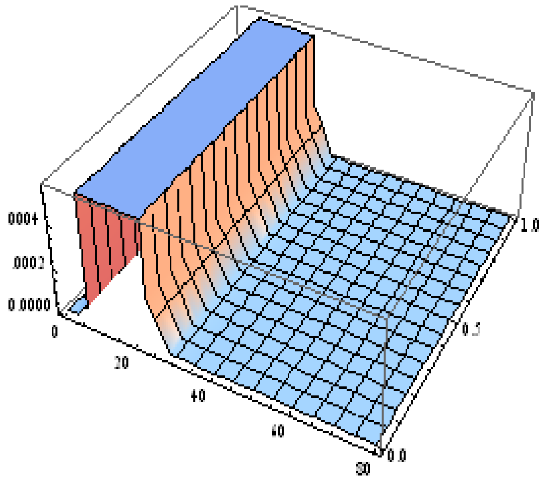

Fig. (1.a): The VIM solution for IMKdV equation for c=0.01 and $0 \leq x \leq 80$, $0 \leq t \leq 1$.

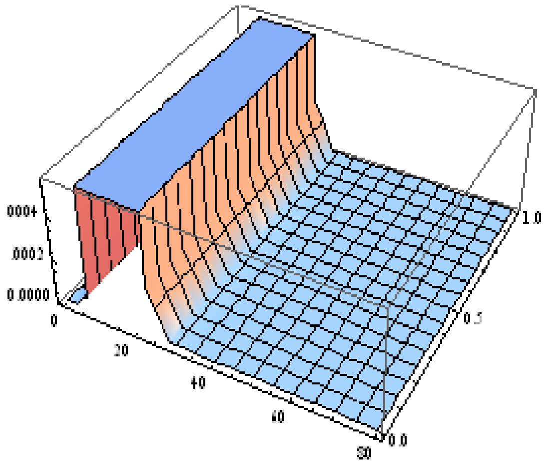

Fig. (1.b): ADM solution for IMKdV equation for c=0.01 and $0 \leq x \leq 80$, $0 \leq t \leq 1$

Table (1): Invariants for the IMKdV equation by VIM and ADM with c=0.01, n=2, [0, 80]

T	I_1VIM	I_2VIM	I_3VIM	I_1 *ADM*	I_1ADM	I_1ADM
0.0	0.78806	0.16087	4.8972E-03	0.78806	0.160868	4.8972E-03
0.2	0.78805	0.16159	4.8938E-03	0.78805	0.161592	4.8945E-03
0.4	0.78802	0.16377	4.8837E-03	0.78804	0.163765	4.8866E-03
0.6	0.78796	0.16739	4.8668E-03	0.78801	0.167386	4.8734E-03
0.8	0.78789	0.17246	4.8433E-03	0.78797	0.172456	4.8549E-03
1	0.78779	0.17899	4.8131E-03	0.78793	0.178975	4.8312E-03

Table (2): Comparison between ADM and VIM with absolute error for the IMKdV (n=2, ε=1, c=0.01)

X	*T=0.2*		*T=0.8*	
	Abs.Error ADM	*Abs. Error VIM*	*Abs.Error ADM*	*Abs. Error VIM*
10	6.49383E-06	6.49386E-06	2.58911E-05	2.5892E-05
20	6.50793E-06	6.508E-06	2.6117E-05	2.6116E-05
30	3.7846E-10	3.7845E-10	1.5187E-9	1.5187E-09
40	2.1702E-14	2.17E-14	8.7088E-14	8.7087E-14
50	1.2445E-18	1.2445E-18	4.9939E-18	4.9939E-18
60	7.1363E-23	7.1363E-23	2.8637E-22	2.8637E-22
70	4.0923E-27	4.0922E-27	1.6422E-26	1.6422E-26
80	2.3467E-31	2.3467E-31	9.4168E-31	9.4168E-31

6.2 Interaction of Two Solitary Waves

The interaction of two IMKdV solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the IMKdV equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes.

For *p = 2*

$$u_0 = u(x,0) = u_1 + u_2 \tag{22}$$

Where

$$u_i = c_i \operatorname{sec} h\{A_i x + x_i\} \tag{23}$$

Where; $c_1 = -0.17, A_1 = c_1\sqrt{\frac{c_1}{\mu + c_1 \nu}}, x_1 = 58, c_2 = -0.34, A_2 = c_2\sqrt{\frac{c_2}{\mu + c_2 \nu}}, x_2 = 23$.

The values of I_1, I_2 and I_3 throughout the simulation are shown in table (3).

Table (3): Invariants for the IMKdV equation with n=2, [0, 80]

T	I_1VIM	I_2VIM	I_3VIM	I_1ADM	I_2ADM	I_3ADM
0	-3.11468	0.705623	0.0522919	-3.11468	0.705623	0.0522919
0.2	-3.11468	0.705616	0.0522894	-3.11468	0.705621	0.0522909
0.4	-3.11468	0.705597	0.0522819	-3.11468	0.705615	0.0522877
0.6	-3.11468	0.705566	0.0522693	-3.11468	0.705604	0.0522825
0.8	-3.11468	0.705522	0.0522518	-3.11468	0.70559	0.0522753
1	-3.11468	0.705466	0.0522294	-3.11468	0.705572	0.0522659

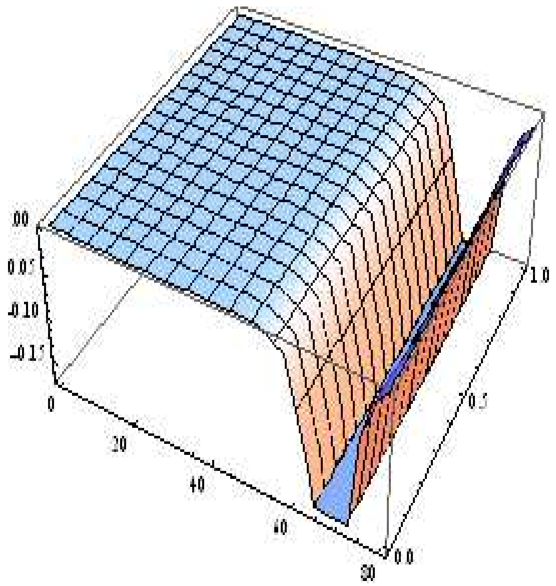

Fig (2.a): VIM interaction two solitary waves of IMKdV equation at times $0 \le t \le 1$

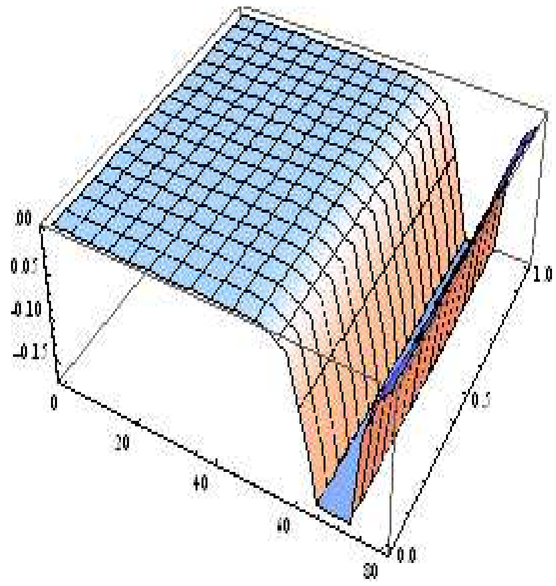

Fig (2.b): ADM interaction two solitary waves of IMKdV equation at times $0 \le t \le 1$

We have repeated this experiment with interacting waves which are two negative solitary waves, we take $c_1 = -0.34, c_2 = 0.17, D_1 = 23$ and $D_2 = 38$ our results are given in table (4).

Table (4): Invariants for the IMKdV equation with n=2, [0, 80]

T	I_1VIM	I_2VIM	I_3VIM	I_1ADM	I_2ADM	I_3ADM
0	-3.11468	0.705623	0.0522919	-3.11468	0.705623	0.0522919
0.2	-3.11468	0.705616	0.0522894	-3.11468	0.705621	0.0522909
0.4	-3.11468	0.705597	0.0522819	-3.11468	0.705615	0.0522877
0.6	-3.11468	0.705566	0.0522693	-3.11468	0.705604	0.0522825
0.8	-3.11468	0.705522	0.0522518	-3.11468	0.70559	0.0522753
1	-3.11468	0.705466	0.0522294	-3.11468	0.705572	0.0522659

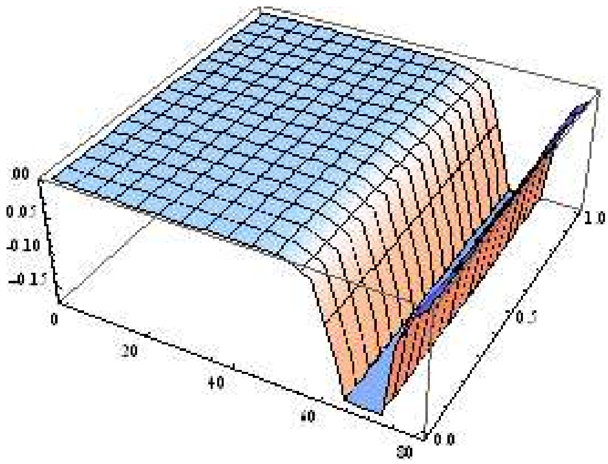

Fig (3.a): VIM interaction two solitary waves of IMKdV equation at times $0 \leq t \leq 1$

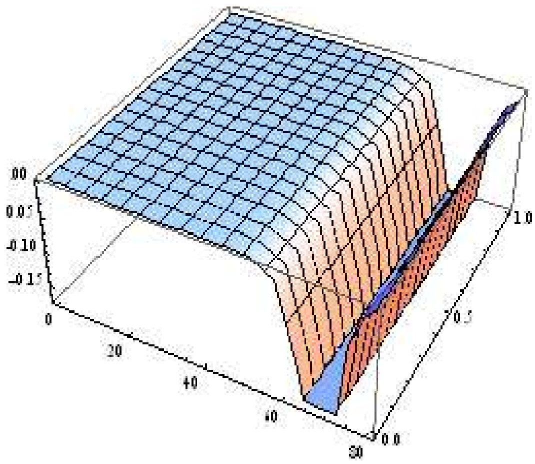

Fig (3.b): ADM interaction two solitary waves of IMKdV equation at times $0 \leq t \leq 1$

7 Conclusions

In this paper, VIM has been successfully applied to finding the solutions of "Improved" Modified Kortweg-de Varies equation. The obtained solutions are compared with those of ADM.the results of the present method are in approximate agreement with those of ADM. The two methods are powerful and efficient methods that both give approximations of higher accuracy. The two methods are powerful mathematical tool for solving linear and nonlinear partial differential equations.

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