Gen. Math. Notes, Vol. 21, No. 1, March 2014, pp. 97-117

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On the Nullity of Expanded Graphs

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(Received: 13-12-13 / Accepted: 22-1-14)

Abstract

The nullity (degree of singularity) $\eta(G)$ of a graph G is the multiplicity of zero as an eigenvalue in its spectrum. It is proved that, the nullity of a graph is the number of non-zero independent variables in any of its high zero-sum weightings. Let u and v be nonadjacent coneighbor vertices of a connected graph G, then $\eta(G) = \eta(G-u) + 1 = \eta(G-v) + 1$. If G is a graph with a pendant vertex (a vertex with degree one), and if H is the subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$. Let H be a graph of order n and G_1, G_2,…, G_n be given vertex disjoint graphs, then the expanded graph $H_n^{G_i}$ is a graph obtained from the graph H by replacing each vertex v_i of H by a graph G_i with extra sets of edges $S_{i,j}$ for each edge v_iv_j of H in which $S_{i,j} = \{uw: u\in V(G_i), w\in V(G_j)\}$. In this research, we evaluate the nullity of expanded graphs, for some special ones, such as null graphs, complete bipartite graphs, star graphs, complete graphs, nut graphs, paths, and cycles.

Keywords: *Graph Theory, Graph Spectra, Nullity of a Graph.*

I Introduction

A graph G is said to be a **singular graph** provided that its adjacency matrix A(G) is a singular matrix. The eigenvalues λ_1, λ_2, …λ_p of A(G) are said to be the eigenvalues of the graph G, which form the spectrum of G. The occurrence of zero as an eigenvalue in the spectrum of the graph G is called its **nullity (degree of singularity)** and is denoted by $\eta(G)$. See [1] and [3].

Definition 1.1[2, 5, 7] *A graph G is said to be* η ***-singular*** *or the nullity of G is* η*, abbreviated* $\eta(G)$ *or* η *if, the multiplicity of zero (as an eigenvalue) in* $S_p(G)$ *is* η.

Definition 1.2[2] *A **vertex weighting** of a graph G is a function* $f: V(G) \rightarrow R$ *where R is the set of real numbers, which assigns a real number (weight) to each vertex.*

*A weighting of G is said to be **non-trivial** if there is at least one vertex* $v \in V(G)$ *for which* $f(v) \neq 0$.

Definition 1.3[2] *A non-trivial vertex weighting of a graph G is called a **zero-sum weighting** provided that for each* $v \in V(G)$, $\sum f(u) = 0$, *where the summation is taken over all* $u \in N_G(v)$.

Clearly, the following weighting for G is a non-trivial zero-sum weighting, where x and y are weights and $(x, y) \neq (0, 0)$, *as indicated in Fig.1.1.*

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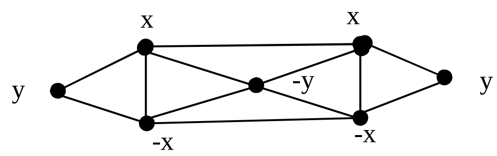


Figure 1.1: A non-trivial zero-sum weighting of a graph

Definition 1.4[5] *Out of all zero-sum weightings of a graph G, a **high zero-sum weighting** of G is one that uses maximum number of non-zero independent variables,* $M_v(G)$.

An important relation between the singularity of a graph, and existence of a zero-sum weighting is, that a graph is singular iff it possesses a non trivial zero-som weighting.[2]

Proposition 1.5[5] *In any graph G, the maximum number of non-zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of G.*

In Fig. 1.1, the weighting for the graph G is a high zero-sum weighting that uses 2 independent variables, hence, $\eta(G) = 2$.

Let r(A(G)) be the rank of A(G). Clearly, $\eta(G) = p - r(A(G))$. The rank r(G) of a graph G is the rank of its adjacency matrix A(G). Then, each of $\eta(G)$ and r(G) determines the other.

The nullity of some known graphs such as cycle C_n , path P_n, complete K_p and complete bipartite $K_{r,s}$ graphs are given in the next lemma.

Lemma 1.6[5, 6, 7]

i) The eigenvalues of the cycle C_n are of the form:

$2\cos\frac{2\pi r}{n}$, *r = 0, 1, …, n-1. According to this,*

$$\eta(C_n) = \begin{cases} 2, & \text{if } n = 0 \,(\text{mod}\, 4), \\ 0, & \text{otherwise.} \end{cases}$$

ii) The eigenvalues of the path P_n are of the form:

$2\cos\frac{\pi r}{n+1}$, *r =1,2, … n. And thus,*

$$\eta(P_n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

iii) The nullity of the complete graph K_p, is:

$$\eta(K_p) = \begin{cases} 1, & \text{if } p = 1, \\ 0, & \text{if } p > 1. \end{cases}$$

iv) The nullity of the complete bipartite graph $K_{r,s}$, is:

$$\eta(K_{r,s}) = \begin{cases} 0 & \text{if } r = s = 1, \\ r + s - 2 & \text{otherwise.} \end{cases}$$

Definition 1.7: *Two vertices of a graph G are said to be of the* ***same type (coneighbors)*** *if they are not adjacent and have the same set of neighbors. Thus, the two vertices v_i, v_j of the same type have the same row vectors $R_i = R_j$ describing them, where R_i and R_j are the* i^{th} *and* j^{th} *row vector of A(G), corresponding to the vertices v_i and v_j, i, j = 1, 2, …, p. Each pair of such (same type) vertices results in two dependent (coincide) rows which yield a zero in spectra of the graph G. It is clear that the occurrence of m equal rows contributes (m-1) to the nullity.*

Corollary 1.8[4] (End Vertex Corollary): *If G is a graph with a pendant vertex, and H is the subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$.*

Applying Corollary 1.8, several times, deleting v_1 and v_3 respectively is illustrated in the next figure.

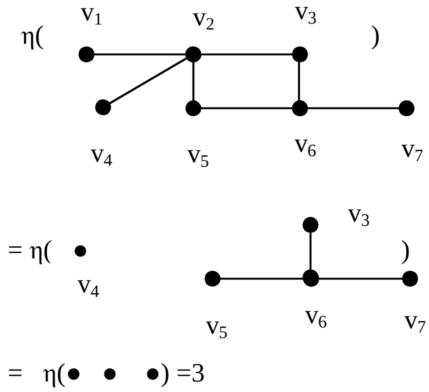


Figure 1.2: Illustration of Corollary 1.8

So (End Vertex Corollary) is a strong tool to determine the nullity of trees.

Operations on Graphs

Many interesting graphs are obtained from combining pairs (or more) of graphs or operating on a single graph in some way. We now discuss a number of operations which are used to combine graphs to produce new ones.

Lemma 1.9[1, 3] *Let $G = G_1 \cup G_2 \cup \ldots \cup G_t$, where $G_1, G_2, \ldots, G_t$ are connected components of G, then* $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$

Definition 1.10[1, 3] *The **join** G_1+G_2 of two graphs G_1 and G_2 is a graph whose vertex set, $V(G_1+G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv:$ for all $u \in G_1$ and all $v \in G_2\}$.*

Definition 1.11[1] *The **sequential join** $G_1+G_2+ \ldots +G_n$ of n disjoint graphs $G_1, G_2, \ldots, G_n$ is the graph $(G_1+G_2) \cup (G_2+G_3) \cup \ldots \cup (G_{n-1}+G_n)$ and denoted by*

$$\sum_{i=1}^{n} G_i \text{ , for i=1,2,…,n and}$$

defined by

$$V(\sum_{i=1}^{n} G_i) = \bigcup_{i=1}^{n} V(G_i), \ E(\sum_{i=1}^{n} G_i) = \{\bigcup_{i=1}^{n} E(G_i)\} \cup$$

$$\{uv: \text{for all } u \in G_i \text{ and all } v \in G_{i+1}, i=1,2,\ldots,n-1\}.$$

As depicted in Fig.1.3.

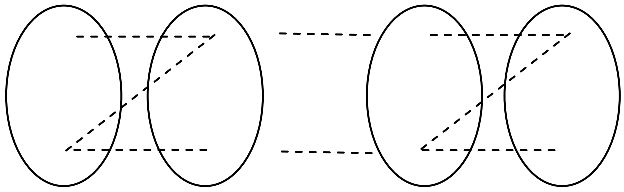


Figure 1.3: The sequential join graph $\sum_{i=1}^{n} G_i$

It is clear that, $p(\sum_{i=1}^{n} G_i) = \sum_{i=1}^{n} p_i$, $q(\sum_{i=1}^{n} G_i) = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1}$,

In which $p_i = p(G_i)$ and $q_i = q(G_i)$.

Definition 1.12[4] *Let P_n be a path with vertex $\{v_1, v_2,\dots,v_p\}$. Replacing each vertex v_i by an empty graph N_{p_i} of order p_i, for $i=1,2,\dots,p$ and joining edges between each vertex of N_{p_i} and each vertex of $N_{p_{i+1}}$ for $i=1,2,\dots,n-1$, we get a graph order $p_1+p_2+\dots+p_n$, denoted by $\sum_{i=1}^{n} N_{p_i}$. Such graph is called a* ***sequential join.***

Definition 1.13[1] *The* ***strong product graph*** *$G_1 \boxtimes G_2$ of G_1 and G_2, is the union of the Cartesian product graph $G_1 \times G_2$ and the Kronecker product graph $G_1 \otimes G_2$.*

Clearly, $p(G_1 \boxtimes G_2) = p(G_1)p(G_2)$ and $q(G_1 \times G_2) = p(G_1)q(G_2) + p(G_2)\, q(G_1) + 2\, q(G_1)\, q(G_2)$.

It is apparent that $K_m \boxtimes K_n = K_{mn}$.

Results, relating the nullity of the graphs G_1 and G_2 *and* their strong product $G_1 \boxtimes G_2$, are not studied widely.

We conclude that, if both G_1 and G_2 are singular graphs, then so is $G_1 \boxtimes G_2$.

Definition 1.14[1] *The* ***corona*** $G = G_1 \circ G_2$ *of two disjoint graphs G_1 and G_2 is defined as the graph obtained from taking one copy of G_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

As illustrated in Figure 1.8, where the copies of G_2 are denoted by G'_1, G'_2, …, G'_{p_1}, $V_1 = V(G_1) = \{v_1, v_2, ..., v_{p_1}\}$, $U^{(i)} = V(G'_i) = \{u_1^{(i)}, u_2^{(i)}, ..., u_{p_2}^{(i)}\}$, for $i = 1, 2, ..., p_1$, , and $V(G) = V_1 \bigcup_{i=1}^{p_1} U^{(i)}$

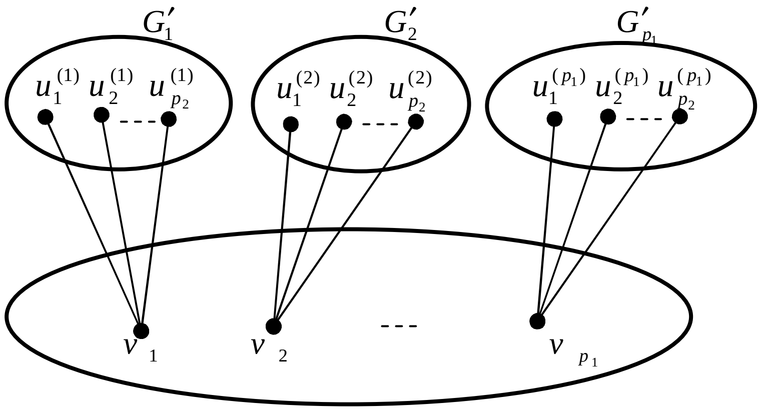


Figure 1.4: The corona G_1 e G_2

From the definition of the corona, it is clear that G_1 e G_2 is connected iff G_1 is connected. Also if G_2 contains at least one edge, then G_1 e G_2 is not bipartite graph.

And $p(G_1 \text{ e } G_2) = p_1(1 + p_2)$,

$q(G_1 \text{ e } G_2) = q_1 + p_1 q_2 + p_1 p_2$, with a diameter

$diam(G_1 \text{ e } G_2) = diam(G_1) + 2$.

Note that $G_1 \text{ e } G_2 \neq G_2 \text{ e } G_1$ unless $G_1 \cong G_2$.

Studying the nullity of the corona graph G_1 e G_2 is one of the main subjects discussed in the present study.

II On the Nullity of the Sequential Join of Some Special Graphs

The nullities of the sequential join of some special graphs, N_p, $K_{r,s}$, S_p, K_p, P_m and C_p are determined in this section.

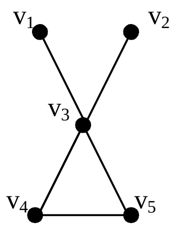
Definition 2.1: *Let the graphs G_1, G_2, …,G_n be given. An* ***expanded graph*** *(expanded join graph) $H_n^{G_i}$ of a labeled graph H of vertex set{v_1, v_2,…, v_n}, is a graph obtained from H by replacing each vertex v_i by the graph G_i, i = 1,2,…,n, with extra sets of edges $S_{i,j}$ for each edge v_iv_j of H in which $S_{i,j}$ = {uw: $u \in G_i$, $w \in G_j$}. We call G_i's inserting graphs and H the base graph. Thus, the order $p(H_n^{G_i}) = \sum_{i=1}^{n} p_i$ and the size $q(H_n^{G_i}) = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1}$, where the last summations is taken over all i for which the vertex v_i is adjacent with v_{i+1} in H. That is $G_j + G_k$ is an induced subgraph of $H_n^{G_i}$ for each pair of adjacent vertices v_j,v_k of H. Moreover, if v_j,v_k,v_i forms a path P_3 in H, then $(G_j \cup G_i)+G_k$ is an induced subgraph of $H_n^{G_i}$. While if v_j,v_k and v_i forms a triangle in H, then $(G_j + G_i) + G_k$ is a subgraph of $H_n^{G_i}$.*

An illustration for Definition 2.1 is given in the next example.

Example 2.2: Let the graphs G_1, G_2, G_3, G_4, G_5 and H be given as follows:

G_1=$\mathbf{P_3}$, G_2=$\mathbf{K_2}$, G_3=$\mathbf{K_1}$, G_4:$\mathbf{C_3}$, G_5=$\mathbf{K_1}$ and,

H:



Then the expanded graph of H by inserting the above G_i graphs is indicated in Figure 2.1, in which $p(G) = \sum_{i=1}^{5} p_i = 10$ and

$$q(G) = \sum_{i=1}^{5} q_i + \sum_{i=1}^{4} p_i p_{i+1} = 18.$$

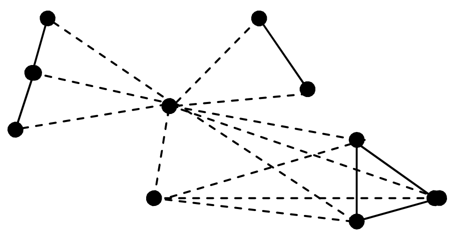


Figure 2.1: The expanded graph G = $H_5^{G_i}$

Moreover, if the base graph H is a path P_n, then the expanded graph $P_n^{G_i}$ is the sequential join of the graphs G_1, G_2,…, G_n. If H = P_5 and G_i's, i = 1,2,3,4, 5 are given as in Example 2.2, then, the sequential join graph $\sum_{i=1}^{5} G_i$ is depicted in Fig. 2.2, with p(G) = 10 and q(G) = 6 + 14 = 20.

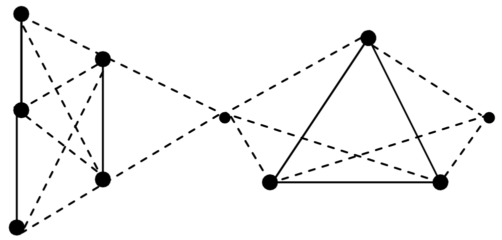


Figure 2.2: The sequential join graph $\sum_{i=1}^{5} G_i$

Next, we determine the nullity of the sequential join of some special graphs such as $G_i \cong N_p$, $K_{r,s}$, S_p, K_p, P_m, or C_p, for i = 1,2,…,n.

If G $= \sum_{i=1}^{n} N_{p_i}$ where $N_{p_i} = N_1$ (the trivial graph) for all i, then the graph G is simply a path of order n.

Proposition 2.3: *For an expanded graph* $\sum_{i=1}^{n} N_2$ *, we have:*

i) If n = 2k, for k = 1,2,…, then $\eta(\sum_{i=1}^{2k} N_2)$ is

$2 + \eta(\sum_{i=1}^{2(k-1)} N_2)$ =2k = n.

ii) If n = 2k+1, for k = 1,2,…, then $\eta(\sum_{i=1}^{2k+1} N_2)$ is

$2 + \eta(\sum_{i=1}^{2k-1} N_2)$ = 2k + 2 = n + 1.

Proof: i) Let $w_{(i,j)}$, i=1,2 and j=1,2,…,2k, be a zero-sum weighting for the vertex

$v_{i,j}$ in the graph $\sum_{i=1}^{2k} N_2$ (or $\sum_{i=1}^{2k+1} N_2$), as indicated in Fig.2.3

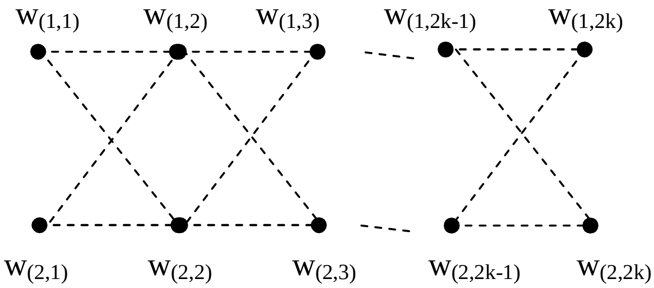


Figure 2.3: A weighting of the graph $\sum_{i=1}^{2k} N_2$

If k = 1, then using the weights technique it is easy to evaluate that $\eta(\sum_{i=1}^{2} N_2) = 2$ and $\eta(\sum_{i=1}^{3} N_2) = 4$ by Lemma 1.6 ii, Since the vertices $v_{1,2k}$ and $v_{2,2k}$ are coneighbors as well as $v_{1,2k-1}$ and $v_{2,2k-1}$ hence removing the vertices $v_{1,2k}$ and $v_{1,2k-1}$ from the graph $\sum_{i=1}^{2k} N_2$, a graph with end vertex (namely $v_{2,2k}$) is obtained. Apply (End Vertex Corollary) to it, we get $\eta(\sum_{i=1}^{2k} N_2) = 2 + \eta(\sum_{i=1}^{2(k-1)} N_2)$.

ii) Similar argument holds for the odd case also. ■

Theorem 2.4: *For $n \geq 2$, if $G = \sum_{i=1}^{n} N_{p_i}$, then*

$$\eta(G) = \begin{cases} \sum_{i=1}^{n} p_i - n & \text{if } n \text{ is even}, \\ \sum_{i=1}^{n} p_i - n + 1 & \text{if } n \text{ is odd}. \end{cases}$$

Proof: The proof is just an extension to that of Proposition 2.3, and hence it is omitted. ■

Corollary 2.5: *In Theorem 2.4 if $p_i = p$ ∀i, then the nullity of $G = \sum^{n} N_p$ is*

$$\eta(G) = \begin{cases} n(p-1) & \text{if } n \text{ is even}, \\ n(p-1)+1 & \text{if } n \text{ is odd}. \end{cases} \blacksquare$$

The Sequential Join of Complete Bipartite Graph

A graph G is said to be a bipartite graph if it contains no odd cycles. Thus, the sequential join of complete bipartite graphs is also a bipartite graph. Moreover, the sequential join of complete bipartite graphs $\sum_{i=1}^{n} K_{r_i,s_i}$ has $p=\sum_{i=1}^{n}(r_i+s_i)$ and $q=\sum_{i=1}^{n} r_i s_i+\sum_{i=1}^{n-1}(r_i+s_i)(r_{i+1}+s_{i+1})$

while, the diameter of $\sum_{i=1}^{n} K_{r_i,s_i}$ is n-1, for n ≥ 3.

We define the next term:

Definition 2.6: *A singular graph G is said to be a* ***completely non stable*** *if* $\sum_{u\in G} w(u) = 0$ *for a high zero-sum weighting of G. It is clear that if the above condition holds for only a high zero-sum weighting then it holds for any other one.*

Thus, complete bipartite graphs with order greater than 2 are completely non stable graphs, while P_{4n+1} *is not.*

Lemma 2.7 (Coneighbor Lemma): *Let u and v be coneighbor vertices of a connected graph G, then*

$\eta(G) = \eta(G-u) + 1 = \eta(G-v) + 1$.

Proof: Label the vertices of G by $u\equiv v_1$, $v = v_2, v_3,\ldots,v_p$. Let A(G), A(G−u), A(G−v) be the adjacency matrices of G, G−u, G−v, respectively. Applying row elimination $R_1 \longrightarrow R_1 - R_2$ and column elimination $C_1 \longrightarrow C_1 - C_2$ to the matrix A, we get zero in each entry of row one and zero in each entry of column one.

And obtain a new matrix A*, where $A^* = A(K_1 \cup (G-u))$.

Hence, $r(G) = r(G-u) \implies p - \eta(G) = (p-1) - \eta(G-u)$,

$\therefore \eta(G) = \eta(G-u) + 1$, similarly $\eta(G) = \eta(G-v) + 1$. ■

Definition 2.8: *Two adjacent vertices v_1 and v_2 in a graph G are said to be* ***semi-coneighbors*** *if $N(v_1) = N(v_2)$ in the graph G-e where $e = v_1v_2$.*

Remark 2.9: Let w be any zero-sum weighting of a graph G. If v_1 and v_2 are semi-coneighbors, then they must be weighted by the same variable (weight), say x, because in any zero-sum weighting for G we have:

$$\sum_{u\in N_G(v_1)} w(u) = w(v_2) + \sum_{u\in N_{G-e}(v_1)} w(u) = 0, \quad (1)$$

$$\sum_{u\in N_G(v_2)} w(u) = w(v_1) + \sum_{u\in N_{G-e}(v_2)} w(u) = 0. \qquad (2)$$

Therefore, from (1) and (2), we get $w(v_1) = w(v_2)$

Proposition 2.10: *The strong product graph* $K_2 \boxtimes P_n = \sum^{n} K_2$ *is non-singular.*

Proof: For n = 2 or 3, it is easy to prove that any zero-sum weighting for $K_2 \boxtimes P_2$ and $K_2 \boxtimes P_3$ is trivial. ■

For any zero-sum weighting of the graph $K_2 \boxtimes P_n$ as indicated in Fig.2.4, we can put $w(v_{i,j}) = x_j$, for i = 1,2 and j = 1,2,…,n. This is possible by Remark 2.9

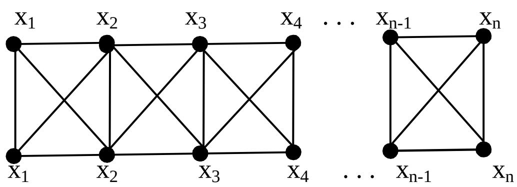


Figure 2.4: The strong product graph $K_2 \boxtimes P_n$

Apply $\sum_{u\in N(v_{ij})} w(u) = 0$, for all i,j, we get:

$x_1 + 2x_2 = 0$ (1)
$2x_1 + x_2 + 2x_3 = 0$ (2)
$2x_2 + x_3 + 2x_4 = 0$ (3)
…
$2x_{n-2} + x_{n-1} + 2x_n = 0$ (n-1)
$2x_{n-1} + x_n = 0$ (n)

Then, from Equation (1), we get:

$$x_2 = -\frac{1}{2} x_1 \qquad (1')$$

From Equations (1') and (2), we get:

$$x_3 = -\frac{1}{2}\left[2x_1 - \frac{1}{2} x_1\right] = -\frac{3}{4} x_1 \qquad (2')$$

From Equations (2') and (3), we get:

$$x_4 = \frac{1}{2}\left[x_1 + \frac{3}{4} x_1\right] = \frac{7}{8} x_1 \qquad (3')$$

And so on, all the values of $x_2, x_3, \ldots, x_n$ are defined in term of x_1.

Finally, put the values of x_{n-1} and x_n in Equation (n) to get:

$ax_1 = 0$, for some number a, this implies that $x_1 = 0$ and hence all the remaining variables are zeros.

Therefore, there exist no non trivial zero-sum weighting for the graph $K_2 \boxtimes P_n$.

Hence, by Proposition 1.5, $\eta(K_2 \boxtimes P_n) = 0$. ■

Since, $K_2 \boxtimes P_n$ is singular graph, then $\lambda\mu + \lambda + \mu$ must equal to zero for some eigenvalue λ of K_2 and μ of P_n. This follows from the relations between the eigenvalues of the strong product graph and the eigenvalues of its product components. See [5].

But $\lambda = 1$ or $\lambda = -1$, hence $\mu + 1 + \mu = 0 \Rightarrow \mu = -\frac{1}{2}$ or $-\mu - 1 + \mu = 0 \Rightarrow -1 = 0$ which is impossible.

Thus, we conclude that $-\frac{1}{2}$, is not an eigenvalue for P_n for any n. This leads that $2\cos\left(\frac{i\Pi}{n+1}\right) \neq -\frac{1}{2}$ for any i, i = 1, 2, …, n, and any n. That is ∃ no such integers i and n that satisfies $\cos\left(\frac{i\Pi}{n+1}\right) = -\frac{1}{4}$.

The nullity of the expanded graph G, G= $\sum_{i=1}^{n} K_{r_i,s_i}$ is determined in the next theorem.

Theorem 2.11: *For* $n \geq 2$, $\eta\left(\sum_{i=1}^{n} K_{r_i,s_i}\right) = \sum_{i=1}^{n}(r_i + s_i - 2)$**.**

Proof: Apply (Coneighbor Lemma) for each pair of coneighbor vertices u and v in $\sum_{i=1}^{n} K_{r_i,s_i}$, that is removing a vertex out of each such a pair which are exactly $\sum_{i=1}^{n}(r_i + s_i - 2)$, we obtain the graph $K_2 \boxtimes P_n$.

Then, $\eta(G) = \sum_{i=1}^{n}(r_i + s_i - 2) + \eta(K_2 \boxtimes P_n)$. ■

The Star graph S_p(or $S_{1,p-1}$) is a complete bipartite graph, with one of its partite sets consisting of exactly one vertex. It is a special type of trees, trees with diameter 2, with only three distinct eigenvalues, namely, $\sqrt{p-1}, 0, -\sqrt{p-1}$.

Moreover, the expanded graph of n stars S_{p_i}, i=1, 2, …,n, G = $\sum_{i=1}^{n} S_{p_i}$ has order, p= $\sum_{i=1}^{n} p_i$ and size q,

$$q = \sum_{i=1}^{n} p_i - n + \sum_{i=1}^{n-1} p_i p_{i+1}$$, with diameter n–1, n≥3.

Corollary 2.12: *For n, $p_i \geq 2$, the nullity of G is* $\sum_{i=1}^{n} p_i - 2n$.

Proof: $S_{p_i} \cong K_{1,p_i-1}$, hence it is a special case of Theorem 2.3.7. ■

Corollary 2.13: *For n ≥ 2, if G =* $\sum^{n} S_p$ *, p ≥2.*

Then, η(G) = n(p−2).

Proof: Put $r_i+s_i = p$ in Theorem 2.3.7, then, the prove is immediate. ■

The Sequential Join of Complete Graphs

The complete graph K_p, p >2, is a simple graph with neither a cut vertex nor a bridge, and has maximum size q, $q = \frac{p(p-1)}{2}$. While the sequential join $\sum_{i=1}^{n} K_{p_i}$, n ≥ 3, is not a complete graph. Its order is $p = \sum_{i=1}^{n} p_i$

and size $q = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n-1} p_i p_{i+1} = \sum_{i=1}^{n} \frac{p_i(p_i-1)}{2} + \sum_{i=1}^{n-1} p_i p_{i+1}$.

The nullity of $\sum_{i=1}^{n} K_{p_i}$ is determined in the next theorem.

Theorem 2.14: *If n ≥ 2, and $p_i \geq 2$ for i=1, 2,…, n, then* $\sum_{i=1}^{n} K_{p_i}$ *is non-singular.*

Proof: For n = 2, then, the sequential join of K_{p1} and K_{p2} is $k_{p_1+p_2}$ which is a complete graph, hence by Lemma 1.4.11(iii), η(G) = 0.

For n > 2, no non zero-sum weighting for G exists. This holds from the fact $\sum^{n} K_2$ is an induced subgraph of $\sum_{i=1}^{n} K_{p_i}$ and each extra vertex u in K_{p_i} is semi-coneighbor with each vertex v ∈ K_{p_j} . By Remark 2.9, the weight of u is the same as the weight of each v, which is zero; hence G is a non-singular graph by Proposition 1.5. ■

The Sequential Join of Paths

Paths are extreme graphs to determine many invariants of the graph such as the diameter, and the average distance. Moreover, the sequential join of path graphs, $\sum_{i=1}^{n} P_{m_i}$, has order $p = \sum_{i=1}^{n} m_i$ and size $q = \sum_{i=1}^{n}(m_i - 1) + \sum_{i=1}^{n-1} m_i m_{i+1}$, and the diameter is n-1, for n > 2, while it is equal to 2, where n=2, and m_1 or m_2 > 2.

Lemma 2.15: *If $G_1 = P_m$ and $G_2 = P_n$, and $G = G_1 + G_2$, then, the nullity of the graph G is given by:*

$$\eta(G) = \begin{cases} 2 & \text{if both } m = 4k-1 \text{ and } n = 4t-1, \text{ for } k, t \in Z^+, \\ 1 & \text{if either } m = 4k-1 \text{ or } n = 4t-1, \text{ but not both}, \\ 0 & \text{if neither } m, \text{nor } n = 4t-1, \text{ for any } k \text{ and any } t. \end{cases}$$

Proof: Label the vertices of G_1 by $v_1, v_2, \dots, v_m$, with a high zero-sum weighting $w_{(1,1)}, w_{(1,2)}, \dots, w_{(1,m)}$, and the vertices of G_2 by $u_1, u_2, \dots, u_n$, with a high zero-sum weighting $w_{(2,1)}, w_{(2,2)}, \dots, w_{(2,n)}$, in G.

By Definition 2.6, if m = 4k – 1 and n = 4t – 1, then both P_m and P_n are completely non stable graphs, and the same weighting can be used for the join graph, hence the nullity of the join graph is 2 in this case. If either of them is completely non stable, say P_m but not P_n, then $w_{(2,1)} = w_{(2,2)} = \dots = w_{(2,n)} = 0$ in any high zero-sum weighting of the join. Finally, if both n and m cannot be written as 4k–1 for any k and any t, then in the join graph $w_{(i,j)} = 0 \ \forall$ i,j. Thus, G is non-singular. ■

Theorem 2.16: *If G = $\sum^{n} P_m$, then nullity of the graph G is*

$$\eta(G) = \begin{cases} n & \text{if } m = 4k-1, \text{ for } k \in Z^+, \\ 0 & \text{if } m \neq 4k-1, \text{ for any } k \in Z^+. \end{cases}$$

Proof: If m = 3, then the proof follows from Lemma 2.15. Moreover, if m = 4k–1, k∈Z^+, then the path graph P_m is a completely non stable graph, and each component of the compound graph $\sum^{n} P_m$ uses exactly one variable, hence there exist exactly n variables in any high zero-sum weighting of G, then, η(G) = n.

If m ≠ 4k–1, k∈Z^+, then each P_m is not a completely non stable, hence there exists no non-trivial zero-sum weighting for G, for n > 1. Thus, by Proposition 1.5, we conclude that G is non-singular. ■

It is clear that $P_5 + P_5$ has no non trivial zero-sum weighting, hence η(G) = 0.

Observation 2.17: If G = $\sum_{i=1}^{n} P_{m_i}$, then η(G) = no. of paths P_{m_i} of order $4k_i$–1, for k_i ∈Z^+.

Example 2.18: Let $G_1 = P_2$, $G_2 = P_3$ and $G_3 = P_4$, then there is a zero-sum weighting of the graph $\sum_{i=1}^{3} P_{m_i}$, where m_i=2,3,4 as indicated in Fig.2.5.

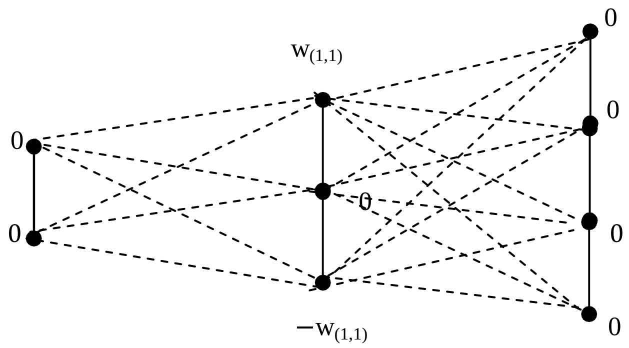


Figure 2.5: A zero-sum weighting of the graph $P_3^{P_{m_i}}$

This zero-sum weighting uses exactly one independent variables, namely $w_{(1,1)}$, hence η($\sum_{i=1}^{3} P_{m_i}$) = 1.

III The Sequential Join of Cycles

Cycles are 2-regular, critical 2-connected graphs, odd cycles are critical 3-colarable graphs. The nullity of the sequential join of n cycles, $\sum_{i=1}^{n} C_{p_i}$, is our goal in the next.

Proposition 3.1: *If G = $C_n + C_m$, then the nullity of the graph G is given by:*

$$\eta(G)=\begin{cases} 4 & \text{if both } n=4k \text{ and } m=4t, \text{ for } k,t\in Z^+, \\ 2 & \text{if } n=4k \text{ or } m=4t \text{ but not the both}, \\ 0 & \text{if neither the order of } C_n \text{ nor of } C_m \text{ is } 0 (\mathrm{mod}\, 4). \end{cases}$$

Proof: The proof is similar to that of Theorem 2.16. ■

Theorem 3.2: *For $n \geq 2$, if $G = \sum_{j=1}^{n} C_{p_j}$, $p_j > 2$, $j=1,2,\ldots,n$, the nullity of G is*

$$\eta(G) = \begin{cases} 2n & \text{if } p_j \equiv 4k_j, k_j \geq 1 \text{ for each } j, j = 1,2,\ldots n, \\ 0 & \text{if no order of } C_{p_j} \text{ is equal to zero } (\mathrm{mod}\, 4). \end{cases}$$

Proof: It is known that $\eta(C_{4k}) = 2$, and using weights, it is easy to show that $\eta(C_{4k_1} + C_{4k_2}) = 2+2 = 4$.

If $w_{(1,j)}$, $w'_{(1,j)}$, $-w_{(1,j)}$, $-w'_{(1,j)}$ …, are weights of C_{p_j}'s, $p_j = 4k_j$, then from the condition $\sum_{\forall v \in N(i,j)} w_{(i,j)} = 0$, there is no relation between $w_{(1,j)}$ and $w'_{(1,j)}$, $\forall j$, $j=1,2,\ldots,n$, if $j=1$

Hence, if $j=n$ where $p_j = 4k_j$, then $\eta(G) = 2n$. For n subgraphs of G, if no non trivial zero-sum weighting exists (that is order of non of them is zero mod 4), then no non trivial zero-sum weighting for G exists, hence by Proposition 1.5, $\eta(G) = 0$. ■

Observation 3.3: If $G = \sum_{i=1}^{n} C_{p_i}$, then the nullity of the graph G is 2j, if j orders of the cycles C_{p_i}'s are of form $p_i \equiv 4k_j$.

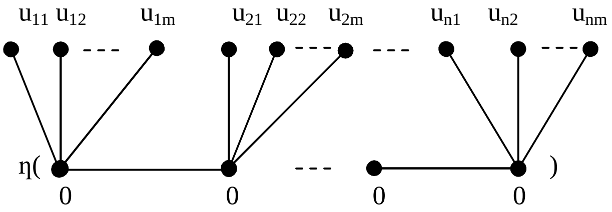
IV Nullity of the Corona of a Path with other Special Graphs

In this section, we study the nullity of the corona of two graphs.

In Definition 1.12, we choose $G_1 = P_n$ and G_2 is a known graph, such as N_m, K_2, P_m, C_4, $K_{r,s}$ or K_p.

Proposition 4.1: *Let $G_1 = P_n$ and $G_2 = N_m$, then the nullity of the corona graph $\eta(G)$, $G = P_n \Theta N_m$ is $n(m-1)$.*

Proof: Follows from applying (End Vertex Corollary) n-times namely to the vertices u_{1j}, $j=1, 2, \ldots, m$, as illustrated in Fig.4.1.



$= \eta(nN_{m-1}) = n(m-1)$

Figure 4.1: The corona graph $P_n \Theta N_m$

Proposition 4.2: *The nullity of the graph $G = P_n \Theta K_2$ is zero.*

Proof: Let $w_{(i,j)}$ be a weighting of the corona graph $P_n \Theta K_2$. From the condition $\sum_{\forall v \in N(i,j)} w_{(i,j)} = 0$ for all v in $P_n \Theta K_2$, the graph $P_n \Theta K_2$ can be weighted as indicated in Fig.4.2.

Where, $w_{(1,1)} = y$

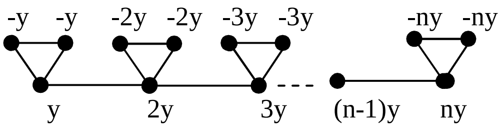


Figure 4.2: The corona graph $P_n \Theta K_2$

Now, the sum of the weights over the neighborhood of the vertex weighted $nw_{(1,1)}$ is $(n-1)w_{(1,1)} - nw_{(1,1)} - nw_{(1,1)} = 0 \Rightarrow w_{(1,1)} = 0$.

The graph $P_n \Theta K_2$ has no non trivial zero-sum weighting. Hence it is non-singular. ■

Proposition 4.3: *For the corona graph $G = P_n \Theta K_m$, is non singular.*

Proof: Is similar to that of Lemma 4.2, hence $\eta(G) = 0$. ■

Proposition 4.4: *Let $G_1 = P_n$ and $G_2 = C_4$, then the nullity of $G = P_n \Theta C_4$ is $\eta(G) = 2n$.*

Proof: Follows from the fact that there exists exactly 2n pairs of coneighbors vertices in G and after removing a vertex out of each such a pair, we obtain the graph indicated in Figure 4.2.

$$\therefore \eta(P_n \Theta C_4) = 2n + \eta(P_n \Theta K_2) = 2n. \blacksquare$$

Proposition 4.5: *For any complete bipartite graph $K_{r,s}$, the nullity of $G = P_n \,\Theta\, K_{r,s}$ is given by*

$\eta(G) = n(r + s - 2)$.

Proof: Follows by applying Coneighbor Lemma, hence it is omitted. ■

Proposition 4.6: Let $G^{\wedge}$ be the graph K_1+P_n, then,

$$\eta(G^{\wedge}) = \begin{cases} 1 & \text{if } n \equiv 4k - 1, \text{ for } k \in Z^{+}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof: Let $w_{(2,1)}$ and $w_{(1,j)}$, j=1,2,…,n be a zero-sum weighting for the graph $G^{\wedge}$, as indicated in Fig.4.3.

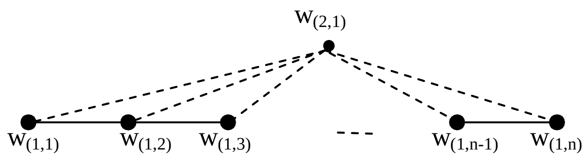


Figure 4.3: The cone graph $G^{\wedge}$ of the path P_n.

From the condition $\sum_{\forall v \in N(i,j)} w_{(i,j)} = 0$, for all v in $G^{\wedge}$

$$w_{(1,2)} + w_{(2,1)} = 0 \quad \Rightarrow \quad w_{(1,2)} = - w_{(2,1)} \tag{1}$$

$$w_{(1,j)} + w_{(1,j+2)} + w_{(2,1)} = 0, \quad j=1,2,\dots,n-2 \tag{2}$$

$$w_{(1,n-1)} + w_{(2,1)} = 0$$

From Equation (2) we get:

$$w_{(1,j+2)} = - w_{(1,j)} - w_{(2,1)}, \text{ for } j = 1,2,..,n-2 \tag{3}$$

Thus, $w_{(1,3)} = - w_{(1,1)} - w_{(2,1)}$ (4)

$$w_{(1,4)} = - w_{(1,2)} - w_{(2,1)} \tag{5}$$

$\therefore \quad w_{(1,4)} = 0$

This gives that $w_{(1,4k)} = 0$, for $k \in Z^{+}$, and $w_{(1,2k-1)} = - w_{(1,1)} + w_{(1,2)}$. Assume that $w_{(1,2)} \neq 0$, then the sum of the weights over the neighborhood of the vertex weighted $w_{(2,1)}$ is $\sum_{j=1}^{n} w_{(1,j)} \neq 0$, which is a contradiction from which we assumed

$w_{(1,2)} = 0$. Now if $n \neq 4k-1$, then we get $w_{(1,1)} = 0$ and hence, no non-trivial zero-sum weighting exists. Thus, any high zero-sum weighting of $G^{\wedge}$ will use only one non-zero variable say $w_{(1,1)}$ where $n = 4k-1$, and hence the prove is complete. ■

If $G^{\wedge\wedge}$ is a cone over the cone $G^{\wedge}$ then $\eta(G^{\wedge\wedge}) = \eta(G^{\wedge})$, and moreover if this process is continued any t times, then $\eta(G^{\wedge t}) = \eta(G^{\wedge})$, because of the fact that the second vertex of $G^{\wedge\wedge}$ which is added and joined to all vertices of $G^{\wedge}$ is semi-coneighbor with the first vertex of $G^{\wedge}$ which is added and joined to all vertices of G. Hence, both must be zero weighted and hence, the result follows for any t.

Proposition 4.7: *Let $G_1 = P_n$ and $G_2 = P_m$, then the nullity of $G = P_n \,\Theta\, P_m$ is*

$$\eta(G) = \begin{cases} n & if\ m = 4k-1,\ k = 1,2...,\\ 0 & otherwise. \end{cases}$$

Proof: If m = 3, then the graph G is $P_n \,\Theta\, P_3$, as indicated in Fig.4.4.

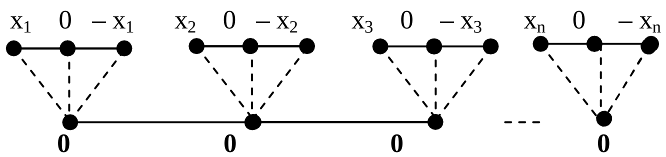


Figure 4.4: The graph $P_n \,\Theta\, P_3$

Applying (Coneighbour Lemma), we have n-pairs of coneighbors. After removing a vertex out of each such a pair, we obtain the graph $P_n \,\Theta\, K_2$ thus, $\eta(P_n \,\Theta\, P_3) = n + \eta(P_n \,\Theta\, K_2)$. But, $\eta(P_n \,\Theta\, K_2) = 0$. Hence, $\eta(P_n \,\Theta\, P_3) = n$.

If m>3, then we have n induced subgraphs of the graph $P_n \Theta P_m$ and each of them is a cone of a path P_m . But the nullity of a cone of a path P_m , $m = 4k-1$ is one , and we have n such a cones.

$$\therefore\ \eta(G) = \begin{cases} n & if\ m = 4k-1,\ k = 1,2...\\ 0 & otherwise. \quad ■ \end{cases}$$

Open Problem: Evaluate $\eta(G_1 \Theta\, G_2)$ in terms of invariants of G_1 and G_2?**.**

Nullity of the Semi-Corona of a Path with other Special Graphs

In the following section we are going to define the semi-corona.

Definition 4.8: *Let H_n be a graph, whose vertices labeled $h_1,h_2,\ldots,h_n$ and $G_1,G_2,\ldots,G_n$ be distinct graphs with orders $p_1,p_2,\ldots,p_n$ and their vertices labeled by u_i^j $1 \le i \le n$,$1 \le j \le p_i$.*

We define the **semi-corona** graph G = $H_n \Theta \prod_{i=1}^{n} G_i$, to be the graph whose vertex set is V(G) = V($H_n \bigcup_{i=1}^{n} G_i$) and edge set E(G)=E(H_n) $\bigcup$ E($\bigcup_{i=1}^{n} G_i$) $\bigcup_{i=1}^{n}$ {$h_i u_i^j$ for each j,1≤j≤ p_i}.

That is the vertex h_1 is adjacent with $u_1^1, u_1^2, \ldots, u_1^{p_1}$, h_2 is adjacent with u_2^1, $u_2^2,\ldots,u_2^{p_2}$ and h_n is adjacent with $u_n^1, u_n^2, \ldots, u_n^{p_n}$, as illustrated in Fig.4.5, where $G_1 = P_2$, $G_2 = P_3$, $G_3 = C_3$, $G_4 = C_4$ and H = C_4

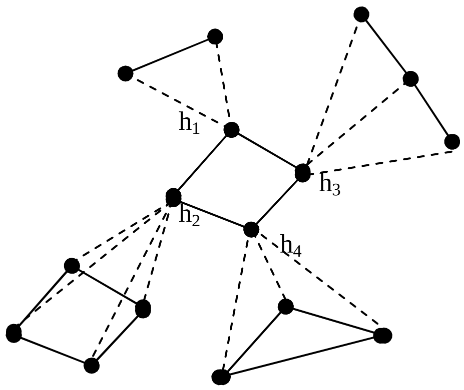


Figure 4.5: The semi-corona graph G = $C_4 \Theta \prod_{i=1}^{n} G_i$

Proposition 4.9:

1) $\eta(P_n \Theta \prod_{i=1}^{n} N_{m_i}) = \sum_{i=1}^{n}(m_i - 1)$.

2) $\eta(P_n \Theta \prod_{i=1}^{n} K_{p_i}) = 0$.

3) $\eta(P_n \Theta \prod_{i=1}^{n} K_{r_i,s_i}) = \sum_{i=1}^{n}(r_i + s_i - 2)$.

Proof: The proof is a generalization for that of Propositions 4.1, 4.2 and 4.5, respectively. ■

References

[1] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, New York, (2012).

[2] M. Brown, J.W. Kennedy and B. Servatius, Graph singularity, *Graph Theory Notes of New York*, 25(1993), 23-32.

[3] D.M. Cvetković, M. Doob, I. Gutman and A. Torgas′ev, *Recent Results in the Theory of Graph Spectra*, North-Holland, Amsterdam, (1988).

[4] B. Liu, Y. Huang and S. Chen, On the characterization of graphs with pendent vertices and given nullity, *Electronic Journal of Linear Algebra*, 18(2009), 719-734.

[5] P.A. Rashid, Characterization for the degree of the singularity of a graph, *M.Sc. Thesis*, Salahuddin University, Arbil, Iraq, (2001).

[6] I. Sciriha, A characterization of singular graphs, *J. of Linear Algebra*, 16(2007), 451-462.

[7] Kh. R. Sharaf and D.H. Mohammed, Degree of singularity of complement of some graphs and graphs with loops, *J. of Duhok Univ.*, 8(2) (2005), 101-110.

[8] Kh. R. Sharaf and D.A. Ali, Nullity and bounds to the nullity of dendrimer graphs, *J. Computer Science and Mathematics*, Univ. of Mosul, (in Press), (2013).