Gen. Math. Notes, Vol. 19, No. 2, December, 2013, pp. 95-103
ISSN 2219-7184; Copyright © ICSRS Publication, 2013
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A Subclass of Multivalent Functions with Negative Coefficients Defined by Integral Operator with Fractional Derivative I

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(Received: 20-7-13 / Accepted: 24-8-13)

Abstract

In this paper, we study a new class $\Re A^*$ *of multivalent functions with negative coefficient defined by integral operator with fractional calculus. We obtain coefficient estimates, radius of* $k-$ *closed – to – convex, integral representation.*

Keywords: *Multivalent Functions, Integral Operator, Coefficient Estimates, Integral Representation,* $k-$ *Closed – to – Convex.*

1 Introduction

Let $\Re AM^*$ denote of a class of multivalent analytic functions of the form:

$$f(z) = mz^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \tag{1}$$

Which are analytic and multivalent functions in the unit disk

$$U = \{z : z \in C \text{ and } |z| < 1 \}.$$

Let $\Re AM_p^*$ a subclass of $\Re AM^*$ consisting of functions of the form:

$$f(z) = mz^p - \sum_{n=p+1}^{\infty} a_n z^n \quad .\,(a_n \geq 0 \ , \ p \in IN \quad m > 0) \tag{2}$$

In the next we defined a new integral operator introduced by R. H. Buti.

Definition (1): *Let the function defined by (2) in the class, then*

$$(RAF)_{\beta,\theta}^{\mu,\zeta}(f(z)) = \frac{1}{\Gamma(\mu\zeta+1)} \int_0^1 t^{-(\beta+\theta)p} \left(\log\frac{1}{t}\right)^{\mu\zeta} f(\frac{z}{t^{-(\beta+\theta)}})dt$$

$$= mz^p - \sum_{n=p+1}^{\infty} \left(\frac{1}{1-\beta(n-1)}\right)^{\mu+1} a_n z^n$$

$$= mz^p - \sum_{n=p+1}^{\infty} \Psi(\beta,\mu,n) a_n z^n$$

Where $\Psi(\beta,\mu,n) = \left(\frac{1}{1-\beta(n-1)}\right)^{\mu+1}$ and $\beta \leq 0 \ , \ 0 \leq \mu \leq 1$.

We need the following definition given by H.M. Srivastava and S.Owa [6].

Definition 3: *The fractional derivative of order* $\delta (0 \leq \delta < 1)$ *is defined by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt, \tag{12}$$

Where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta}$ is removed by required $log(z-t)$ to be real when $(z-t) > 0$.

The fractional derivative of the integral operator is defined by

Definition (2): *A function* f *defined by (1) belonging to the class* $\Re^*$ *is in the class* $\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$ *if it satisfies the condition:*

$$\left| \frac{z^2(1-\alpha)\left(RAF_{\beta}^{\mu}(f(z))\right)'' - z(p-1)\left(RAF_{\beta}^{\mu}(f(z))\right)'}{\alpha z^2\left(RAF_{\beta}^{\mu}(f(z))\right)'' + z(1-\lambda\theta)\left(RAF_{\beta}^{\mu}(f(z))\right)'} \right| < 1 , \tag{3}$$

Where $\beta \le 0$, $0 < \alpha < 1$, $0 \le \lambda < 1$, $0 \le \theta < 1$, $0 \le \mu \le 1$.

For a given real number z_0 $(0 < z_0 < 1)$. Let $\Re^{pi}(i = 0,1)$ be a subclass of $\Re_p^*$ satisfies the condition $z_0^{-p} f(z_0) \le 1$ and $p^{-1} z_0^{1-p} f(z_0) \le 1$ respectively.

Let

$$\Re^{*pi}(\alpha,\lambda,\mu,\theta,\beta,z_0) = \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta) \cap \Re^{pi}(i = 0, 1) \tag{4}$$

Some another classes studied by M.K. Aouf, A. Shamandy and M.F. Yassen [1], W.G. Atshan and S.R. Kulkarni [3], N.E. Cho and M.K. Aouf [4], S.R. Kulkarni and Mrs. S.S. Joshi [5], consisting of multivalent and meromorphic univalent functions respectively.

2 Coefficient Estimates:

In the next theorem, we obtain a necessary and sufficient condition for function to be in the class $\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$.

Theorem (1): *A function f defined (2) be in the class $\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$ if and only if*

$$\sum_{n=p+1}^{\infty} n[1 + (n-p) - \lambda\theta]\Psi(\beta,\mu,n)a_n \le mp(1-\lambda\theta) . \tag{5}$$

Under the parametric restraints given (3), the result is sharp.

Proof: Let the inequality (5) holds true. For $|z| = 1$, we have

$$\begin{aligned} &\left| z^2(1-\alpha)\left(RAF_{\beta}^{\mu}(f(z))\right)'' - z(p-1)\left(RAF_{\beta}^{\mu}(f(z))\right)' \right| \\ &- \left| \alpha z^2\left(RAF_{\beta}^{\mu}(f(z))\right)'' + z(1-\lambda\theta)\left(RAF_{\beta}^{\mu}(f(z))\right)' \right| \\ &= \Bigg| m(1-\alpha)p(p-1)z^p - (1-\alpha)\sum_{n=p+1}^{\infty} n(n-1)\Psi(\beta,\mu,n)a_n z^n \\ &- mp(p-1)z^p + (p-1)\sum_{n=p+1}^{\infty} n\Psi(\beta,\mu,n)a_n z^n \Bigg| \end{aligned}$$

$$-\left|mp(p-1)\alpha z^p - \alpha\sum_{n=p+1}^{\infty} n(n-1)\Psi(\beta,\mu,n)a_n z^n\right.$$

$$\left. + mp(1-\lambda\theta)z^p - (1-\lambda\theta)\sum_{n=p+1}^{\infty} n\Psi(\beta,\mu,n)a_n z^n\right|$$

$$=\left|m\alpha p(p-1)z^p + \sum_{n=p+1}^{\infty} n\left[(1-\alpha)(n-1)-(p-1)\right]\Psi(\beta,\mu,n)a_n z^n\right|$$

$$-\left|mp\left[(p-1)\alpha+(1-\lambda\theta)\right]z^p - \sum_{n=p+1}^{\infty} n\left[(\alpha(n-1)+(1-\lambda\theta)\right]\Psi(\beta,\mu,n)a_n z^n\right|$$

$$\leq m\alpha p(p-1)|z|^p + \sum_{n=p+1}^{\infty} n\left[(1-\alpha)(n-1)-(p-1)\right]\Psi(\beta,\mu,n)a_n|z|^n$$

$$-\left[mp\left[(p-1)\alpha+(1-\lambda\theta)\right]\right]|z|^p + \sum_{n=p+1}^{\infty} n\left[(\alpha(n-1)+(1-\lambda\theta)\right]\Psi(\beta,\mu,n)a_n|z|^n$$

$$=\sum_{n=p+1}^{\infty} n\left[(1+(n-p)-\lambda\theta\right]\Psi(\beta,\mu,n)a_n - mp(1-\lambda\theta).$$

Hence by principle of the maximum modulus, $f(z)\in\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$.

Conversely, assume that $f(z)\in\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$. Then

$$\left|\frac{z^2(1-\alpha)\left(RAF_\beta^\mu(f(z))\right)'' - z(p-1)\left(RAF_\beta^\mu(f(z))\right)'}{\alpha z^2\left(RAF_\beta^\mu(f(z))\right)'' + z(1-\lambda\theta)RAF_\beta^\mu(f(z))'}\right|$$

$$\left|\frac{m\alpha p(p-1)z^p + \sum_{n=p+1}^{\infty} n\left[(1-\alpha)(n-1)-(p-1)\right]\Psi(\beta,\mu,n)a_n z^n}{mp\left[(p-1)\alpha+(1-\lambda\theta)\right]z^p - \sum_{n=p+1}^{\infty} n\left[(\alpha(n-1)+(1-\lambda\theta)\right]\Psi(\beta,\mu,n)a_n z^n}\right|<1,$$

for all $z\in U$. Using the fact $\operatorname{Re}(z)\leq|z|$ for z it follows that

$$\operatorname{Re}\left\{\frac{m\alpha p(p-1)z^p + \sum_{n=p+1}^{\infty} n\left[(1-\alpha)(n-1)-(p-1)\right]\Psi(\beta,\mu,n)a_n z^n}{mp\left[(p-1)\alpha+(1-\lambda\theta)\right]z^p - \sum_{n=p+1}^{\infty} n\left[(\alpha(n-1)+(1-\lambda\theta)\right]\Psi(\beta,\mu,n)a_n z^n}\right\}<1. \qquad (6)$$

Now choose the values of z on the real axis so that $\dfrac{\left(RAF_{\beta}^{\mu}(f(z))\right)''}{\left(RAF_{\beta}^{\mu}(f(z))\right)'}$ is real. Upon clearing the dominator in (6) and letting $z \to 1^{-}$ through positive values, we obtain

$$\sum_{n=p+1}^{\infty} n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)a_n \le mp(1-\lambda\theta).$$

The function

$$f_n(z) = m\left[z^p - \frac{p(1-\lambda\theta)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)} z^n\right],$$

Show that the inequality (5) is sharp.

Corollary (1): *Let* $f(z) \in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$ *. Then*

$$a_n \le \frac{mp(1-\lambda\theta)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)} \quad . \tag{7}$$

Theorem (2): *Let* $f(z) \in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$.Then $f(z) \in \Re^{*p0}(\alpha,\lambda,\mu,\theta,\beta,z_0)$ if *and only if*

$$\sum_{n=p+1}^{\infty} \left(\frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{p(1-\lambda\theta)} - z_0^{n-p} \right) a_n \le 1. \tag{8}$$

Proof: Since $f(z) \in \Re^{*p0}(\alpha,\lambda,\mu,\theta,\beta,z_0)$, we have

$$z_o^{-p} f(z_0) = m - \sum_{n=p+1}^{\infty} a_n z_o^{n-p} \quad ,$$

Which gives

$$m \le 1 + \sum_{n=p+1}^{\infty} a_n z_o^{n-p} \quad .$$

Put m in (5), we get

$$\sum_{n=p+1}^{\infty} n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)a_n \le p(1-\lambda\theta)\left[1 + \sum_{n=p+1}^{\infty} a_n z_o^{n-p}\right]$$

Which is equivalent to

$$\sum_{n=p+1}^{\infty} \left(\frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{p(1-\lambda\theta)} - z_0^{n-p} \right) a_n \le 1.$$

Conversely, let the inequality holds true, then we have

$$\sum_{n=p+1}^{\infty}\left(\frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{p(1-\lambda\theta)}-z_0^{n-p}\right)a_n$$

$$=m\sum_{n=p+1}^{\infty}\left(\frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda\theta)}a_n\right)-\sum_{n=p+1}^{\infty}a_n z_0^{n-p}$$

$$=m-\sum_{n=p+1}^{\infty}a_n z_o^{n-p}=z_o^{-p}f(z_0)\leq 1.$$

Then $f(z)\in\Re^{*p0}(\alpha,\lambda,\mu,\theta,\beta,z_0)$.

Corollary (2): Let $f(z)\in\Re^{*p0}(\alpha,\lambda,\mu,\theta,\beta,z_0)$. Then

$$a_n\leq\frac{p(1-\lambda\theta)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)-p(1-\lambda\theta)z_o^{n-p}}\ . \qquad (10)$$

Theorem (3): Let $f(z)\in\Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$. Then $f(z)\in\Re^{*p1}(\alpha,\lambda,\mu,\theta,\beta,z_0)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{p(1-\lambda\theta)}-(np^{-p})z_0^{n-p}\right)a_n\leq 1. \qquad (11)$$

Proof: Since $f(z)\in\Re^{*p1}(\alpha,\lambda,\mu,\theta,\beta,z_0)$, we have

$$p^{-1}z_o^{1-p}f'(z_0)=m-\sum_{n=p+1}^{\infty}np^{-1}a_n z_o^{n-p}, \qquad (12)$$

Which gives

$$m\leq 1+\sum_{n=p+1}^{\infty}np^{-1}a_n z_o^{n-p} \qquad (13)$$

Substituting this value of m (given (13)) in Theorem 1 we get desire assertion.

Corollary (3): Let $f(z)\in\Re^{*p0}(\alpha,\lambda,\mu,\theta,\beta,z_0)$. Then

$$a_n\leq\frac{p(1-\lambda\theta)}{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)-n(1-\lambda\theta)z_o^{n-p}}\ . \qquad (14)$$

3 Integral Representation:

In the next theorem, we obtain the integral representation for

$f(z)\in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$.

Theorem (4): *Let* $f(z)\in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$. *Then*

$$\left(RAF_{\beta}^{\mu}(f(z))\right)' = \exp\left[\int_{0}^{z} \frac{(p-1)+\Phi(\tau)(1-\lambda\theta)}{\tau[1-\alpha(1+\Phi(\tau))]}\,d\tau\right] \quad \text{where } |\Phi(\tau)|<1 \ .$$

Proof: By putting $\dfrac{\left(RAF_{\beta}^{\mu}(f(z))\right)''}{\left(RAF_{\beta}^{\mu}(f(z))\right)'} = \Omega(z)$ in (3) we have

$$\left|\frac{\Omega(z)(1-\alpha)-(1-p)}{\Omega(z)\alpha+(1-\lambda\theta)}\right|<1,$$

or equivalently

$$\frac{\Omega(z)(1-\alpha)-(1-p)}{\Omega(z)\alpha+(1-\lambda\theta)} = \Phi(\tau)$$

Where

$$\frac{\left(RAF_{\beta}^{\mu}(f(z))\right)''}{\left(RAF_{\beta}^{\mu}(f(z))\right)'} = \frac{(p-1)+\Phi(z)(1-\lambda\theta)}{z[1-\alpha(1+\Phi(z))]}$$

After integration obtain

$$\log\left(RAF_{\beta}^{\mu}(f(z))\right)' = \int_{0}^{z} \frac{(p-1)+\Phi(\tau)(1-\lambda\theta)}{\tau[1-\alpha(1+\Phi(\tau))]}\,d\tau .$$

Thus

$$\left(RAF_{\beta}^{\mu}(f(z))\right)' = \exp\left[\int_{0}^{z} \frac{(p-1)+\Phi(\tau)(1-\lambda\theta)}{\tau[1-\alpha(1+\Phi(\tau))]}\,d\tau\right],$$

And this gives the result.

Definition (3): *Let* $f(z)$ *defined by the form*

$$f(z) = kz + \sum_{n=p+1}^{\infty} a_n z^n \quad (a_n \geq o, k > 0, p \in IN) .$$

Then $f(z)$ is said $k-$closed – to – convex of order $\alpha, (0 \leq \alpha < 1)$ if and only if

$$\left| \frac{f'(z)}{z^{p-1}} - kp \right| \leq 1 - \alpha \ . \tag{15}$$

4 Radius of *k* - Closed – to – Convex

In the next theorem, we obtain the radius of $k-$closed – to – convex for $f(z) \in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$.

Theorem (5): *If* $f(z) \in \Re^{*p}(\alpha,\lambda,\mu,\theta,\beta)$ *, then* $f(z)$ *is* $k-$*closed – to – convex in the disk* $|z| < r_{p^1}(\alpha,\lambda,\mu,\theta,\beta)$,

Where

$$r_{p^1}(\alpha,\lambda,\mu,\theta,\beta) = \inf_n \left\{ \frac{(1-\alpha)[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda\theta)} \right\}^{\frac{1}{n-1}} \tag{16}$$

Proof: For $|z| < r_{p^1}(\alpha,\lambda,\mu,\theta,\beta)$ we have

$$\left| \frac{f'(z)}{z^{p-1}} - mp \right| \leq \sum_{n=p+1}^{\infty} n a_n |z|^{n-p}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - mp \right| \leq 1 - \alpha$ if $\sum_{n=p+1}^{\infty} \left(\frac{n}{1-\alpha} \right) a_n |z|^{n-p} \leq 1.$ (17)

According to Theorem (1), we have

$$\sum_{n=p+1}^{\infty} \frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda\theta)} a_n \leq 1. \tag{18}$$

Hence (17) will be true if

$$\frac{n|z|^{n-p}}{1-\alpha} \leq \frac{n[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda\theta)}$$

Equivalently if

$$|z| \leq \left\{ \frac{(1-\alpha)[1+(n-p)-\lambda\theta]\Psi(\beta,\mu,n)}{mp(1-\lambda\theta)} \right\}^{\frac{1}{n-1}} \quad n \geq 2 \ . \tag{19}$$

The proof is completes.

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