Gen. Math. Notes, Vol. 16, No. 2, June, 2013, pp.93-102
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Coefficient Estimates for Certain Subclasses of Bi-Univalent Functions

Oana Crişan

Department of Mathematics, Babeş-Bolyai University
400084 Cluj-Napoca, Romania
E-mail: oana.crisan@ubbcluj.com

(Received: 8-3-13 / Accepted: 12-4-13)

Abstract

In this paper we introduce some new subclasses of the class σ of bi-univalent functions and obtain bounds for the initial coefficients of the Taylor series expansion of functions from the considered classes.

Keywords: *Bi-univalent functions, Starlike functions with respect to symmetric points, Convex functions with respect to symmetric points, Close-to-convex functions, Quasi-convex functions.*

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\},$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and let $\mathcal{S}$ be the class of all functions from $\mathcal{A}$ which are univalent in $\mathcal{U}$.

If the functions f and g are analytic in $\mathcal{U}$, then f is said to be subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω (i.e. analytic in $\mathcal{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$) such that $f(z) = g(\omega(z))$.

The Koebe one-quarter theorem states that the image of $\mathcal{U}$ under every function f from $\mathcal{S}$ contains a disk of radius 1/4. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad z \in \mathcal{U}$$

and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f),\ r_0(f) \geq \frac{1}{4}.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and the analytic extension of f^{-1} to $\mathcal{U}$ are univalent in $\mathcal{U}$. We denote by σ the class of all bi-univalent functions in $\mathcal{U}$.

In [7], the authors introduced the class $\mathcal{S}^*(\phi)$ of the so-called Ma-Minda starlike functions and the class $\mathcal{C}(\phi)$ of Ma-Minda convex functions, unifying several previously studied classes related to those of starlike and convex functions. The class $\mathcal{S}^*(\phi)$ consists of all the functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \phi(z)$, whereas $\mathcal{C}(\phi)$ is formed with functions $f \in \mathcal{A}$ for which the subordination $1 + zf''(z)/f'(z) \prec \phi(z)$ holds. The function ϕ considered here is analytic with positive real part in $\mathcal{U}$, $\phi(0) = 1$, $\phi'(0) > 0$ and with the property that ϕ maps $\mathcal{U}$ onto a domain starlike with respect to 1 and symmetric with respect to the real axis.

Lewin [6] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. In recent papers, several authors considered a series of subclasses of the bi-univalent function class σ, similar to the above mentioned classes of starlike and convex functions (see [1], [4], [11] and also [2]). In these papers, bounds of the initial coefficients $|a_2|$ and $|a_3|$ of the Taylor expansion (1) were investigated.

In [10], Sakaguchi introduced and investigated the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points in $\mathcal{U}$, consisting of functions $f \in \mathcal{A}$ that satisfy the condition $\Re\frac{zf'(z)}{f(z)-f(-z)} > 0$, $z \in \mathcal{U}$. The class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd starlike functions.

Similarly, in [13], Wang et al. introduced the class C_s of convex functions with respect to symmetric points, formed with all functions $f \in \mathcal{A}$ for which the inequality $\Re\frac{(zf'(z))'}{f'(z)+f'(-z)} > 0$ holds for all $z \in \mathcal{U}$. In the style of Ma and Minda, Ravichandran (see [12]) defined the classes $\mathcal{S}_s^*(\phi)$ and $\mathcal{C}_s(\phi)$. A function $f \in \mathcal{A}$ is said to be in $\mathcal{S}_s^*(\phi)$ if the subordination $\frac{2zf'(z)}{f(z)-f(-z)} \prec \phi(z)$ holds, whereas $f \in \mathcal{A}$ belongs to $\mathcal{C}_s(\phi)$ if the relation $\frac{2(zf'(z))'}{f'(z)+f'(-z)} \prec \phi(z)$ is true.

In view of our following investigation, we give now the definitions of close-to-convex and quasi-convex functions. A function $f \in \mathcal{A}$ is called close-to-convex if there exists a convex function h such that $\Re(f'(z)/h'(z)) > 0$, $z \in \mathcal{U}$, or equivalently, if there exists h starlike such that the inequality $\Re(zf'(z)/h(z)) > 0$ holds true for all $z \in \mathcal{U}$. A function $f \in \mathcal{A}$ is said to be quasi-convex if $\Re([zf'(z)]'/h(z)) > 0$, $z \in \mathcal{U}$. The classes of close-to-convex and quasi-convex functions were first introduced and studied by Kaplan [5] and Noor [8], respectively.

In the present paper, we define first two new subclasses of bi-univalent functions by using combinations of starlike and convex functions with respect to symmetric points. We also introduce a subclass of bi-univalent functions defined using a combination of close-to-convex and quasi-convex functions. For functions belonging to each of the considered classes, we investigate the bounds of the initial coefficients of their series expansions.

In order to derive our main results, we require the following lemma.

Lemma 1.1. *([9]) If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in $\mathcal{U}$ with positive real part, then*

$$|p_n| \leq 2 \quad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_2|^2}{2}. \tag{2}$$

2 Main Results

In the following, let ϕ be an analytic function with positive real part in $\mathcal{U}$, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(\mathcal{U})$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad B_1 > 0. \tag{3}$$

Definition 2.1. *A function $f \in \sigma$ is said to be in the class $\mathcal{S}^*_{s,\sigma}(\alpha, \phi)$ if the following subordinations hold:*

$$\frac{2[(1-\alpha)zf'(z) + \alpha z(zf'(z))']}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f'(z) + f'(-z))} \prec \phi(z)$$

and

$$\frac{2[(1-\alpha)wg'(w) + \alpha w(wg'(w))']}{(1-\alpha)(g(w) - g(-w)) + \alpha w(g'(w) + g'(-w))} \prec \phi(z),$$

where g is the extension of f^{-1} to $\mathcal{U}$.

Remark 2.2. *When $\alpha = 0$, the class $\mathcal{S}^*_{s,\sigma}(0, \phi)$ represents the class of all bi-univalent Ma-Minda starlike functions with respect to symmetric points, whereas when $\alpha = 1$, $\mathcal{S}^*_{s,\sigma}(1, \phi)$ is the class of all bi-univalent Ma-Minda convex functions with respect to symmetric points, introduced in [3].*

Theorem 2.3. *If $f \in \mathcal{S}^*_{s,\sigma}(\alpha,\phi)$ is given by (1) then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|(1+2\alpha)B_1^2 + 2(1+\alpha)^2(B_1 - B_2)|}} \tag{4}$$

and

$$|a_3| \le \frac{1}{2}B_1\left(\frac{1}{1+2\alpha} + \frac{1}{2(1+\alpha)^2}B_1\right). \tag{5}$$

Proof. Let $f \in \mathcal{S}^*_{s,\sigma}(\alpha,\phi)$ and g be the analytic extension of f^{-1} to $\mathcal{U}$. Then there exist two functions u and v, analytic in U, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in U$, such that

$$\frac{2(zf'(z) + \alpha z^2 f''(z))}{(1-\alpha)(f(z) - f(-z)) + \alpha(f'(z) + f'(-z))} = \phi(u(z)), \quad (z \in \mathcal{U}), \tag{6}$$

$$\frac{2(wg'(w) + \alpha w^2 g''(w))}{(1-\alpha)(g(w) - g(-w)) + \alpha(g'(w) + g'(-w))} = \phi(v(w)), \quad (w \in \mathcal{U}). \tag{7}$$

Next, define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{8}$$

and

$$p_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + b_1 w + b_2 w^2 + \cdots . \tag{9}$$

Since u and v are Schwarz functions, p_1 and p_2 are analytic functions in U, with $p_1(0) = p_2(0) = 1$ and which have positive real part in U. The equations (8) and (9) then give

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left[c_2 - \frac{c_1^2}{2}\right] z^2 + \cdots\right) \tag{10}$$

and

$$v(w) = \frac{p_2(w) - 1}{p_2(w) + 1} = \frac{1}{2}\left(b_1 w + \left[b_2 - \frac{b_1^2}{2}\right] w^2 + \cdots\right). \tag{11}$$

Using (6) and (7), we have

$$\frac{2(zf'(z) + \alpha z^2 f''(z))}{(1-\alpha)(f(z) - f(-z)) + \alpha(f'(z) + f'(-z))} = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{12}$$

and

$$\frac{2(wg'(w) + \alpha w^2 g''(w))}{(1-\alpha)(g(w) - g(-w)) + \alpha(g'(w) + g'(-w))} = \phi\left(\frac{p_2(w) - 1}{p_2(w) + 1}\right). \tag{13}$$

Next, the equations (3), (10) and (11) lead to

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right]z^2+\cdots \tag{14}$$

and

$$\phi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1+\frac{1}{2}B_1b_1w+\left[\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2\right]w^2+\cdots \tag{15}$$

Because the inverse g of f is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots,$$

we find that

$$\frac{2(zf'(z)+\alpha z^2f''(z))}{(1-\alpha)(f(z)-f(-z))+\alpha(f'(z)+f'(-z))} = 1+2(1+\alpha)a_2z+2(1+2\alpha)a_3z^2+\cdots \tag{16}$$

and

$$\begin{aligned}&\frac{2(wg'(w)+\alpha w^2g''(w))}{(1-\alpha)(g(w)-g(-w))+\alpha(g'(w)+g'(-w))}\\ &= 1-2(1+\alpha)a_2z+2(1+2\alpha)(2a_2^2-a_3)z^2+\cdots.\end{aligned} \tag{17}$$

Therefore, from the combination of equations (12)-(17), we deduce

$$2(1+\alpha)a_2 = \frac{1}{2}B_1c_1, \tag{18}$$

$$2(1+2\alpha)a_3 = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{19}$$

$$-2(1+\alpha)a_2 = \frac{1}{2}B_1b_1 \tag{20}$$

and

$$2(1+2\alpha)(2a_2^2-a_3) = \frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{21}$$

Equations (18) and (20) evidently show

$$c_1 = -b_1 \tag{22}$$

and from (20) we get

$$b_1^2 = \frac{16(1+\alpha)^2a_2^2}{B_1^2}. \tag{23}$$

By adding (19) with (21) and also using (22) and (23), it follows that

$$a_2^2 = \frac{B_1^3(b_2+c_2)}{8[(1+2\alpha)B_1^2+2(1+\alpha)^2(B_1-B_2)]},$$

and now, by applying Lemma 1.1 for the coefficients b_2 and c_2, the last equations gives the bound of $|a_2|$ from (4).

For the estimation of $|a_3|$, we subtract (21) from (19) and, in view of (18) and (22), it follows that

$$a_3 = \frac{1}{16(1+\alpha)^2}B_1^2b_1^2 + \frac{1}{8(1+2\alpha)}B_1(b_2-c_2).$$

The bound of $|a_3|$, as asserted in (5), is now a consequence of Lemma 1.1, and this completes our proof. □

Definition 2.4. *A function $f \in \sigma$ is said to be in the class $\mathcal{L}_{s,\sigma}(\alpha,\phi)$ if the following subordinations hold:*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f'(z)+f'(-z))}\right)^{1-\alpha} \prec \phi(z)$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2(wg'(w))'}{(g'(w)+g'(-w))}\right)^{1-\alpha} \prec \phi(w)$$

where g is the extension of f^{-1} to $\mathcal{U}$.

Theorem 2.5. *If $f \in \mathcal{L}_{s,\sigma}(\alpha,\phi)$ is given by (1) then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|(\alpha^2-3\alpha+3)B_1^2+2(2-\alpha)^2(B_1-B_2)|}} \tag{24}$$

and

$$|a_3| \le \frac{1}{2}B_1\left(\frac{1}{2(2-\alpha)^2}B_1 + \frac{1}{3-2\alpha}\right). \tag{25}$$

Proof. Let $f \in \mathcal{L}_{s,\sigma}(\alpha,\phi)$ and $g = f^{-1}$. We have

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f'(z)+f'(-z))}\right)^{1-\alpha}$$
$$= 1 + 2(2-\alpha)a_2z + [2(3-2\alpha)a_3 - 2\alpha(1-\alpha)a_2^2]z^2 + \cdots$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2(wg'(w))'}{(g'(w)+g'(-w))}\right)^{1-\alpha}$$
$$= 1 - 2(2-\alpha)a_2z + [2(3-2\alpha)(2a_2^2-a_3) - 2\alpha(1-\alpha)a_2^2]z^2 + \cdots .$$

Since $f \in \mathcal{L}_{s,\sigma}(\alpha, \phi)$, there are two Schwarz functions u and v such that

$$\frac{2(zf'(z))'}{f'(z)+f'(-z)} = \phi(u(z)) \quad \text{and} \quad \frac{2(wg'(w))'}{g'(w)+g'(-w)} \prec \phi(u(w)).$$

Proceeding now in the same manner as in the proof of Theorem 2.3, we obtain

$$2(2-\alpha)a_2 = \frac{1}{2}B_1c_1, \tag{26}$$

$$2(3-2\alpha)a_3 - 2\alpha(1-\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{27}$$

$$-2(2-\alpha)a_2 = \frac{1}{2}B_1b_1 \tag{28}$$

and

$$2(3-2\alpha)(2a_2^2 - a_3) - 2\alpha(1-\alpha)a_2^2 = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{29}$$

Equations (26) and (28) obviously yield $c_1 = -b_1$, and after some further calculations using (27)-(29) we find

$$a_2^2 = \frac{B_1^3(b_2+c_2)}{8[(\alpha^2-3\alpha+3)B_1^2 + 2(2-\alpha)^2(B_1-B_2)]}$$

and

$$a_3 = \frac{1}{16(20-\alpha)^2}B_1^2c_1^2 + \frac{1}{8(3-2\alpha)}B_1(c_2-b_2).$$

After applying Lemma (1.1), the estimates in (24) and (25) follow.

□

Definition 2.6. *A function $f \in \sigma$ is said to be in the class $\mathcal{Q}_{s,\sigma}(\alpha, \phi)$ if following subordinations hold:*

$$\frac{(1-\alpha)zf'(z) + \alpha z(zf'(z))'}{(1-\alpha)h(z) + \alpha zh'(z)} \prec \phi(z) \tag{30}$$

and

$$\frac{(1-\alpha)wg'(w) + \alpha w(wg'(w))'}{(1-\alpha)h(w) + \alpha wh'(w)} \prec \phi(w) \tag{31}$$

where h satisfies

$$\Re\left[\frac{(1-\alpha)zh'(z) + \alpha z(zh'(z))'}{(1-\alpha)h(z) + \alpha zh'(z)}\right] > 0 \tag{32}$$

and g is the analytic continuation of f^{-1} to $\mathcal{U}$.

Remark 2.7. *When $\alpha = 0$, the class $\mathcal{Q}_{s,\sigma}(0,\phi)$ consists of all bi-univalent close-to-convex functions of Ma-Minda type, whereas when $\alpha = 1$, $\mathcal{Q}_{s,\sigma}(1,\phi)$ represents the class of all bi-univalent quasi-convex functions of Ma-Minda type.*

Theorem 2.8. *If $f \in \mathcal{Q}_{s,\sigma}(\alpha,\phi)$ is given by (1), then*

$$|a_2| \leq \sqrt{\frac{B_1^2 + B_1^3 + 4|B_1 - B_2|}{|3(1+2\alpha)B_1^2 + 4(1+\alpha)^2(B_1 - B_2)|}} \tag{33}$$

and

$$|a_3| \leq B_1\left(\frac{1}{1+2\alpha} + \frac{B_1}{4(1+\alpha)^2}\right) + \frac{4|B_1 - B_2|}{3(1+2\alpha)B_1} \tag{34}$$

Proof. If $f \in \mathcal{Q}_{s,\sigma}(\alpha,\phi)$, then there exist two Schwarz functions u and v such that

$$\frac{(1-\alpha)zf'(z) + \alpha z(zf'(z))'}{(1-\alpha)h(z) + \alpha zh'(z)} = \phi(u(z))$$

and

$$\frac{(1-\alpha)wg'(w) + \alpha w(wg'(w))'}{(1-\alpha)h(w) + \alpha wh'(w)} = \phi(v(w)).$$

A computation gives

$$\begin{aligned}\frac{(1-\alpha)f'(z) + \alpha(zf'(z))'}{h'(z)} &= 1 + (1+\alpha)(2a_2 - h_2)z \\ &+ [3(1+2\alpha)a_3 - 2(1+\alpha)^2a_2h_2 - (1+2\alpha)h_3 + (1+\alpha)^2h_2^2]z^2 + \cdots\end{aligned} \tag{35}$$

and

$$\begin{aligned}\frac{(1-\alpha)g'(w) + \alpha(wg'(w))'}{h'(w)} &= 1 - (1+\alpha)(2a_2 + h_2) \\ &+ [3(1+2\alpha)(2a_2^2 - a_3) + 2(1+\alpha)^2h_2a_2 - (1+2\alpha)h_3 + (1+\alpha)^2h_2^2]z^2 +\end{aligned} \tag{36}$$

From (35) and (36) in combination with (10), (11), (14) and (15) it now follows that

$$(1+\alpha)(2a_2 - h_2) = \frac{1}{2}B_1c_1, \tag{37}$$

$$3(1+2\alpha)a_3 - 2(1+\alpha)^2a_2h_2 - (1+2\alpha)h_3 + (1+\alpha)^2h_2^2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2, \tag{38}$$

$$-(1+\alpha)(a_2 + h_2) = \frac{1}{2}B_1b_1, \tag{39}$$

and

$$3(1+2\alpha)(2a_2^2 - a_3) + 2(1+\alpha)^2a_2h_2 - (1+2\alpha)h_3 + (1+\alpha)^2h_2^2 = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2b_1^2. \tag{40}$$

By squaring and adding equations (37) and (39) we obtain

$$c_1^2 + b_1^2 = \frac{8(1+\alpha)^2(4a_1^2 + h_2^2)}{B_1^2}. \tag{41}$$

Because h satisfies (32), there exists an analytic function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ with $\Re p(z) > 0$ such that

$$\frac{(1-\alpha)zh'(z) + \alpha z(zh'(z))'}{(1-\alpha)h(z) + \alpha z h'(z)} = p(z).$$

The above relation yields

$$h_2 = \frac{p_1}{1+\alpha} \quad \text{and} \quad h_3 = \frac{p_2 + p_1^2}{2(1+2\alpha)}. \tag{42}$$

From relations (38), (40), (41) and (42) it now follows that

$$a_2^2 = \frac{2(p_2 - p_1^2)B_1^2 + (c_2 + b_2)B_1^3 - 4p_1^2(B_1 - B_2)}{4[3(1+2\alpha)B_1^3 + 4(1+\alpha)^2(B_1 - B_2)]}. \tag{43}$$

By applying Lemma 1.1, we get the desired estimate of $|a_2|$ from (33).

For the estimation of $|a_3|$, observe first that from (37) and (39) we obtain

$$a_2 = \frac{(c_1 - b_1)B_1}{8(1+\alpha)} \quad \text{and} \quad c_1^2 - b_1^2 = -\frac{4(1+\alpha)(c_1 - b_1)h_2}{B_1}. \tag{44}$$

Using (38), (40), (42) and (44) lead to

$$a_3 = \frac{(c_1 - b_1)^2 B_1^2}{64(1+\alpha)^2} + \frac{[(c_1 - b_1)p_1 + c_2 - b_2]B_1}{12(1+2\alpha)} + \frac{(c_1 - b_1)(B_1 - B_2)p_1}{6(1+2\alpha)B_1},$$

which, in view of Lemma 1.1, yields the bound of $|a_3|$ as asserted in (34) □

3 Acknowledgements

This work was possible with the financial support of the Sectoral Operation Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/S/76841 with the title "Modern Doctoral Studies: Internationalization and Interdisciplinarity".

References

[1] R.M. Ali, S.K. Lee, V. Ravichandran and S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, 25(2012), 344-351.

[2] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babeş-Bolyai Math.*, 31(2) (1986), 70-77.

[3] O. Crişan, On some subclasses of bi-univalent functions involving starlikeness with respect to symmetric points *(Submitted)*.

[4] B.A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.*, 24(2011), 1569-1573.

[5] W. Kaplan, Close-to-convex schlicht functions, *Mich. Math. J.*, 1(1952), 169-185.

[6] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Am. Math. Soc.*, 18(1967), 63-68.

[7] W.C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *In Proceedings of the Conference on Complex Analysis, Tianjin, 1992, Conf. Proc. Lecture Notes Anal., I*, Int. Press, Cambridge, MA, (1994).

[8] K.I. Noor, On quasi-convex functions and related topics, *Int. J. Math. Math. Sci.*, 10(2) (1987), 241-258.

[9] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, (1975).

[10] K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan*, 11(1) (1959), 72-75.

[11] H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, 23(2010), 1188-1192.

[12] V. Ravichandran, Starlike and convex functions with respect to conjugate points, *Acta Math. Acad. Paedagog. Nyireg.*, 20(2004), 31-37.

[13] Z.G. Wang, C.Y. Gao and S.M. Yuan, On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points, *J. Math. Anal. Appl.*, 322(2006), 97-106.