Gen. Math. Notes, Vol. 25, No. 1, November 2014, pp.50-61
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Fuzzy Set-Valuation on Graphs

A.T. Shahida[1] and M.S. Sunitha[2]

[1,2]Department of Mathematics, National Institute of Technology
Calicut 673601, India
[1]E-mail: Shahisajid@gmail.com
[2]E-mail: sunitha@nitc.ac.in

(Received: 5-8-14 / Accepted: 20-9-14)

Abstract

In this paper the concepts of fuzzy set-indexer,fuzzy segregation, topological fuzzy set-indexer, topogenic fuzzy set-indexer are introduced. Some properties of fuzzy set-indexer,topologically fuzzy set-indexer, topogenic fuzzy set-indexers are studied.

Keywords: *Fuzzy set-indexer, Fuzzy segregation, Topological fuzzy set-indexer, Topogenic fuzzy set-indexer.*

1 Introduction

B.D.Acharya introduced the notion of set-valuations of graphs and their applications [2] in MRI, Allahabad during the year 1979-1983 in series of lectures, and proved that every graph admits a set-indexer. In the recent literature the notion of set-valuation has been changed by Hegde [8] using the term set-coloring. However we follow the terminology used by Acharya in [2] and Acharya viewed the notion of set-graceful graphs as a set analogue of the well known graceful graphs, which was introduce by Alexander Rosa [1].

In 1965, Zadeh published his seminal paper on "Fuzzy sets" which described fuzzy set theory and, consequently, fuzzy logic. The purpose of Zadeh's paper was to develop a theory which could deal with ambiguity and imprecision of certain classes of sets in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction. Labelled graphs are becoming an increasingly useful family of Mathematical Models for a broad range of applications in Coding Theory, X-ray crystallography, communication network broadcasting and addressing, database management, radar, circuit design and so on. Use of fuzzy sets in graph labelling is a good area to research. In this paper we introduced a new term 'fuzzy set indexer of a graph as a fuzzy set analogue of the well known set-indexer of a graph and initiate the study of Fuzzy set-indexer of a graph G and its related properties.

2 Preliminaries

The following definitions are from [2], [3],[4], [5], [6] and [7].
Motivated from 'number valuations' of graph elements as well as from certain social network analysis B.D. Acharya defined 'a set-indexer' of a graph as follows.

Definition 2.1 *By assigning subsets of a set to the vertices and the symmetric difference of sets associated with end vertices of an edge to the corresponding edge, a set-indexer of a graph $G = (V, E)$ is an injective set-valued function $f : V(G) \to 2^X$,called a 'vertex set-valuation' of G that assigns subsets of a nonempty set X to the vertices of G such that the edge set-valuation $f^{\oplus} : E(G) \to 2^X$ induced by the set $E(G)$ of the edges of G , defined by the rule $f^{\oplus}(uv) = \{(f(u) \setminus f(v)) \cup (f(v) \setminus f(u))\} = f(u) \oplus f(v)$ for all $uv \in E(G)$ is injective , where $\oplus$ denotes symmetric difference of subsets of X.*

The original objective of introducing the notion of a set-indexer was to generate optimal automatic coding of edges merely by coding of vertex labels.

Definition 2.2 *A graph $G = (V, E)$ is said to be topological if there exists a nonempty set X and a set-indexer $f : V(G) \to 2^X$ such that the family $f(V)$ form a topology on the ground set X.*

Definition 2.3 *A set-indexer f of a graph G, is called a segregation of the ground set X of G if the sets $f(V(G)) = \{f(u), u \in V(G)\}$ and $f^{\oplus}(E(G)) = \{f^{\oplus}(e) : e \in E(G)\}$ are disjoint.*

Definition 2.4 *A graph $G = (V, E)$ is said to be topogenic if their exists a nonempty set X and a set-indexer $f : V(G) \to 2^X$ such that $f(V) \cup f^{\oplus}(E)$,is a topology on X.*

Definition 2.5 *Let X be a finite universal set ,a fuzzy subset A of X is defined by membership function $\mu_A : X \to [0, 1]$, $0\mu_A(x)1$, denoted by $\mu_A = \{(x, \mu_{A(x)})/x \in X\}$.*

Definition 2.6 *Let A and B be fuzzy subsets of X with the grade of membership of x in A and B denoted by $\mu_A(x)$ and $\mu_B(x)$ respectively. Then*

$A = B \iff \mu_A(x) = \mu_B(x) \forall x \in X$
$A \subseteq B \iff \mu_A(x)\mu_B(x) \forall x \in X$
$C = A \cup B \iff \mu_c(x) = Max[\mu_A(x), \mu_B(x)] \forall x \in X$
$C = A \cap B \iff \mu_c(x) = Min[\mu_A(x), \mu_B(x)] \forall x \in X$
$E = A' \iff \mu_E(x) = 1 - \mu_A(x) \forall x \in X$.

More generally for the family of fuzzy subsets , $A = \{A_i, i \in I\}$,the union , $C = \bigcup_I A_i$ and intersection, $D = \bigcap_I A_i$ are defined by $\mu_C(x) = Sup_I\{\mu_{A_i}(x)\}, x \in X$, $\mu_D(x) = Inf_I\{\mu_{A_i}(x)\}, x \in X$. The symbol ϕ will be used to denote an empty set ($\mu_\phi(x) = 0 \forall x \in X$). For X , we have by definition $\mu_X(x) = 1 \forall x \in X$.
The set of all fuzzy subsets of a set X is denoted by 2^{X_f} and is the fuzzy powerset of X.

Definition 2.7 *A fuzzy topology is a family τ of fuzzy subsets of X which satisfy the following conditions:*
$(a) X, \phi \in \tau$
(b) If $A, B \in \tau$ then $A \bigcap B \in \tau$
(c) If $A_i \in \tau$ for each $i \in I$, then $\bigcup_I A_i \in \tau$
τ is called a fuzzy topology for X ,and the pair (X, τ) is a fuzzy topological space.

3 Fuzzy Set-Indexer of a Graph G

Definition 3.1 *Let $G = (V, E)$ be a simple graph and an injective fuzzy set-valued function $f_f : V(G) \to 2^{X_f}$ is called a vertex fuzzy set-valuation that assigns distinct fuzzy subsets of a ground set X to the vertices of G such that the induced edge fuzzy set-valued function $f_f^{\oplus} : E(G) \to 2^{X_f}$ that assigns to each edge uv of G, the fuzzy symmetric difference $f_f^{\oplus} = f_f(u) \oplus f_f(v)$ where 2^{X_f} is the set of all fuzzy subsets of X , $\oplus$ is the operation defined by $f_f^{\oplus}(uv) = (f_f(u) \setminus f_f(v)) \bigcup (f_f(v) \setminus f_f(u))$ is injective .Then f_f is called a fuzzy set-indexer of G.*

Remark 3.2 *In this paper we take $X = \{x_1, x_2, ..., x_n\}, n$ is a positive integer.*

Example 3.3 *In Fig-1 take $X = \{x_1, x_2\}$, the vertices are labelled as $\phi, \{x_1/.1\}, \{x_1/.3, x_2/.5\}, \{x_2/.7\}$ .The edges will get the labels as follows.*

Consider the edge between the vertices $\{x_1/.1\}$ and $\{x_1/.3, x_2/.5\}$,the edge is labelled as follows
$Max[\{x_1/.1\}, \{x_1/.3, x_2/.5\}] - Min[\{x_1/.1\}, \{x_1/.3, x_2/.5\}]$

$= \{x_1/.3, x_2/.5\} - \{x_1/.1, x_2/0\}$
$= \{x_1/.3, x_2/.5\} \cap \{x_1/.1\}^c$
$= \{x_1/.3, x_2/.5\} \cap \{x_1/.9, x_2/1\}$
$= Min[\{x_1/.3, x_2/.5\}, \{x_1/.9, x_2/1\}]$
$= \{x_1/.3, x_2/.5\}$.
similarly we get the remaining edge labels as follows.
The edge between the vertices $\{x_1/.3, x_2/.5\}$ and $\{x_2/.7\}$ is labelled as $\{x_1/.7, x_2/.5\}$.
The edge between the vertices $\{x_2/.7\}$ and ϕ ,the edge is labelled as $\{x_2/.7\}$.
The edge between the vertices $\{x_1/.1\}$ and ϕ , is labelled as $\{x_1/.1\}$.

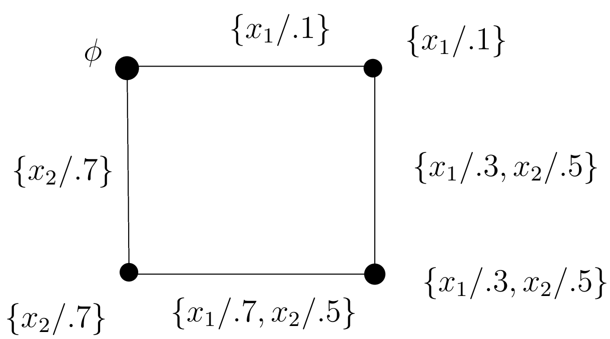

Figure-1

Proposition 3.4 *Every graph has a fuzzy set-indexer.*

Proof: Let $G = (V, E)$ be a simple graph with n vertices , $n \geq 0$.
Let $X = \{x_1, x_2, ..., x_n\}$, and $v_1, v_2, v_3, ..., v_n$ be the vertices of G.
label the vertices of G as follows.
$v_1 \to \{x_1/.1\}$
$v_2 \to \{x_2/.2\}$
$v_3 \to \{x_3/.3\}$
$. . . v_n \to \{x_n/.n\}$
The edges are labelled as follows
Consider any two vertices v_i and $v_j, i \neq j$ labelled as $\{x_i/.i\}$, $\{x_j/.j\}$, then the edge between them if it exist is labelled as $\{x_i/.i, x_j/.j\}$,
since $\{x_i/.i\} \oplus \{x_j/.j\} = (\{x_i/.i\} \cup \{x_j/.j\}) \setminus (\{x_i/.i\} \cap \{x_j/.j\})$
$= Max(\{x_i/.i\}, \{x_j/.j\}) \setminus Min(\{x_i/.i\}, \{x_j/.j\})$
$= (\{x_i/.i, x_j/.j\}) \setminus \phi$
$= (\{x_i/.i, x_j/.j\}) \cap \phi^c$
$= Min(\{x_i/.i, x_j/.j\}, X)$
$= \{x_i/.i, x_j/.j\}$
Then the fuzzy set-valued function $f_f : V(G) \to 2^{X_f}$ defined by $f_f(v_i) =$

$\{x_i/.i\}, i = 1, 2, 3, ..., n$ is injective and the induced edge fuzzy set valued function $f_f^{\oplus} : E(G) \to 2^{X_f}$ is also injective.
Hence f_f is a fuzzy set-indexer of G.

Definition 3.5 *[2] Let G be a simple finite graph, the set-indexing number $\sigma(G)$ of a graph G is the least cardinality of the ground set X with respect to which G admit a set-indexer.*

Definition 3.6 *Let G be a simple finite graph, the fuzzy set-indexing number $\sigma_f(G)$ of a graph G is the least cardinality of the set of fuzzy subsets of X with respect to which G admit a fuzzy set-indexer and the fuzzy set indexing number is denoted by $\sigma_f(G)$.*

Example 3.7 *In example 3.3, the fuzzy set-indexing number is 5, that is the cardinality of the set*
$\{\phi, \{x_1/.1\}, \{x_1/.3, x_2/.5\}, \{x_1/.7, x_2/.5\}, \{x_2/.7\}\}$.

Proposition 3.8 *For any (p,q)-graph $G = (V, E)$,with $|V| = p$, $|E| = q$ and f_f is a fuzzy set-indexer of G with respect to X, then $p \leq \sigma_f(G) \leq p+q-k$,where k is the number of vertices of G that are adjacent to the vertex with label ϕ, if it exist.*

Proof: The inequality $p \leq \sigma_f(G)$ is trivial ,since the fuzzy set-indexer is injective and hence no two vertices can have the same label.
The maximum choice of $\sigma_f(G)$ is $p + q$.But if we label ϕ to any vertex of G then $\sigma_f(G) < p + q$.
Let v_i be a vertex labelled as ϕ and $u_1, u_2, ..., u_k$ are the vertices adjacent to v_i, then the edges between v_i and $u_j, j = 1, 2, ..., k$ have the same labels as $u_j, j = 1, 2, ..., k$.
so $\sigma_f(G) \leq p + q - k$.

Definition 3.9 *A fuzzy set-indexer f_f of a graph G, is called a fuzzy segregation of the fuzzy ground set X of G if the families $f_f(V(G)) = \{f(u), u \in V(G)\}$ and $f_f^{\oplus}(E(G)) = \{f^{\oplus}(e) : e \in E(G)\}$ have no common members.*

Example 3.10 *Fig-2 is an illustration of fuzzy set-indexer which is a fuzzy segregation. In the Fig-2 take $X = \{x_1, x_2, x_3\}$ the vertices are labelled by the set*
$\{\{x_1/.2, x_2/.3, x_3/.4\}, \{x_1/.3\}, \{x_2/.7\}, \{x_1/.5, x_2/.4\}, \{x_3/.8\}\}$

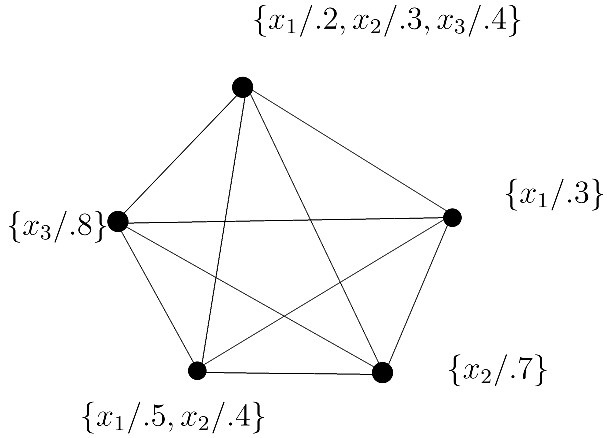

Figure-2

Therefore the edges are labelled as
$\{x_1/.3, x_2/.3, x_3/.4\}, \{x_1/.3, x_2/.7\}, \{x_1/.5, x_2/.6\}, \{x_1/.5, x_2/.4, x_3/.8\},$
$\{x_1/.2, x_2/.3, x_3/.6\}$ So the vertices and edges have distinct labels, hence the fuzzy set-indexer is a fuzzy segregation of G.

4 Fuzzy Topological Fuzzy Set-Indexer of a Graph

Definition 4.1 *A graph $G = (V, E)$ is said to be fuzzy topological if there exists a nonempty ground set X and a fuzzy set-indexer $f_f : V(G) \to 2^{X_f}$ such that the family $f_f(V)$ forms a fuzzy topology on the ground set X.*

Example 4.2 *Consider a graph G with five vertices ,and set $X = \{x_1, x_2\}$, the vertices are labelled as $\phi, X, \{x_1/.4\}, \{x_2/.3\}, \{x_1/.7, x_2/.6\}$. Then the family $f_f(V(G))$ form a fuzzy topology on X.*

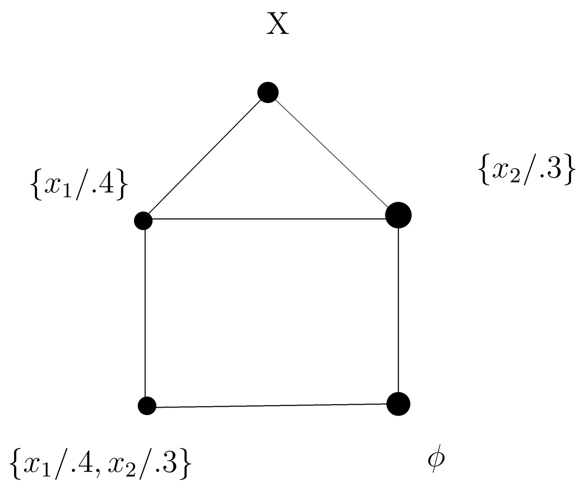

Figure.3

5 Fuzzy Topogenic Fuzzy Set-Indexer of a Graph

In this section , we introduce fuzzy topogenic fuzzy set indexer and investigate some foundational results on fuzzy topogenic fuzzy set-indexers of graphs. We also establish the existence of non-fuzzy topogenic graphs and identify certain classes of graphs that admit fuzzy topogenic set-indexers.

Definition 5.1 *A fuzzy topogenic fuzzy set-indexer of a graph $G = (V, E)$ is a fuzzy set-indexer $f_f : V(G) \to 2^{X_f}$ such that the family $f_f(V) \bigcup f_f^{\oplus}(E(G))$ is a fuzzy topology on the ground set X.*

Example 5.2 *In Fig-4, let $X = \{x_1, x_2\}$ be the ground set and $f_f : V(G) \to 2^{X_f}$ be defined as follows. The vertices are labelled $\phi, X, \{x_1/.3\}, \{x_2/.7\}$ so that the edges get the labellings as $\{x_1/.3\}$, $\{x_1/.3, x_2/.7\}, \{x_2/.7\}, X$ then the family $f_f(V(G)) \cup f_f^{\oplus}(E(G))$ form a fuzzy topology on X.*

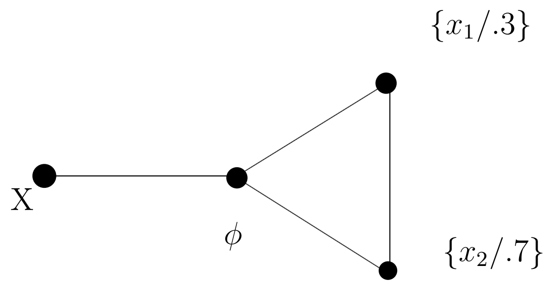

Figure-4

Remark 5.3 *Note that a fuzzy topogenic graph may have non-fuzzy topogenic set-indexer. Consider graph in Fig-4 which is fuzzy topogenic, however if we rearrange the label of vertices as* $\{x_1/.1, x_2/.7\}, \phi$, $\{x_1/.1\}, \{x_2/.7\}$ *then* f_f *become a non-fuzzy topogenic set-indexer of* G *(Fig-5).*

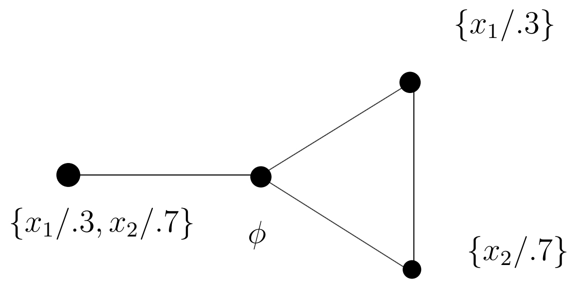

Figure-5
Non-fuzzy topogenic set-indexer of a fuzzy topogenic graph G.

Remark 5.4 *Now the very first basic question is whether every graph is fuzzy topogenic.That is, given any graph* $G = (V, E)$ *,can we invariably find a ground set* X *and a fuzzy set-indexer* f_f *of* G *such that* $f_f(V(G)) \bigcup f_f^{\oplus}(E(G))$ *is a fuzzy topology on* X*. Here we establish that all graphs with at most* 3 *vertices are fuzzy topogenic.*

Theorem 5.5 *All graphs with at most* 3 *vertices are fuzzy topogenic.*

Proof: Let $G = (V, E)$ be a graph with p vertices, where $p = 1, 2, 3$. We have the following cases.

Case.1: $p = 1$
Trivial graph K_1 is the only graph of order 1; it has a unique fuzzy topogenic fuzzy set-indexer that assigns the empty set , ϕ to its unique vertex.

Case.2: $p = 2$
There are two graphs of order 2, K_2 and $\bar{K}_2 = K_1 \cup K_1$.
Each of these graphs has a fuzzy topogenic set-indexer obtained by assigning the empty set to one vertex and the entire set X to the other vertex.

Case.3: $p = 3$.
There are four graphs of order 3, they are , $G_1 \cong \bar{K}_3, G_2 \cong K_1 \cup K_2, G_3 \cong K_{1,2} \cong P_3$ and $G_4 \cong K_3$.
Let v_1, v_2, v_3 be the vertices of $G_i, i = 1, 2, 3, 4$.

Subcase.1:.$G_1 \cong \bar{K}_3$
Let the ground set $X = \{x_1\}$. Then the vertices of $\bar{K}_3$ are labelled as follows.

$f_f(v_1) = \phi, f_f(v_2) = X, f_f(v_3) = \{x_1/.1\}$, then f_f is a fuzzy topogenic fuzzy set-indexer of G_1 and the family $f_f(V(G_1)) \cup f_f^{\oplus}(E(G_1)) = \{\phi,\mathrm{X}, \{x_1/.1\}\}$ is a fuzzy topology on X.

Subcase.2: $G_2 \cong K_1 \cup K_2$.
Let the ground set $X = \{x_1\}$. Then the vertices of G_2 are labelled as follows. Assume that the vertex of K_1 is v_1 and $f_f(v_1) = \{x_1/.1\}$, the vertices of K_2 are v_2, v_3 which are labelled as $f_f(v_2) = X, f_f(v_3) = \phi$, Then the family $f_f(V(G_2)) \cup f_f^{\oplus}(E(G_2)) = \{\phi,\mathrm{X}, \{x_1/.1\}\}$ is a fuzzy topology on X

Subcase.3: $G_3 \cong K_{1,2}$

Let the ground set $X = \{x_1\}$. The vertices are labelled as follows.
The two pendant vertices be assigned the sets $X, \{x_1/.1\}$ and the internal vertex be assingned the label ϕ. Then the family $f_f(V(G_3)) \cup f_f^{\oplus}(E(G_3)) = \{\phi,\mathrm{X}, \{x_1/.1\}\}$ is a fuzzy topology on X.

Subcase.4: $G_4 \cong K_3$
Let the ground set $X = \{x_1\}$. Consider the fuzzy set-valuation $f_f : V(G_4) \to 2^{X_f}$. Let $f_f V(G_4) = \{\phi, X, \{x_1/.1\}\}$. Then
$f_f^{\oplus}(E(G_4)) = \{\phi, X, \{x_1/.1\}\}$, a fuzzy topology on X. Hence f_f is a fuzzy topogenic set-indexer of G_4.

Theorem 5.6 *For every positive integer n , there exist*
(a) *A connected fuzzy topogenic graph of order n.*
(b) *A totally disconnected fuzzy topogenic graph of order n.*
(c) *A disconnected fuzzy topogenic graph of order n.*

Proof:
(a). Let S_n be a star whose vertices are labeled $u_1, u_2, ..., u_n$ so that u_n is the internal vertex of the star. Let $X = \{x_1, x_2, x_3, ..., x_n\}$ be the ground set. Define $f_f : V(S_n) \to 2^{X_f}$ such that
$f_f(u_{n\})=\phi}$,
$f_f(u_1) = X$
$f_f(u_2) = \{x_1/.1\}$
$f_f(u_3) = \{x_1/.1, x_2/.2\}$
.
.
.
$f_f(u_{n-1}) = \{x_1/.1, x_2/.2, ..., x_{n-2}/.(n-2)\}$ (Fig-5).

Then $f_f(V) \cup f_f(E)$ form a fuzzy topology on X, hence f_f is a fuzzy topogenic fuzzy set-indexer of S_n.

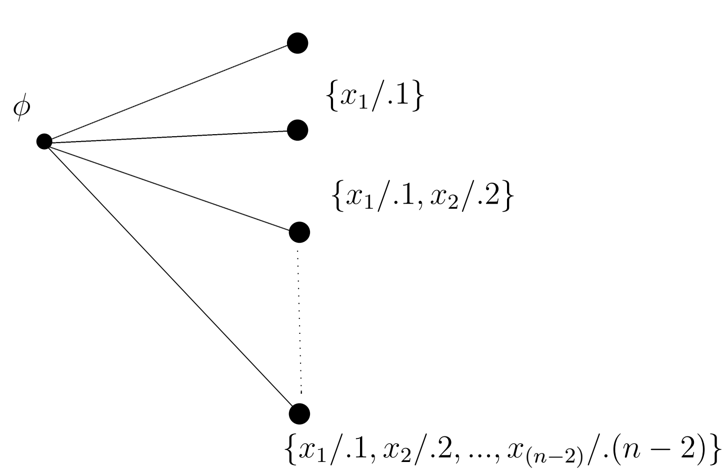

Figure-6

(b) Let $G = \bar{K}_n$. Label the n vertices of G as in Case (a). Then $f_f(V(G))$ is a fuzzy topology on X.

(c) From S_n of case (a), remove any $k(< n)$ edges, then the remaining collection $f_f(V(G)) \cup f_f(E(G))$ form a fuzzy topology on X.

Definition 5.7 ***Fuzzy Topogenic Strength of a Graph***
Consider a topogenic set-indexer $f_f : V(G) \to 2^{X_f}$ of a (p, q)-graph $G = (V, G)$ and let $\tau_{f_f} = f_f(V(G)) \cup f_f^{\oplus}(E(G))$ is a fuzzy topology on X. The number of distinct f_f-open sets,viz , $|\tau_{f_f}|$, is called the fuzzy topogenic strength of f_f over G, denoted by ϱ_f. If G is finite ,the minimum (respectively maximum)of $|\tau_{f_f}|$ taken over all possible fuzzy topogenic fuzzy set-indexers f_f of G is denoted $\varrho_f^0(G)$(respectively $\varrho_f^1(G)$).Because of the injectivity of f_f and $f_f^{\oplus}$, we must have
$\varrho_f^0(G) \leq |f_f(V(G)) \cup f_f(E(G))| \leq \varrho_f^1(G)) \leq p + q - k$, where k is the number of vertices of G that are adjacent to the vertices with label as ϕ(such a vertex exists since τ_{f_f} is a fuzzy topology on X).

Moreover, $p \leq \varrho_{\substack{0\\f}}$ and $q + 1 \leq \varrho_{\substack{0\\f}}$ (since $\phi \notin f_f^{\oplus}(E(G))$). From these observations we get following theorems.

Theorem 5.8 *For any fuzzy topogenic $(p, q)-$ graph G,*
max $\{p, q + 1\} \leq \varrho_{\substack{0\\f}} \leq \varrho_{\substack{1\\f}} \leq p + q - \delta$ where $\delta = \delta(G)$ is the minimum vertex degree of G.

Theorem 5.9 *For a fuzzy topogenic path P_n, $n \leq \varrho_f^0 \leq 2n - 2$.*

Proof: By theorem.5.9, we have $max\{p, q+1\} \leq \varrho_f^0(G) \leq p+q-\delta$. For the path P_n, we have $p = n, q = n-1$. Thus, we have $n \leq \varrho_f^0 \leq 2n-2$.

Theorem 5.10 *For a fuzzy topogenic cycle $C_n, n+1 \leq \varrho_f^0 \leq 2n-2$.*

Proof: By theorem.5.9, we have $\max\{p, q+1\} \leq \varrho_f^0 \leq \varrho_{\underset{f}{1}} \leq p+q-\delta$. For the case of cycle ,we have $p = n, q = n$ and $\delta = 2$. Thus we have $n+1 \leq \varrho^0 \leq 2n-2$.

Theorem 5.11 *For a fuzzy topogenic complete bipartite graph $K_{m,n}$, $mn+1 \leq \varrho_0 \leq m(n+1)$ where $n \leq m$.*

Proof: This follows from theorem.5.9 and the fact that for $K_{m,n}$,we have , $p = m+n, q = mn$ and $\delta = n$.

6 Conclusion

The concepts of fuzzy set-indexer, fuzzy segregation, topological fuzzy set-indexer, topogenic fuzzy set-indexer are introduced and some of their properties are studied.

Acknowledgements: The first author is grateful to the National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Mumbai, India for providing the financial assistance.

References

[1] A. Rosa, On certain valuations of the vertices of a graphs, *International Symposium*, Rome, July(1966), Gordon and Breach, New York and Dunod Paris, (1967), 349-355.

[2] B.D. Acharya, Set-valuations and their applications, *MRI Lecture Notes in Applied Mathematics*, The Mehta Research Institute of Mathematics and Mathematics Physics, Allahabad, (1983).

[3] C.L. Chang, Fuzzy topological space, *Journal of Mathematical Analysis and Applications*, 24(1968), 182-190.

[4] G.J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic Theory and Applications*, Published by Prentic Hall PTR, Prentice-Hall inc., (1995).

[5] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, (1969).

[6] J.E. Joy, A study on topologies arising from graphs and diagraphs, *Ph.D Thesis*, Mathematics Research Center (Kannur University), Mary Matha Arts and Science College, Mananthavady, Kerala, India, 2011.

[7] L.A. Zadeh, Fuzzy sets, *Information and Control*, 8(1965), 338-353.

[8] S.M. Hegde, Set colourings of graphs, *Europian Journal of Combinatorics*, 30(2009), 986-995.