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Spectrum of Positive Definite Functions on Product Hypergroups

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Abstract

This paper aims to show that the amenability of $K_1 \times K_2$ is equivalent to the following condition: "If φ is a continuous positive definite function defined on $K_1 \times K_2$ and $\varphi \geq 0$ then the constant function $1_{K_1 \times K_2}$ belongs to the spectrum of φ", which K_1 and K_2 are locally compact hypergroups as defined by R. Jewett [1], with convolutions $*_1, *_2$ respectively.Our study deals with the cases of exponentially bounded product hypergroups and discrete solvable product hypergroups. And study of conditionally exponential convex functions.

Keywords: Product hypergroups, Positive definite functions, Exponentially bounded, Discrete solvable, Conditionally exponential convex functions.

1 Introduction

Let K be a locally compact Hausdorff space, $M(K)$ denote the space of all bounded radon measures, $M^1(K)$ be the subset of all probability measures and ε_x be the point mass measure of $x \in K$. The support of a measure μ is denoted by supp μ. $C(K)$ denotes the space of continuous functions on K. The space K is called a hypergroup if the following conditions are satisfied:

(H1) There exists a map: $K \times K \to M^1(K)$, $(x,y) \to \varepsilon_x * \varepsilon_y$, called convolution, which is continuous, where $M^1(K)$ bears the vague topology.

(H2) supp $\varepsilon_x * \varepsilon_y$ is compact.

(H3) There exists a homomorphism $K \to K$, $x \to x^-$, called involution, such that $x = (x^-)^-$ and $(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{y^-} * \varepsilon_{x^-}$.

(H4) There exists an element $e \in K$, called unit element, such that $\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$.

(H5) $e \in$supp $\varepsilon_x * \varepsilon_{y^-}$ if and only if $x = y$.

(H6) The map $(x,y) \to$ supp $\varepsilon_x * \varepsilon_y$ of $K \times K$ into the space of nonvoid compact subset of K is continuous, the latter space with topology as given in [2,7].

Let K_1 and K_2 are locally compact hypergroups, with convolutions $*_1, *_2$ respectively. The cartesian product of K_1 and K_2 will take the form

$$K_1 \times K_2 = \{(x_1, x_2) : x_1 \in K_1, and\ x_2 \in K_2\}$$

with convolution $*$ defined on $M(K_1 \times K_2)$by

$$\varepsilon_{(x_1,x_2)} * \varepsilon_{(y_1,y_2)} = (\varepsilon_{x_1} *_1 \varepsilon_{y_1}) \times (\varepsilon_{x_2} *_2 \varepsilon_{y_2})$$

where $\varepsilon_{(x_1,x_2)}$ is the one point mass measure. And the involution of the product hypergroups is defined by

$$(x_1, x_2)^- = (x_1^-, x_2^-), \forall (x_1, x_2) \in K_1 \times K_2$$

finally, the identity element of the product hypergroups is (e_1, e_2), which e_1 and e_2 are the identites of K_1 and K_2 respectively.

A map φ define on $(K_1 \times K_2)^2$ on to $\mathbb{R}^+$ is called positive definite function if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(\,(x_1, x_2)_i * (x_1, x_2)_j^-\,) \geq 0.$$

where $\{c_1, c_2, ..., c_n\} \in \mathbb{C}$, $\{(x_1, x_2)_1, (x_1, x_2)_2, ..., (x_1, x_2)_n\} \in K_1 \times K_2$.

For an example of positive, positive definite functions on a product hypergroups $K_1 \times K_2$ are given by a functions of the form $f * f^\sim$, where f is a positive function on $K_1 \times K_2$ with compact support, $f^\sim$ is defined by $f^\sim(x_1, x_2) = \overline{f(x_1, x_2)^{-1}}$ and $*$ is the convolution, it is easy to see that the function $f * f^\sim$ is positive definite.

If $P(K_1 \times K_2)$ be the convex set of all continuous positive-definite functions φ on $K_1 \times K_2$ with $\varphi(e_1, e_2) = 1$. The spectrum $sp\varphi$ of $\varphi \in P(K_1 \times K_2)$ can be defined as the set of all indecomposible $\psi \in P(K_1 \times K_2)$ which are limits, in the sense of the topology of uniform converges on compact subsets of $K_1 \times K_2$, of functions of the form

$$(x_1, x_2) \to \sum_{i,j=1}^{n} c_i \overline{c_j}\ \varepsilon_{(x_1,x_2)_i} * \varepsilon_{(x_1,x_2)_j^-}\ \ \psi(x_1, x_2)$$

where $\{c_1, ..., c_n\} \in \mathbb{C}$,$\{(x_1, x_2)_1, (x_1, x_2)_{2,} ..., (x_1, x_2)_n\} \in K_1 \times K_2$.

If π_φ denotes the cyclic unitary representation of $K_1 \times K_2$ associated with φ, then $sp\varphi$ consists of all $\psi \in P(K_1 \times K_2)$ for which π_ψ is irreducible and weakly contained in π_ψ [2].

Our main subject here is to prove that exponentially bounded product hypergroups and solvable discete hypergroups satisfy the followig property (which we denote by (P)):

(P) If $\varphi \in P(K_1 \times K_2)$ and if φ is positive in usual sense, then the constant positive- definite function 1 on $K_1 \times K_2$, $1_{K_1 \times K_2}$, belongs to $sp\varphi$. For connected hypergroups we show that the condition that the hypergroup is amenable is equivalent to the following weaker version (P*) of P:

(P*) if $\varphi \in P(K_1 \times K_2)$ and if φ is positive, then $1_{K_1 \times K_2} \in sp_d(\varphi)$, where $sp_d(\varphi)$ is the spectrum of φ when the domain of φ is $(K_1 \times K_2)_d$ (the discrete product hypergroups).

2 Exponentially Bounded Hypergroups

Let π be a continuous unitary representation of $K_1 \times K_2$ in the Hilbert space $(H_\pi, \langle ., . \rangle)$. A unit vector $\xi \in H_\pi$ will be called a positive vector for π, if

$$Re\ \ \langle \pi(x_1, x_2)\xi, \xi \rangle\ \geq 0$$

for all $(x_1, x_2) \in K_1 \times K_2$.

So,

$$Re\ \ \langle \pi(.)\xi, \xi \rangle \in P(K_1 \times K_2)$$

Now, it is easy to translate (P) into a property of unitary representations with positive vectors. In fact, condsider the following property (P') of $K_1 \times K_2$ which is formally stronger than (P):

(P') If π is a unitary representation of $K_1 \times K_2$ with a positive vector, then π contains weakly $1_{K_1 \times K_2}$.

Theorem 2.1 *(P) and (P') are equivalent for every product hypergroups $K_1 \times K_2$.*

Proof: Let π be a unitary representation of $K_1 \times K_2$ with a positive vector $\xi \in H_\pi$. Let $\varphi(x_1, x_2) = Re\ \langle \pi(x_1, x_2)\xi, \xi\rangle$, $(x_1, x_2) \in K_1 \times K_2$. If (P) holds, then $1_{K_1 \times K_2}$ is weakly contained in π_φ which is the subrepresentation of $\pi \oplus \pi$ and this implies that $1_{K_1 \times K_2}$ is weakly contained in π.

A locally compact product hypergroups is called Exponentially bounded if

$$\lim_n |G^n|^{\frac{1}{n}} = 1$$

for each compact neighbourhood G of (e_1, e_2), where $|.|$denotes the Haar measure and $G^n = \{g_1, ..., g_n; g_i \in G\}$. Exponentially bounded hypergroups are amenable[4].

Theorem 2.2 *Exponentially bounded product hypergroups satisfy property (P).*

Proof: Let $K_1 \times K_2$ be an exponentially bounded product hypergroups and let $\varphi \in P(K_1 \times K_2)$, with $\varphi \geq 0$. Let G be a compact neighbourhood of (e_1, e_2) with the condition $G = G^{-1}$, and $\epsilon > 0$. Then there is an $n \in N$ such that

$$\begin{aligned} &\int_{G^{n+1} \times G^{n+1}} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\, d(y_1, y_2) d(z_1, z_2) \\ \leq\ & (1+\epsilon) \int_{G^n \times G^n} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\, d(y_1, y_2) d(z_1, z_2) \qquad (1) \end{aligned}$$

where $d(y_1, y_2)$, and $d(z_1, z_2)$are Haar measures on $K_1 \times K_2$.

In fact, otherwise

$$\begin{aligned} \left|G^{n+1}\right|^2 &\geq \int_{G^{n+1} \times G^{n+1}} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\, d(y_1, y_2) d(z_1, z_2) \\ &> (1+\epsilon)^n \int_{G^n \times G^n} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\, d(y_1, y_2) d(z_1, z_2) \end{aligned}$$

for all $n \in N$.

Since

$$\int_{G^n \times G^n} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\, d(y_1, y_2) d(z_1, z_2) > 0,$$

this would be a contradiction with

$$\lim\ |G^n|^{\frac{1}{n}} = 1.$$

Now choose $n \in N$ such that (1) holds.

Let $f = \chi_{G^n}$ be the characteristic function of G^n. Let π be the unitary representation of $K_1 \times K_2$ associated to φ with Hilbert space H_π. Let $\xi \in H_\pi$ be such that $\varphi(x_1, x_2) = \langle \pi(x_1, x_2)\xi, \xi\rangle$, $(x_1, x_2) \in K_1 \times K_2$.

Then

$$\|\pi(f)\xi\|^2 = \int_{K_1}\int_{K_2} f^- * f(x_1, x_2)\ \ \varphi(x_1, x_2)\ \ d(x_1, x_2) > 0,$$

since $f^- * f(e_1, e_2)\ \ \varphi(e_1, e_2) > 0$ and $f^- * f(x_1, x_2)\varphi(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in K_1 \times K_2$.

Now let

$$\psi(x_1, x_2) = \frac{1}{\|\pi(f)\xi\|^2}\langle \pi(x_1, x_2)\pi(f), \pi(f)\xi\rangle, \qquad (x_1, x_2) \in K_1 \times K_2.$$

Thenψ is associated to π.moreover, for each $(x_1, x_2) \in K_1 \times K_2$

$$|\psi(x_1, x_2) - 1|^2 = \frac{1}{\|\pi(f)\xi\|^4}|\langle \pi((x_1, x_2)f - f)\xi, \pi(f)\xi\rangle|^2$$

$$\leq \frac{\|\pi((x_1, x_2)f - f)\xi\|^2}{\|\pi(f)\xi\|^2}$$

$$= \frac{\int_{(K_1\times K_2)^2}((x_1, x_2)f - f)(y_1, y_2)((x_1, x_2)f - f)(z_1, z_2)\,\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)}{\int_{(K_1\times K_2)^2} f(y_1, y_2) f(z_1, z_2)\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi) d(y_1, y_2)d(z_1, z_2)}$$

$$= \frac{\int_{((x_1,x_2)G^n\Delta G^n)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)}{\int_{(G^n)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)}$$

where Δ is the symmetric difference.

Now (1) implies that for $(x_1, x_2) \in G$.

$$\int_{((x_1,x_2)G^n\Delta G^n)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)$$

$$\leq \int_{\left(\frac{G^{n+1}}{G^n}\right)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2) + \int_{\left(\frac{G^n}{(x_1,x_2)G^n}\right)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)$$

$$\leq \epsilon\int_{(G)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2) + \int_{\left(\frac{(x_1,x_2)^{-1}G^n}{(x_1,x_2)G^n}\right)^2}\varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-}(\varphi)\,d(y_1, y_2)d(z_1, z_2)$$

$$\leq 2\epsilon \int_{(G^n)^2} \varepsilon_{(y_1,y_2)} * \varepsilon_{(z_1,z_2)^-} (\varphi)\, d(y_1,y_2) d(z_1,z_2)$$

since $(x_1,x_2)^{-1} \in G$. Hence $|\psi(x_1,x_2) - 1|^2 \leq 2\epsilon$ for all $(x_1,x_2) \in G$.

It is to be noted that last Theorem can be reformulate in the form: " If φ is positive and $\varphi \in P(K_1 \times K_2)$ where $(K_1 \times K_2)$ is an exponentially bounded product hypergroups, then the constant function $1_{K_1 \times K_2}$ is the uniform limit on compact subsets of $K_1 \times K_2$ of functions of the form

$$(x_1,x_2) \to \sum_{i,j=1}^{n} \varepsilon_{(x_1,x_2)_i} * \varepsilon_{(x_1,x_2)_j^-} (\varphi(x_1,x_2))\, c_i \bar{c}_j$$

where $c_l \geq 0$ and $(x_1,x_2)_l \in K_1 \times K_2$ for all $1 \leq l \leq n$.

Theorem 2.3 *Discrete solvable product hypergroups satisfy property (P).*

Proof: Let $K_1 \times K_2$ be a discrete solvable product hypergroups and let $\varphi \in P(K_1 \times K_2)$ with $\varphi \geq 0$. Let $(K_1 \times K_2) = (K_1 \times K_2)_n \supseteq (K_1 \times K_2)_{n-1} \supseteq \ldots. \supseteq (K_1 \times K_2)_0 = \{(e_1,e_2)\}$, be a composition series with abelian factor $(K_1 \times K_2)_i/(K_1 \times K_2)_{i-1}$, $1 \leq i \leq n$. First we show by induction on i that: for each $0 \leq i \leq n$ there is a net $(\psi_\alpha)_\alpha$ in $P(K_1 \times K_2)$ with $\psi \geq 0$ such that $\lim \psi(x_1,x_2) = 1$ for all $(x_1,x_2) \in (K_1 \times K_2)_i$ and such that π_{ψ_α} is weakly contained in π for all α.

For $i = 0$, the assertion is trivial (take $\psi_\alpha = \varphi$). For any i suppose that a net $(\psi_\alpha)_{\alpha \in N}$ exists. Let ψ be a limit point of $\{\psi_\alpha\}_{\alpha \in N}$ in the weak *-topology $\sigma(l^\infty(K_1 \times K_2), l^1(K_1 \times K_2))$. Then $\psi \in P(K_1 \times K_2)$ and $\psi \geq 0$.

Moreover

$$\psi(x_1,x_2) = \lim_\alpha \psi_\alpha(x_1,x_2) = 1$$

for all $(x_1,x_2) \in (K_1 \times K_2)_i$.

Hence $\psi \mid (K_1 \times K_2)_{i-1}$ factors to a positive definite function of $(K_1 \times K_2)_{i+1}/(K_1 \times K_2)_i$. Thus by last theorem in its reformulated form there is a net $\left(\psi'_\beta\right)_\beta$ in $P((K_1 \times K_2)_{i+1}/(K_1 \times K_2)_i)$ of the form

$$\psi'_\beta(x_1,x_2) = \sum c_k c_l \varepsilon_{(x_1,x_2)} * \varepsilon_{(x_1,x_2)^-}(\psi(x_1,x_2)), \quad (x_1,x_2) \in (K_1 \times K_2)_{i+1}$$

where all $c_k \geq 0$ and $(x_1,x_2) \in (K_1 \times K_2)_{i+1}$, such that

$$\lim \psi'_\beta(x_1,x_2) = 1$$

for all $(x_1,x_2) \in (K_1 \times K_2)_{i+1}$.

It is clear that $\psi'_\beta \in P(K_1 \times K_2)$ and $\psi'_\beta \geq 0$. Moreover $\pi_{\psi'_\beta} = \pi_\psi$. Hence each $\pi_{\psi'_\beta}$ is weakly contained in $\{\pi_{\psi_\alpha} \mid \alpha \in A\}$ which is weakly contained in

π_φ. So, we get a net $(\psi_\alpha)_\alpha \in P(K_1 \times K_2)$ such that $\lim \psi_\alpha(x_1, x_2) = 1$ for all $(x_1, x_2) \in (K_1 \times K_2)_n = (K_1 \times K_2)$ and such that each π_{ψ_α} is weakly contained in π_φ. Hence $1_{K_1 \times K_2}$ is weakly contained in π_φ.

Now we reformulate property (P*), defined earlier, as follows: If π is a unitary representaion of $K_1 \times K_2$ with positive vectors, then $1_{K_1 \times K_2}$ is weakly contained in π, when π and $1_{K_1 \times K_2}$ is viewed as representations of the discrete product hypergroups $K_1 \times K_2$.

Theorem 2.4 *For a connected product hypergroups $K_1 \times K_2$,the following statements are equivalent:*

i) $K_1 \times K_2$ has property (P*).
ii) $K_1 \times K_2$ is amenable.

Proof: Suppose $K_1 \times K_2$ is amenable. Let N be the closure of the commutative subhypergroup of $K_1 \times K_2$, by [8] proposition 3, N has polynomial growth hence it is exponentially bounded [4].Let $\varphi \in P(K_1 \times K_2)$, $\varphi \geq 0$. By last theorem in its reformulated form there is a net $(\psi_\alpha)_\alpha$in $P(K_1 \times K_2)$ with $\psi_\alpha \geq 0$ such that $\lim \psi_\alpha(x_1, x_2) = 1$ for all $(x_1, x_2) \in N$ and such that π_{ψ_α} is weakly contained in π_φ for all α. Considering $K_1 \times K_2$ as a discrete product hypergroups we can apply the method of proof of the last theorem to get some $\psi \in P(K_1 \times K_2)$, $\psi \geq 0$ with $\psi \mid N = 1$ and such that π_ψ is weakly contained in π_φ. Since $K_1 \times K_2/N$ is abelian, $1_{K_1 \times K_2}$ is weakly contained in π_ψ and the result follows.

Now if $K_1 \times K_2$ has property (P*), then $1_{K_1 \times K_2}$ is weakly contained in the regular representation $\lambda_{K_1 \times K_2}$,when both representations are considered as representations of $K_1 \times K_2$. This is equivalent to the amenability of $K_1 \times K_2$ [4].

3 Conditionally Exponential Convex Functions on Product Dual Hypergroups

In this section we will give some properties of the class of conditionally exponential convex functions defined on product dual hypergroups.

Definition 3.1 *Let K^* be the dual of the hypergroup K the function $\psi : K^* \to \mathbb{C}$ is said to be conditionally exponential convex if for all $n \in \mathbb{N}$ and any $y_1, y_2, ..., y_n \in K^*$ and $c_1, c_2, ..., c_n \in \mathbb{C}$ we have:*

$$^{n}_{i,j=1}[\psi(y_i) + \overline{\psi(y_j)} - \psi(y_i + y_j)]c_i\overline{c_j} \geq 0$$

for all $n \in \mathbb{N}$, $c_1, c_2, ..., c_n \in \mathbb{C}$ and any $y_1, y_2, ..., y_n \in K^*$.

Theorem 3.2 *If $\psi : K_1^* \to \mathbb{C}, \psi : K_2^* \to \mathbb{C}$ are conditionally exponential convex functions respectively, then $\psi : K_1^* \times K_2^* \to \mathbb{C}$ defined by*

$$\psi(y_1, y_2) = \psi(y_1) + \psi(y_2)$$

is conditionally exponential convex function.

Proof: Let $\psi : K_1^* \to \mathbb{C}$, and $\psi : K_2^* \to \mathbb{C}$,
then

$$\begin{aligned} {}^{n}_{i,j=1}[\psi(y_1)_i + \overline{\psi(y_1)_j} - \psi((y_1)_i + (y_1)_j)]c_i\overline{c_j} &\geq 0 \\ {}^{n}_{i,j=1}[\psi(y_2)_i + \overline{\psi(y_2)_j} - \psi((y_2)_i + (y_2)_j)]c_i\overline{c_j} &\geq 0 \end{aligned}$$

then we have

$$\begin{aligned} \psi(y_1, y_2) &= {}^{n}_{i,j=1}[\psi(y_1, y_2)_i + \overline{\psi(y_1, y_2)_j} - \psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \\ &= {}^{n}_{i,j=1}[\psi(y_1)_i + \psi(y_2)_i + \overline{\psi(y_1)_j + \psi(y_2)_j} - \psi[(y_1)_i + (y_1)_j] - \psi[(y_2)_i + (y_2)_j]]c_i\overline{c_j} \\ &= {}^{n}_{i,j=1}[\psi(y_1)_i + \overline{\psi(y_1)_j} - \psi[(y_1)_i + (y_1)_j]c_i\overline{c_j} \\ &\quad +{}^{n}_{i,j=1}\psi(y_2)_i + \overline{\psi(y_2)_j} - \psi[(y_2)_i + (y_2)_j]c_i\overline{c_j} \\ &\geq 0 \\ &= \psi(y_1) + \psi(y_2). \end{aligned}$$

there for $\psi(y_1, y_2)$ is conditionally exponential convex function.

Theorem 3.3 *A continuous function $\psi : K_1^* \times K_2^* \to \mathbb{C}$ is conditionally exponential convex iff the following conditions are satisfied: (i) $\psi(0, 0) \geq 0$, (ii) $\Psi_t(y_1, y_2) = \exp[-t\psi(y_1, y_2)]$ is conditionally exponential covex for all t.*

Proof: Suppose that ψ is continuous conditionally exponential convex function, then (i) is easly satisfied. To establish (ii) we have:

$${}^{n}_{i,j=1}[\psi(y_1, y_2)_i + \overline{\psi(y_1, y_2)_j} - \psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \geq 0$$

which implies that

$${}^{n}_{i,j=1}\exp[\psi(y_1, y_2)_i + \overline{\psi(y_1, y_2)_j} - \psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \geq 0$$

So, we have for $t = 1$,

$$\begin{aligned} &{}^{n}_{i,j=1}\Psi_1((y_1, y_2)_i + (y_1, y_2)_j)c_i\overline{c_j} \\ = &\sum_{i,j=1}^{n} \exp[-\psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \\ = &\sum_{i,j=1}^{n} \exp[\psi(y_1, y_2)_i + \overline{\psi(y_1, y_2)_j} - \psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i'\overline{c_j'} \end{aligned}$$

where $c'_k = c_k \exp[-\psi(y_1, y_2)_k]$. Hence, $\Psi_1(y_1, y_2)$ is conditionally exponential convex.

Since $t\psi(t)$ is conditionally exponential convex, then its clear that $\Psi_t(y_1, y_2)$ is conditionally exponential convex all $t > 0$.

To prove the converse, let (i) and (ii) be satisfied. By (i) we have $\exp[-t\psi(0,0)] \leq 1$ for all $t > 0$. So $\Psi_t(y_1, y_2) = \frac{1}{t}[1 - \exp(-t\psi(y_1, y_2))]$ is conditionally exponential convex for all $t > 0$. Using Fattou's lemma we can easily get that $\psi_t(y_1, y_2) = \lim \Psi_t(y_1, y_2)$ is conditionally exponential convex.

Theorem 3.4 *Let $\psi : K_1^* \times K_2^* \to \mathbb{C}$ be a conditionally exponential convex and suppose that $\psi(0,0) \geq 0$ then $\frac{1}{\psi}$ is conditionally exponential convex.*

Proof: Since ψ is conditionally exponential convex function, then the function $\exp[-t\psi(y_1, y_2)]$ is coditionally exponential convex for all $t > 0$. The function $\frac{1}{\psi}$ can be written in the form:

$$\frac{1}{\psi(y_1, y_2)} = \int_0^\infty \exp[-t\psi(y_1, y_2)]dt$$

Hence,

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{1}{\psi((y_1, y_2)_i + (y_1, y_2)_j)} c_i\overline{c_j} \\
= &\sum_{i,j=1}^{n} c_i\overline{c_j} \int_0^\infty \exp[-t\psi((y_1, y_2)_i + (y_1, y_2)_j)]dt \\
= &\int_0^\infty \left\{ \sum_{i,j=1}^{n} \exp[-t\psi((y_1, y_2)_i + (y_1, y_2)_j)]c_i\overline{c_j} \right\} dt \geq 0.
\end{aligned}$$

Thus, $\frac{1}{\psi}$ is conditionally exponential convex.

References

[1] R.I. Jeweet, Space with an abstract convolution of measures, *Advances in Math.*, 18(1975), 1-101.

[2] J. Dixmier, *C^* Algebras*, North-Holland, (1977).

[3] C.F. Dunkl, The measure algebra of a locally compact hypergroup, *Trans. Amer. Math. Soc.*, 179(1973), 331-348.

[4] Y. Guivarc'II, Croissance polynomiale et periodes des fonctions harmoniques, *Bull. Soc. Math. France*, 101(1973), 333-379.

[5] A.S.O. El-Bab and F.M. Bayoumi, Spectrum of positive definite functions on hypergroups, *Kyungpook Math. J*, 35(1995), 17-23.

[6] J.P. Pier, *Amenable Locally Compact Groups*, Oxford University Press, (1984).

[7] R. Spector, Apercu de la theorie des hypergroups, *Lecture Notes in Math.*, Springer, 497(1975), 643-673.

[8] J. Boidol, Group algebras with a unique c*-norm, *J. Funct. Anal.*, 56(1984), 220-232.

[9] M.B. Bekka, On a question of R. Godment about the spectrum of positive definite functions, *Math. Proc. Camb. Phi. Soc.*, 110(1991), 137-142.

[10] A.S.O. El Bab and M.S. Shazly, Characterisation of convolution semi-groups, *Indian Journal of Theoretical Physics*, 4(1987), 300-311.