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On Graceful Labeling of Some Graphs with Pendant Edges

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Abstract

A graceful labeling of a graph G of size q is an injective assignment of labels from the set $\{0,1,...,q\}$ to the vertices of G such that when each edge of G has been assigned a label defined by the absolute difference of its end-vertices, the resulting edge labels are distinct. In this paper we have shown that the graphs which are obtained by adding the pendant edges to the vertices of $\overline{K}_n$ or P_2 or both in $P_2 + \overline{K}_n$ are graceful. We have also shown that the graph which obtains by adding one pendant edge to each pendant vertices of $C_n \odot 1K_1$, $n \equiv 3 \bmod (4)$ is graceful.

Keywords: *Graceful Labeling, Graceful Graphs, Join of Graphs, Corona of Graphs.*

1 Introduction

Graceful labeling problem is an assignment of integers (mostly non-negative integers) to the vertices or edges or both of a graph satisfying certain conditions. At present there are several types of graph labeling (see Gallian [9]) and graceful labeling is the oldest of them. The concept of graceful graph labeling was introduced by Rosa [15] in 1966. Let $G(V,E)$ be a simple undirected graph with order p and size q, if there exist an injective mapping $f:V(G)\rightarrow\{0,1,\ldots,q\}$ that induces a bijective mapping $f^*:E(G)\rightarrow\{1,2,...,q\}$ defined by $f^*(u,v)=|f(u)-f(v)|\ \forall\ (u,v)\in E(G)\ and\ u,v\in V(G)$, then f is called graceful labeling of graph G. A graph G is called graceful if it has a graceful labeling. The notation and terminology used in this paper are taken from Gallian [9].

2 Join of Two Graphs

The join of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ denoted by G_1+G_2, consists of $G_1\cup G_2$ and all edges joining V_1 with V_2 [10].

If G is connected graceful graph then $G+\bar{K}_n$ is graceful [1]. If G is connected graceful graph with order $q+1$ and size q vertices then the join of G and $\bar{K}_n$ and the join of G and any star graph are graceful [16]. Ramirez-Alfunsin [13] has proved that if G is graceful and $|V(G)|=|E(G)|=q$ and either 1 or q is not a vertex label then $G+\bar{K}_n$ is graceful for all n. Barrientos [5] proved that $G+nK_1$ is graceful for a graceful graph G of order q and size $q-1$. Redl [14] showed that the double cones $C_n+\bar{K}_2$ are graceful for $n=3, 4, 5, 7, 8, 9, 11$ and all double cones $C_n+\bar{K}_2$ are not graceful for $n\equiv 2(\text{mod}\,4)$. Balakrishnan and Sampathkumar [3] have shown that graph $mK_2+\bar{K}_n$ is graceful for all n and $m\geq 3$. Bhat-Nayak and Gokhale [6] have proved that $2K_2+\bar{K}_2$ is not graceful, while Amutha and Kathiresan [2] proved that the graph obtained by attaching a pendant edge to each vertex of $2K_2+\bar{K}_2$ is graceful. Obviously P_2 is graceful and therefore $P_2+\bar{K}_n$ is also graceful.

Motivated by above results, it has been shown in the following theorem that adding pendant edges to each vertices of $\bar{K}_n$ in the graph $P_2+\bar{K}_n$ is graceful.

Theorem 1: *The graph obtained by adding r - pendant edges to each vertex of $\bar{K}_n$ in the graph $P_2+\bar{K}_n$ admits graceful labeling.*

Proof: The order and size of the graph G obtained by adding r - pendant edges to each vertex of $\bar{K}_n$ in the graph $P_2+\bar{K}_n$ are respectively $n(r+1)+2$ and

$n(r+2)+1$. Let v_1 and v_2 be the vertices of P_2, $u_i\,(1\le i\le n)$ be the vertices of $\overline{K}_n$. Obviously u_{it} $(1\le t\le r)$ will be the pendant vertices corresponding to u_i.

Consider a labeling map $f:V(G)\to\{0,1,\ldots,n(r+2)+1\}$ defined as follows:

$$f(v_1)=0,$$
$$f(v_2)=n(r+2)+1,$$
$$f(u_i)=i\ ,\ 1\le i\le n,$$

And

$$f(u_{it})=n+(i-1)(r+1)+t+1,\quad 1\le i\le n \text{ and } 1\le t\le r.$$

Clearly f is injective.

Now we prove that the induced labeling map $f^*:E(G)\to\{1,2,\ldots,n(r+2)+1\}$ defined as $f^*(u,v)=|f(u)-f(v)|\ \forall\ (u,v)\in E(G)$ and $u,v\in V(G)$, where u and v are adjacent vertices of G, is bijective.

The edge labeling induced by f^* is as follows:

$$\begin{aligned} f^*(v_1u_i)&=\{|f(v_1)-f(u_i)|:i=1,2,\ldots,n\}\\ &=\{1,2,...,n\}\\ f^*(v_2u_i)&=\{|f(v_2)-f(u_i)|:i=1,2,\ldots,n\}\\ &=\{n(r+2),n(r+2)-1,\ldots,n(r+1)+1\}\\ f^*(u_iu_{it})&=\{|f(u_i)-f(u_{it})|:i=1,2,\ldots,n \textit{ and } t=1,2,\ldots,r\}\\ &=\{n+1,n+2,\ldots,n(r+1)\}\end{aligned}$$

And

$$\begin{aligned} f^*(v_1v_2)&=\{|f(v_1)-f(v_2)|\}\\ &=\{n(r+2)+1\}\end{aligned}$$

Hence

$$\begin{aligned} f^*(uv)&=\{|f(u)-f(v)|:uv\in E(G)\}\\ &=\{1,2,\ldots,n(r+2)+1\}.\end{aligned}$$

Therefore f^* is bijective. Thus f is graceful labeling of the graph G.

Example 1: A graph obtained by adding 3- pendant edges to each ve the graph $P_2 + \bar{K}_7$ and its graceful labeling are shown in figure 1(1(b) respectively.

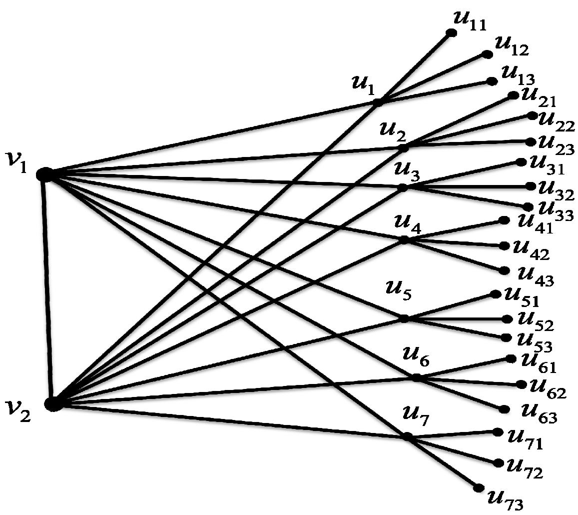

Figure 1(a)

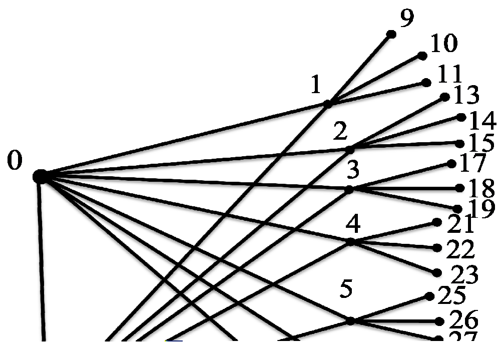

Figure 1(b)

In the following theorem we have given graceful labeling for the g by adding pendant edges to each vertices of P_2 in the graph $P_2 + \bar{K}_n$.

Theorem 2: *The graph obtained by adding* r *- pendant edges to eac in the graph* $P_2 + \bar{K}_n$ *admits graceful labeling.*

Proof: The order and size of the graph G obtained by adding r - per each vertex of P_2 in the graph $P_2 + \bar{K}_n$ are respectively $n + 2r + 2$ ar

. Let v_1 and v_2 be the vertices of P_2 , $u_i\,(1 \le i \le n)$ be the vertices of $\bar{K}_n$. Clearly v_{1t} , v_{2t} $(1 \le t \le r)$ are the pendant vertices adjacent to v_1, v_2 respectively.

Consider a labeling map $f : V(G) \to \{0,1,\ldots,2(n+r)+1\}$ defined as follows:

$$\begin{aligned} f(v_1) &= 0, \\ f(v_2) &= 2n+r+1, \\ f(v_{1t}) &= 2(n+r+1)-t \ , 1 \le t \le r, \\ f(v_{2t}) &= n+t \ , 1 \le t \le r, \end{aligned}$$

And

$$f(u_i) = i \ , 1 \le i \le n.$$

Obviously f is injective.

Now we show that the induced labeling map $f^* : E(G) \to \{1,2,\ldots,2(n+r)+1\}$ defined as $f^*(u,v) = |f(u) - f(v)|\ \forall (u,v) \in E(G)$ and $u, v \in V(G)$, where u and v are adjacent vertices of G , is bijective.

The edge labeling induced by f^* is as follows:

$$\begin{aligned} f^*(v_1u_i) &= \{|f(v_1) - f(u_i)| : i = 1,2,\ldots,n\} \\ &= \{1,2,\ldots,n\} \\ f^*(v_2u_i) &= \{|f(v_2) - f(u_i)| : i = 1,2,\ldots,n\} \\ &= \{2n+r, 2n+r-1,\ldots,n+r+1\} \\ f^*(v_1v_{1t}) &= \{|f(v_1) - f(v_{1t})| : t = 1,2,\ldots,r\} \\ &= \{2(n+r)+1, 2(n+r),\ldots,2n+r+2\} \\ f^*(v_2v_{2t}) &= \{|f(v_2) - f(v_{2t})| : t = 1,2,\ldots,r\} \\ &= \{n+r, n+r-1,\ldots,n+1\} \end{aligned}$$

And

$$\begin{aligned} f^*(v_1v_2) &= \{|f(v_1) - f(v_2)|\} \\ &= \{2n+r+1\} \end{aligned}$$

Hence

$$\begin{aligned} f^*(uv) &= \{|f(u) - f(v)| : uv \in E(G)\} \\ &= \{1,2,\ldots,2(n+r)+1\}. \end{aligned}$$

Therefore f^* is bijective. Thus f is graceful labeling of the graph G .

Example 2: The graph obtained by adding 5- pendant edges to each vertex of P_2 in the graph $P_2 + \bar{K}_8$ and its graceful labeling have shown in figure 2(a) and figure 2(b) respectively.

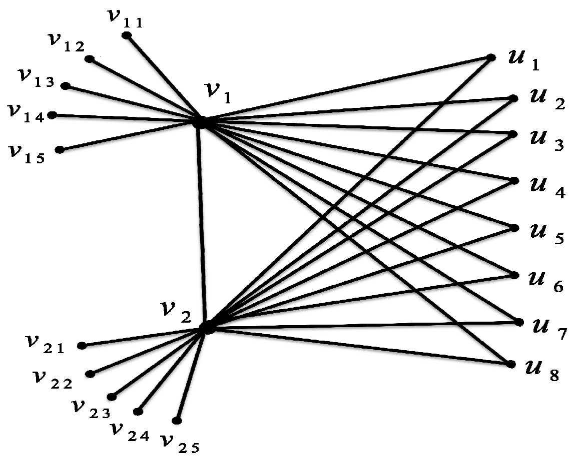

Figure 2(a)

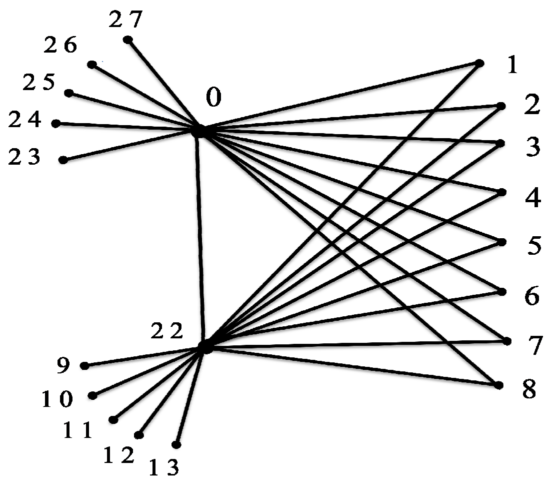

Figure 2(b)

3 Corona of Two Graphs

A new operation, named corona of two graphs, was presented by Frucht and Harary [7] in 1970. Let G be a graph of order p and H be another graph of finite order. The corona of the graph G and H, denoted by $G \odot H$ is a graph that is obtained by taking one copy of G and p copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H by an edge.

i.e. $V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i)$

And

$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} E(H_i) \cup \{(i, u_i) : i \in V(G) \text{ and } u_i \in V(H_i)\}.$$

Frucht [8] proved that the corona $C_m \odot 1K_1$ is graceful. Barrientos [4] proved that if G is graceful graph of order m and size $m-1$ then corona $G \odot H$ is graceful. He also proved that all hairy cycle $C_n \odot rK_1$ are graceful. Pradhan and Kumar [12] have proved that a graph obtained by adding pendant edge to each pendant vertex of hairy cycle $C_n \odot 1K_1$ is graceful if $n \equiv 0 (mod\, 4m)$ and $m \in N$ (set of natural numbers).

In the following theorem we have shown that the graph obtained by adding r – pendant edges to each vertices of graph $P_2 + \bar{K}_n$ denoted by $(P_2 + \bar{K}_n) \odot rK_1$, is graceful.

Theorem 3: *The graph* $(P_2 + \bar{K}_n) \odot rK_1$ *admits graceful labeling where* $n \geq 2(r-1)$ *when* $r > 1$ *and* $n > 0$ *when* $r = 1$.

Proof: The order and size of the graph $(P_2 + \bar{K}_n) \odot rK_1$ obtained by adding r – pendant edges to each vertex of $P_2 + \bar{K}_n$ are respectively $(r+1)(n+2)$ and $(r+2)(n+1) + r - 1$. Let v_1 and v_2 be the vertices of P_2 , $u_i (1 \leq i \leq n)$ be the vertices of $\bar{K}_n$. Clearly v_{1t} , v_{2t} $(1 \leq t \leq r)$ are the pendant vertices adjacent to v_1, v_2 respectively and u_{it} $(1 \leq i \leq n \text{ and } 1 \leq t \leq r)$ are the pendant vertices adjacent to $u_i (1 \leq i \leq n)$ respectively.

Consider the labeling map $f : V\left((P_2 + \bar{K}_n) \odot rK_1\right) \to \{0, 1, \ldots, (r+2)(n+1) + r - 1\}$ defined as follows:

$$f(v_1) = 0,$$
$$f(v_2) = (r+2)(n+1) - 1,$$
$$f(v_{1t}) = (r+2)(n+1) + r - t, \; 1 \leq t \leq r,$$
$$f(v_{2t}) = n + (r+1)t - r, \; 1 \leq t \leq r,$$
$$f(u_i) = i, \; 1 \leq i \leq n,$$

And

$$f(u_{it}) = \begin{cases} n + (r+1)i + t - r, & 1 \leq i \leq n+1-r \text{ and } 1 \leq t \leq r, \\ n + k + (r+1)i + t - r, & n+2-r \leq i \leq n, \; 1 \leq t \leq r \text{ and } k = r + i - (n+1) \end{cases}$$

Clearly f is injective.

Now we show that the induced labeling map

$$f^*: E\left((P_2+\bar{K}_n)\odot rK_1\right)\to\left\{1,2,\ldots,(r+2)(n+1)+r-1\right\}$$

defined as $f^*(u,v)=\left|f(u)-f(v)\right| \ \forall (u,v)\in E\left((P_2+\bar{K}_n)\odot rK_1\right)$
and $u,v\in V\left((P_2+\bar{K}_n)\odot rK_1\right)$,
where u and v are adjacent vertices of $(P_2+\bar{K}_n)\odot rK_1$, is bijective.

The edge label induced by f^* is as follows:

$$\begin{aligned}
f^*(v_1u_i) &= \{\left|f(v_1)-f(u_i)\right| : i=1,2,\ldots,n\}\\
&= \{1,2,\ldots,n\}\\
f^*(v_2u_i) &= \{\left|f(v_2)-f(u_i)\right| : i=1,2,\ldots,n\}\\
&= \{(r+2)(n+1)-2,(r+2)(n+1)-3,\ldots,(r+2)(n+1)\}\\
f^*(v_1v_{1t}) &= \{\left|f(v_1)-f(v_{1t})\right| : t=1,2,\ldots,r\}\\
&= \{(r+2)(n+1)+r-1,(r+2)(n+1)+r-2,\ldots,(r+2)(n+1)\}\\
f^*(v_2v_{2t}) &= \{\left|f(v_2)-f(v_{2t})\right| : t=1,2,\ldots,r\}\\
&= \{(r+1)(n+1)-1,(r+1)n-1,\ldots,(r+1)(n-r+2)-1\}\\
f^*(v_1v_2) &= \{\left|f(v_1)-f(v_2)\right|\}\\
&= \{(r+2)(n+1)-1\}\\
f^*(u_iu_{it}) &= \{\left|f(u_i)-f(u_{it})\right| : i=1,2,\ldots,n+1-r \text{ and } t=1,2,\ldots,r\}\\
&= \{n+1,n+2,\ldots,n+(n+1-r)r\}\\
f^*(u_iu_{it}) &= \{\left|f(u_i)-f(u_{it})\right| : i=n+2-r\ ,n+3-r\ ,\ldots,n \text{ and } t=1,2,\ldots,r\}\\
&= \{(n+2-r)(r+1),(n+2-r)(r+1)+1,\ldots,(n+2-r)(r+1)+r-1,\\
&\quad (n+3-r)(r+1),(n+3-r)(r+1)+1,\ldots,(n+3-r)(r+1)+r-1,\\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\quad n(r+1),n(r+1)+1,\ldots,n(r+1)+r-1\}.
\end{aligned}$$

Hence

$$\begin{aligned}
f^*(uv) &= \left\{\left|f(u)-f(v)\right| : uv\in E\left((P_2+\bar{K}_n)\odot rK_1\right)\right\}\\
&= \left\{1,2,\ldots,(r+2)(n+1)+r-1\right\}.
\end{aligned}$$

So f^* is bijective. Thus f is graceful labeling of the graph $(P_2 + \overline{K}_n) \odot rK_1$.

Example 3: The graph $(P_2 + \overline{K}_6) \odot 3K_1$ and its graceful labeling are shown in figure 3(a) and figure 3(b) respectively.

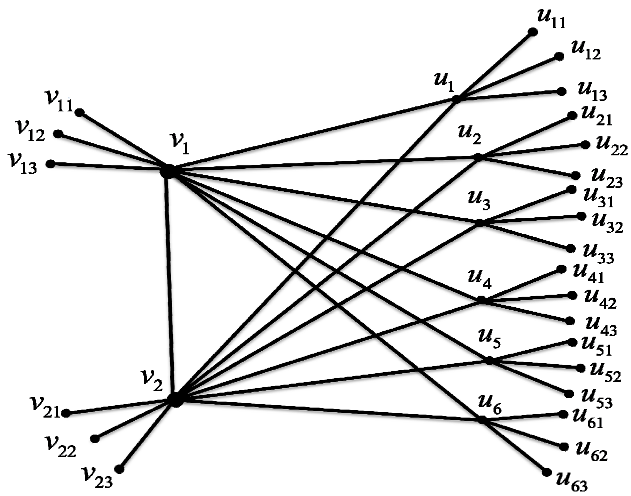

Figure 3(a)

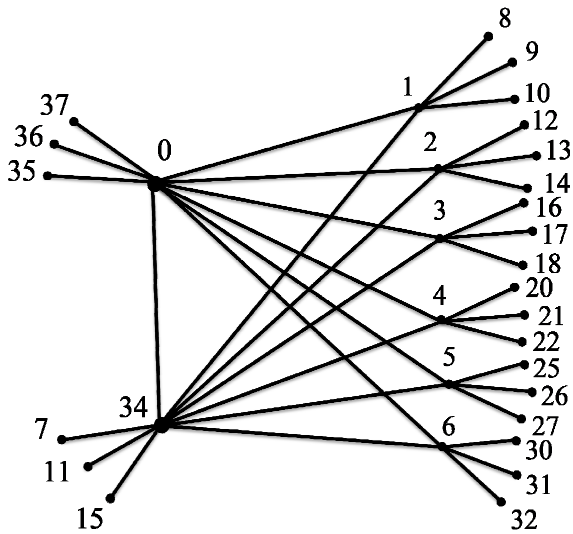

Figure 3(b)

Pradhan, P. and Kumar, A. [12] have proved that a graph obtained by adding pendant edge to each pendant vertex of $C_n \odot 1K_1$ is graceful if $n \equiv 0(mod\, 4m)$ and $m \in N$ (set of natural numbers).

In the following theorem above result has been extended for $n \equiv 3(mod\, 4)$.

Theorem 4: *The graph obtained by adding pendant edge to each pendant vertex of hairy cycle* $C_n \odot 1K_1$, $n \equiv 3(mod\, 4)$ *admits graceful labeling.*

Proof: The order and size of the graph G obtained by adding pendant edge to each pendant vertex of hairy cycle $C_n \odot 1K_1$, $n \equiv 3(mod\, 4)$ are respectively $3n$ and $3n$. Let $u_1, u_2, \ldots, u_n$ be the cycle vertices of $C_n \odot 1K_1$, $v_1, v_2, \ldots, v_n$ be the vertices adjacent to $u_1, u_2, \ldots, u_n$ and $w_1, w_2, \ldots, w_n$ be the vertices adjacent to $v_1, v_2, \ldots, v_n$. Obviously

$$d(u_i) = 3, \ i = 1, 2, \ldots, n$$
$$d(v_i) = 2, \ i = 1, 2, \ldots, n$$
$$d(w_i) = 1, \ i = 1, 2, \ldots, n.$$

Consider a labeling map $f : V(G) \to \{0, 1, \ldots, 3n\}$ defined as follows:

$$f(v_i) = \begin{cases} \dfrac{3(i-1)}{2}, & i \, is \, odd, \\ 3n + 1 - \dfrac{3i}{2}, & i \, is \, even \, and \, i \le \dfrac{n+1}{2}, \\ 3n - \dfrac{3i}{2}, & i \, is \, even \, and \, i > \dfrac{n+1}{2}. \end{cases}$$

$$f(u_i) = \begin{cases} f(v_{i-1}) + 2, & i \, is \, even, \\ f(v_{i+1}) + 2, & i \, is \, odd \, and \, i \le n-1, \\ f(v_i) + 2, & i \, is \, odd \, and \, i = n. \end{cases}$$

And

$$f(w_i) = f(v_i) + 1.$$

It is clear that f is injective and the induced labeling map

$$f^* : E(G) \to \{1, 2, \ldots, 3n\}$$

Defined as $f^*(u,v) = |f(u) - f(v)| \ \forall \ (u,v) \in E(G) \, and \, u, v \in V(G)$, where u and v are adjacent vertices of G, is bijective. Thus f is graceful labeling of the graph G.

Example 4: The graph obtained by adding pendant edge to each pendant vertex of $C_{15} \odot 1K_1$ and its graceful labeling are shown in figure 4(a) and figure 4(b) respectively.

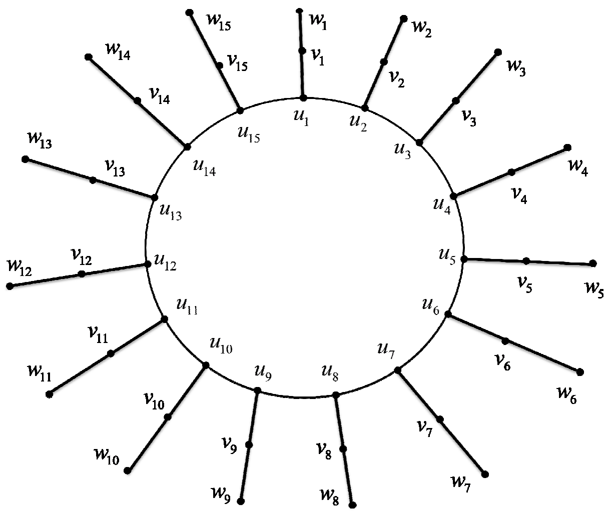

Figure 4(a)

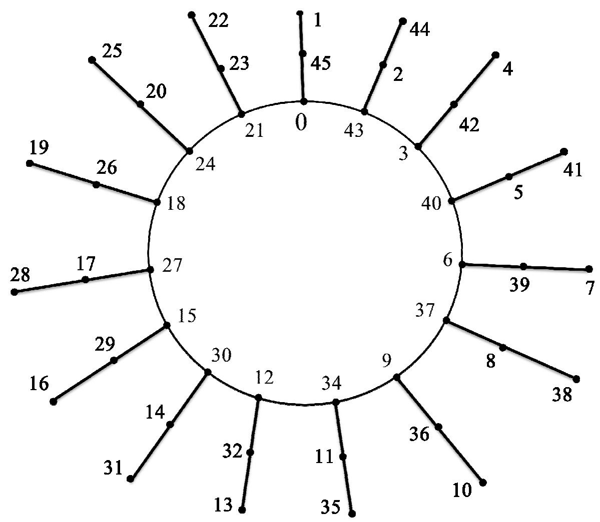

Figure 4(b)

Acknowledgement

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