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Soft Graphs

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Abstract

Soft set theory, introduced by Molodtsov has been considered as an effective mathematical tool for modeling uncertainties. In this paper we introduce the notion of soft graph and investigate some of their properties.

Keywords: Soft sets, Soft graph, Soft graph homomorphism, Soft tree, Soft complete graph.

1 Introduction

Soft set theory [7] was introduced by Molodtsov in 1999 as a general mathematical tool for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which make the theory easily applied to many different fields. The operations of soft sets are defined by Maji et al. [6]. At present, work on soft set theory is progressing rapidly. The algebraic structure of soft set Theory has also being studied in more detail [1], [3], [5], [8]. In this paper we introduce the notion of soft graphs, we also define soft graph homomorphism, soft tree, soft complete graph and investigate some of their properties.

2 **Preliminaries**

For notation definition and facts from Graph theory we refer to [2], [4], [9] and that of Soft set theory to [6], [7].

2.1 Graphs

A graph G = (V, E) consists of a non-empty set of objects V, called vertices and a set E of two element subset of V called edges. Two vertices x and y are *adjacent* if $\{x, y\} \in E$. A graph G = (V', E') is said to be a subgraph of G = (V, E) if $V' \subseteq V$ and $E' \subseteq E$. For any subset S of the vertex set of the graph G, the *induced* subgraph \mathscr{G} is the subgraph of G whose vertex set is S and two vertices are adjacent in S if and only if they are adjacent in G. A graph is *complete* if every vertex is connected to every other vertex and we denote the complete graph on nvertices by K_n . A *tree* is a connected acyclic graph; where by a cycle we mean a closed path. The *distance* between two vertices u and v of a connected graph Gdenoted by d(u, v) is the length of the shortest u-v path. The *diameter* of the graph G is the maximum of vertex eccentricities, where the *eccentricity* of a vertex u, $e(u) = \max\{d(u, v) | v \in V\}$. A homomorphism from a graph G to a graph H is defined as a mapping $h: G \to H$ such that $(x, y) \in E(G) \Longrightarrow (h(x), h(y)) \in E(H)$. A hypergraph H is a pair H = (V, E) where V is a set of elements called vertices and E is a set of non-empty subsets of V called edges or hyperedges, E is a subset of $P(V) - \{\emptyset\}$ where P(V) the power set of V.

2.2 Soft Sets

Definition 2.2.1 Let U be a nonempty finite set of objects called Universe and let E be a nonempty set called parameters. An ordered pair (F, E) is said to be a Soft set over U, where F is a mapping from E into the set of all subsets of the set U. That is $F: E \to P(U)$. The set of all Soft sets over U is denoted by S(U).

Definition 2.2.2 Let (F, A) and (G, B) be two soft sets over the common universe U. We say that (F, A) is a soft subset of (G, B) if $(1) A \subseteq B$, (2) For all $e \in A$, $F(e) \subseteq G(e)$.

Definition 2.2.3 Let (F, A) and (G, B) be two soft sets over the common universe U. The union of two soft sets (F, A) and (G, B) is the soft set (H, C) where $C = A \cup B$ and H is defined as follows:

$$H(e) = \begin{cases} F(e), & \text{for } e \in A - B \\ G(e), & \text{for } e \in B - A \\ F(e) \cup G(e), & \text{for } e \in A \cap B \end{cases}$$

Definition 2.2.4 Let (F, A) and (G, B) be two soft sets over the common universe U such that $A \cap B = \emptyset$. The intersection (F, A) and (G, B) is the soft set (H, C) where $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for all $e \in C$

Definition 2.2.5 Let (F, A) and (G, B) be two soft sets over the common universe U. Then (F, A) and (G, B) denoted by $(F, A) \land (G, B)$ is defined by $(F, A) \land (G, B) = (H, A \times B)$ where $H((\alpha, \beta)) = F(\alpha) \cap G(\beta)$ for all $(\alpha, \beta) \in A \times B$

3 Soft Graphs

Let G = (V, E) be a simple graph, A any nonempty set. Let R an arbitrary relation between elements of A and elements of V. That is $R \subseteq A \times V$. A set valued function $F : A \to P(V)$ can be defined as $F(x) = \{y \in V | xRy\}$. The pair (F, A) is a soft set over V.

Definition 3.1 Let (F, A) be a soft set over V. Then (F, A) is said to be a Soft graph of G if the subgraph induced by F(x) in G, f'(x) is a connected subgraph of G for all $x \in A$.

The set of all soft graph of G is denoted by SG(G).

Example 3.2 Consider the graph G = (V, E) as shown in Fig.1. Let $A = \{v_1, v_3, v_5\}$.

Define the set valued function F by,

 $F(x) = \{ y \in V | xRy \Leftrightarrow x \text{ is adjacent to } y \text{ in } G \}.$

Then $F(v_1) = \{v_2, v_5\}, F(v_3) = \{v_2, v_4\}, F(v_5) = \{v_1, v_2, v_4\}.$

Here subgraph induced by F(x) in G, P(x) is a connected subgraph of G, for all $x \in A$.

Hence $(F, A) \in SG(G)$.



Fig. 1

Example 3.3 Consider the graph G = (V, E) as shown in Fig.2. Let $A = \{v_1, v_5\}$. Define the set valued function F by, $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \le 2\}$. Then $F(v_1) = \{v_1, v_2, v_3\}, F(v_5) = \{v_3, v_4, v_5\}$.

Here f'(x) is a connected subgraph of G, for all $x \in A$. Hence $(F, A) \in SG(G)$.



Example 3.4 Consider the graph G = (V, E) as shown in Fig.3. Let $A = \{v_1\}$. Define the set valued function F by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) = 1\}$. Then $F(v_1) = \{v_2, v_4\}$. Here $f'(v_1)$ is not a connected subgraph of G. Hence (F, A) is not a soft graph.



Fig. 3

Remark 3.5 If (F, A) is a soft graph of the simple graph G = (V, E) then H = (V, E') where $E' = \{F(x) | x \in A\}$ is a hypergraph called induced hypergraph of G.

Proposition 3.6 *Every simple graph can be considered as a soft graph.*

Proof: Let *A* be the set of all edges of the graph *G*. Define a set valued function $F: A \to P(V)$ by, $F(e) = \{y \in V | eRy \Leftrightarrow y \text{ is an end vertex of the edge e} \}$.

Then (F, A) is a soft graph, as f'(e) is a connected subgraph of G for all $e \in A$. Intersection of two soft graphs need not be a soft graph.

Example 3.7 Consider the graph G = (V, E) as shown in Fig.3. Let $A = \{v_1\}$. Define the set valued function F by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \le 1\}$. Then $F(v_1) = \{v_1, v_2, v_4\}$. Define the set valued function H by $H(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \ge 1\}$. Then $H(v_1) = \{v_2, v_3, v_4\}$. Now $F(v_1) \cap H(v_1) = \{v_2, v_4\}$ which does not induce a connected subgraph of G.

Proposition 3.8 Let $(F, A), (H, B) \in SG(G)$ be such that $A \cap B \neq \emptyset$ and $F(e) \cap H(e)$ is a connected subgraph of G, for all $e \in A \cap B$. Then their intersection $(F, A) \cap (H, B) \in SG(G)$.

Proof: The intersection of two soft set (F, A) and (H, B) is given by $(F, A) \cap (H, B) = (U, C)$ where $C = A \cap B \neq \emptyset$ and $U(e) = F(e) \cap H(e) \neq \emptyset$ for all $e \in C$. Let $x \in C = A \cap B$. Since by assumption $\mathcal{O}(x) = \mathcal{F}(x) \cap \mathcal{H}(x)$ is a connected sub graph of G, $(U, C) = (F, A) \cap (H, B) \in SG(G)$.

Proposition 3.9 Let $(F, A), (H, B) \in SG(G)$. If $A \cap B = \emptyset$ then their union $(F, A) \cup (H, B) \in SG(G)$.

Proof: The union of two soft set (F, A) and (H, B) is given by $(F, A) \cup (H, B) = (U, C)$ where $C = A \cup B$ and

$$U(e) = \begin{cases} F(e), & \text{for } e \in A - B \\ H(e), & \text{for } e \in B - A \\ F(e) \cup H(e), & \text{for } e \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, $A \cdot B = A$, $B \cdot A = A$, either $e \in A$ or $e \in B$ for all $e \in A \cup B$.

Hence $U(e) = \begin{cases} F(e), & \text{for } e \in A \\ H(e), & \text{for } e \in B \end{cases}$

Since $(F, A) \in SG(G)$, f'(e) is a connected subgraph of *G* for all $e \in A$. Since $(H, B) \in SG(G)$, f'(e) is a connected subgraph of *G* for all $e \in B$. Thus U'(e) is a connected subgraph of *G* for all $e \in C$ and hence $(U, C) \in SG(G)$. That is $(F, A) \cup (H, B) \in SG(G)$. If $(F,A), (F,B) \in SG(G)$ be such that $F(x) \cap F(y) \neq \emptyset$ for all $x \in A, y \in B$. Then $(F,A) \land (F,B) \notin SG(G)$

Example 3.10 Consider the graph G = (V, E) as shown in Fig.4.

Let $A = \{v_1\}$ and $B = \{v_5\}$. Define the set valued function F by, $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \le 1\}.$

Then $F(v_1) = \{v_1, v_2, v_5\}, F(v_5) = \{v_2, v_3, v_4, v_5, v_6\}.$ Here $F(v_1) \wedge F(v_5) = \{v_2, v_6\} \neq \emptyset$ but does not induce a connected subgraph of G.

Hence $(F, A) \land (F, B) \notin SG(G)$.





Proposition 3.11 Let $(F, A), (H, B) \in SG(G)$ be such that $f'(x) \cap H'(y)$ is a connected subgraph of G for all $x \in A, y \in B$. Then $(F, A) \land (H, B) \in SG(G)$.

Proof:

We have $(F, A) \land (H, B) = (U, C)$ where $C = A \times B$ and $U((x, y)) = F(x) \cap H(y) \neq \emptyset$ for all $(x, y) \in A \times B$.

By assumption we have $f'(x) \cap H'(y)$ is a connected sub graph of G for all

 $x \in A, y \in B$. Hence $(U, C) = (F, A) \land (H, B) \in SG(G)$.

Definition 3.12 Let (F, A) and (H, K) be a soft graph of G. Then (H, K) is a soft subgraph of (F, A) if

(1) $K \subseteq A$ (2) H'(x) is a connected subgraph of F'(x) for all $x \in K$ To illustrate the definition we consider the following example

Example 3.13 Consider the graph G = (V, E) as shown in Fig.4. Let $A = \{v_2, v_3, v_4, v_5\}$ and $K = \{v_3, v_4\}$.

Define a set valued function $F : A \to P(V)$ by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \le 1\}$ Then $F(v_2) = \{v_1, v_2, v_3, v_5\}, F(v_3) = \{v_2, v_3, v_4, v_5\}, F(v_4) = \{v_3, v_4, v_5\}, F(v_5) = \{v_2, v_3, v_4, v_5, v_6\}$

Define a set valued function $H: K \to P(V)$ by $H(x) = \{y \in V | xRy \Leftrightarrow d(x, y) = 1\}$ Then, $H(v_3) = \{v_2, v_4, v_5\}, H(v_4) = \{v_3, v_5\}.$ Therefore $(F, A), (H, K) \in SG(G)$. Here $K \subseteq A$ and H(x) is a subgraph of F(x) for all $x \in K$.

Therefore (H, K) is a soft subgraph of (F, A).

Proposition 3.14 Let $(F, A), (H, A) \in SG(G)$. Then (F, A) is a soft subgraph of (H, A) if and only if $F(x) \subseteq H(x)$ for all $x \in A$.

Proof: Suppose (F, A) is a soft subgraph of (H, A), then clearly $F(x) \subseteq H(x)$ for all $x \in A$.

Conversely let $F(x) \subseteq H(x)$ for all $x \in A$. Since $(F, A) \in SG(G)$, F'(x) is a connected subgraph of G for all $x \in A$. Since $(H, A) \in SG(G)$, H'(x) is a connected subgraph of G for all $x \in A$. Hence F'(x) is a connected subgraph of H'(x) for all $x \in A$. Also $A \subseteq A$.

Thus (F, A) is a soft subgraph of (H, A)

Corollary 3.15 Every soft graph is a soft graph of itself. That is if (F, A) is a soft graph of G then (F, A) is a soft subgraph of itself.

Definition 3.16 Let (F, A) be a soft graph of G_1 and let f be a graph homomorphism from G_1 to G_2 . Then (f(F))(x) = f(F(x)) for all $x \in A$.

Proposition 3.17 Let (F, A) and (H, B) be soft graphs of G_1 such that (F, A) is a soft subgraph of (H, B). If f is a graph homomorphism from G_1 to G_2 , then (f(F), A) is a soft subgraph of (f(H), B) **Proof:** Given (F, A) is a soft subgraph of (H, B), then $A \subseteq B$ and F'(x) is a subgraph of H'(x) for all $x \in A$.

Since f is a homomorphism from G_1 to G_2 and homomorphic image of a connected subgraph in G_1 is a connected subgraph in G_2 we have f(F'(x)) and f(F'(y)) are connected subgraphs of G_2 for all $x \in A$ and $y \in B$.

Also f(F'(x)) is a subgraph of f(H'(x)) for all $x \in A$.

Hence (f(F), A) is a soft subgraph of (f(H), B).

Definition 3.18 Let (F, A) and (H, B) be soft graphs of the graphs G_1 and G_2 respectively. Let $f: G_1 \to G_2$ and $g: A \to B$. Then (f, g) is said to be a soft graph homomorphism if,

- (1) f is a graph homomorphism from G_1 to G_2
- (2) g is a mapping from A to B
- (3) f(F(x)) = H(g(x)) for all $x \in A$

Then (F, A) is said to be a soft graph homomorphic to (H, B).

Definition 3.19 If f is a graph isomorphism from G_1 to G_2 and g is a bijection from A to B. Then (f,g) is said to be a soft graph isomorphism and (F,A) the soft graph isomorphic to (H,B).

Example 3.20 Consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as shown in Fig.5. Let $A = \{v_1, v_2, v_3, v_4\}$ and $B = \{u, v\}$ Define a set valued function $F : V_1 \rightarrow P(V_1)$ by $F(x) = \{y \in V_1 | xRy \Leftrightarrow d(x, y) \le 1\}$ Then $(F, A) \in SG(G_1)$ Define set valued function $F : V_2 \rightarrow P(V_2)$ by $H(x) = \{y \in V_2 | xRy \Leftrightarrow d(x, y) \le 1\}$ Then $(H, B) \in SG(G_2)$ Define $f : G_1 \rightarrow G_2$ by $f(v_1) = u$, $f(v_2) = v$, $f(v_3) = u$, $f(v_4) = v$. Then f is a graph homomorphism from G_1 and G_2 . Define $g : A \rightarrow B$ by $g(v_1) = u$, $g(v_2) = v$, $g(v_3) = u$, $g(v_4) = v$. Then g is a mapping from A to B. Also f(F(x)) = H(g(x)) for all $x \in A$ Hence (F, A) is a soft graph homomorphic to (H, B).



Fig. 5

4 Soft Trees

Definition 4.1 Let G = (V, E) a graph and (F, A) be a soft graph of G. Then (F, A) is said to be a soft tree if F(x) is a tree for all $x \in A$.

Example 4.2 Consider the graphs G = (V, E) as shown in Fig.3. Let $A = \{v_2, v_4\}$ Define a set valued function $F : A \to P(V)$ by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \le 1\}$ Then $F(v_2) = \{v_1, v_2, v_3\}, F(v_4) = \{v_1, v_3, v_4\}$

Here f'(x) is a tree for all $x \in A$. Therefore (F, A) is a soft tree.

Proposition 4.3 Let the graphs G = (V, E) be a tree. Then every $(F, A) \in SG(G)$ is a soft tree.

Proof: Let $(F, A) \in SG(G)$. Then F'(x) is a connected subgraph of G for all $x \in A$.

Since G is a tree every connected subgraph of G is a tree. Therefore (F, A) is a soft tree.

Remark 4.4 The converse of above theorem is not true. That is if (F, A) is a soft tree of G then G need not be a tree.

For from example 4.2 (F, A) is a soft tree but G is not a tree as it contains a cycle.

Theorem 4.5 A soft graph (F, A) of G is a soft tree if and only if F'(x) is acyclic for all $x \in A$.

Theorem 4.6 Let (F, A) be a soft tree of the graph G_1 then the following holds

(1) If (H,B) is a soft subgraph of (F,A) then (H,B) is a soft tree of G_1 .

(2) If (F,A) is a soft tree of G_1 and if (H,B) the soft graph homomorphic image of (F,A) in the acyclic graph G_2 . Then (H,B) is a soft tree of G_2 .

Proof

- (1) Given (H, B) is a soft subgraph of (F, A). Then B ⊆ A and H(x) is a connected subgraph of F(x) for all x∈ B. Since (F, A) is a soft tree, F(x) is a tree for all x∈ A. Now since H(x) is a connected subgraph of F(x), H(x) is a tree for all x∈ B. Therefore (H, B) is a soft tree of G₁
- Given (H, B) the soft graph homomorphic image of (F, A). Then there is a soft graph homomorphism (f, g) where f:G₁→G₂ a graph homomorphism and g: A→B such that f(F(x)) = H(g(x)) for all x∈ A. Since (F, A) is a soft tree, P(x) is a tree for all x∈ A. Since f is a homomorphism and G₂ is acyclic, the induced subgraph of f(F(x)) is also a tree for all x∈ A. That is H(g(x)) induces a tree for all x∈ A. Since g is onto, for all y∈ B there exist x∈ A such that g(x) = y. Hence H(y) is a tree for all y∈ B. Therefore (H, B) is a soft tree of G₂

5 Soft Complete Graphs

Definition 5.1 Let G = (V, E) a graph and (F, A) be a soft graph of G. Then (F, A) is said to be a soft complete graph if F'(x) is a complete graph for all $x \in A$.

Example 5.2 Consider the graphs G = (V, E) as shown in Fig.4. Let $A = \{v_2, v_4\}$ Define the set valued function $F : A \to P(V)$ by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) = 1\}$ Then $F(v_1) = \{v_2, v_3, v_5\}, F(v_3) = \{v_1, v_2, v_4\}$

Here F'(x) is a complete graph for all $x \in A$. Therefore (F, A) is a soft complete graph.

Proposition 5.3 Let the graphs G = (V, E) be a complete graph. Then every $(F, A) \in SG(G)$ is a soft complete graph of G.

Proof: Let $(F, A) \in SG(G)$. Then F'(x) is a complete subgraph of G for all $x \in A$ as every induced subgraph of a complete graph is complete.

Therefore (F, A) is a soft complete graph.

Remark 5.4 The converse of above theorem is not true. That is if (F, A) is a soft complete graph of G then G need not be a complete graph.

To illustrate above remark we consider the following example

Example 5.5 Consider the graphs G = (V, E) as shown in Fig.4. Let $A = \{v_1, v_4, v_5\}$ Define the set valued function F by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) = 1\}$ Then $F(v_1) = \{v_2\}, F(v_4) = \{v_3, v_5, v_6\}, F(v_5) = \{v_3, v_4, v_6\}$ Here F(x) is a complete graph for all $x \in A$.

Therefore (F, A) is a soft complete graph. But G is not a complete graph.

Theorem 5.6 A soft graph (F, A) of G is a soft complete graph if and only if $\widetilde{F}(x)$ is complete for all $x \in A$.

Theorem 5.7 Let (F, A) be a soft complete graph of the graph G_1 then the following holds

- (1) If (H,B) is a soft subgraph of (F,A) then (H,B) is a soft complete graph of G_1 .
- (2) If (F,A) is a soft complete graph of G_1 and if (H,B) the soft graph homomorphic image of (F,A) in G_2 . Then (H,B) is a soft complete graph in G_2 .

Proof:

Given (H, B) is a soft subgraph of (F, A).
Then B⊆A and H(x) is a connected subgraph of F(x) for all x∈ B.
Since (F, A) is a soft complete graph, F(x) is a complete graph for all x∈ A.

Since H'(x) is a subgraph of F'(x), we have $H(x) \subseteq F(x)$. Now as every induced subgraph of a complete graph is complete, H'(x) is a complete graph for all $x \in B$. Therefore (H, B) is a soft complete graph of G_1 .

Given (H, B) the soft graph homomorphic image of (F, A). Then there is a soft graph homomorphism (f, g) where f:G₁→G₂ a graph homomorphism and g: A→B such that f(F(x)) = H(g(x)) for all x∈ A. Since (F, A) is a soft complete graph, F'(x) is a complete graph for all x∈ A. Since f is a homomorphism, the induced subgraph of f(F(x)) is also a complete graph for all x∈ A.

That is H(g(x)) induce a complete graph for all $x \in A$. Since g is onto, for all $y \in B$ there exist $x \in A$ such that g(x) = y. Hence H(y) is a complete graph for all $y \in B$. Therefore (H, B) is a soft complete graph of G_2 .

6 Conclusion

In this paper, we introduced soft graphs, soft subgraph, soft graph homomorphism, soft tree and soft complete graph and studied some of their properties.

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