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Some Geometric Properties of a Certain Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract

In this paper , we have studied a certain subclass of univalent functions defined by linear operator $\theta_{\lambda,\delta,m}^{\gamma,\alpha_1}$ by using differential subordination property.We obtain some geometric properties, like , coefficient inequality , neighborhoods of the class $K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$, convolution properties and integral mean inequalities for the fractional integral for this class.

Keywords: *Univalent Function, Hypergeometric Function, Linear Operator, Differential Subordination, Convolution, Integral Mean, Fractional Integral.*

1 Introduction

Let S denote the class of functions of the form:-

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \tag{1}$$

which are analytic in the open unit disk U={z∈ ¢: $|z| < 1$}. Also, denoted by K the subclass of S consisting of functions of the form:-

$$f(z) = z - \sum_{n=2}^{\infty} a_n |z|^n \ , \tag{2}$$

which are univalent and normalized in U.

For $f \in$ S, and of the form (1)and $g(z) \in$ S given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,

We define the Hadamard product (or convolution)

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{3}$$

For positive real values of $\alpha_1, \dots, \alpha_\delta$ and $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, \dots, j = 1,2, \dots . m$),

The generalized hypergeometric function $_\delta F_m(z)$ is defined by

$$_\delta F_m(z) \equiv \ _\delta F_m\,(\alpha_1, \dots, \alpha_\delta; \beta_1, \dots, \beta_m; z)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_\delta)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \tag{4}$$

$(\delta \leq m + 1; \delta, m \in N_0 = N \cup \{0\}; z \in U)$,

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2) \dots (a+n-1), & n \in N. \end{cases} \tag{5}$$

The notation $_\delta F_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial.

$$\mathrm{H}[\alpha_1, \alpha_2, \alpha_3 \ldots., \alpha_\delta; \beta_1, \beta_2, \beta_3 \ldots., \beta_m]: \mathrm{K} \to \mathrm{K}$$

be a linear operator defined by

$$\mathrm{H}[\alpha_1, \ldots, \alpha_\delta; \beta_1, \ldots, \beta_m]f(z) = z\, {}_\delta F_m(\alpha_1, \alpha_2, \ldots, \alpha_\delta; \beta_1, \beta_2, \ldots, \beta_m; z) * f(z)$$
$$= z - \sum_{n=2}^{\infty} y_n(\alpha_1; \delta; m)|a_n|z^n , \quad (6)$$

where,

$$y_n(\alpha_1; \delta; m) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_\delta)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{1}{(n-1)!} . \quad (7)$$

For notational simplicity, we use shorter notation

$$H_m^\delta[\alpha_1] \quad \text{for} \quad \mathrm{H}[\alpha_1, \ldots, \alpha_\delta; \beta_1, \ldots, \beta_m].$$

In the sequel .It follows from (6) that

$$H_0^1[1]f(z) = f(z), H_0^1[2]f(z) = zf'(z).$$

The linear operator $H_m^\delta[\alpha_1]$ is called Dziok-Srivastava operator (see [5]) introduced by Dziok and Srivastava which was subsequently extended Dziok and Raina [4] by using the generalized hypergeometric function, recently Srivastava et. al. [13] defined the linear operator $\theta_{\lambda,\delta,m}^{\gamma,\alpha_1}$ as follows:-

$$\theta_{\lambda,\delta,m}^{0,\alpha_1} f(z) = f(z)$$

$$\theta_{\lambda,\delta,m}^{1,\alpha_1} f(z) = (1-\lambda) H_m^\delta[\alpha_1] f(z) + \lambda \left(H_m^\delta[\alpha_1] f(z) \right)'$$
$$= \theta_{\lambda,\delta,m}^{\alpha_1} f(z), \quad (8)$$

$$\theta_{\lambda,\delta,m}^{2,\alpha_1} f(z) = \theta_{\lambda,\delta,m}^{\alpha_1} \left(\theta_{\lambda,\delta,m}^{1,\alpha_1} f(z) \right), \quad (9)$$

and in general,

$$\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z) = \theta_{\lambda,\delta,m}^{\alpha_1} \left(\theta_{\lambda,\delta,m}^{\gamma-1,\alpha_1} f(z) \right), (0 \le \lambda \le 1, \delta \le m+1; \delta, m \in N_0$$
$$= N \cup \{0\} \ ; z \in U). \quad (10)$$

If the function f(z) is given by (2), then we see from (6),(7),(8) and (10) that

$$\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z) = z - \sum_{n=2}^{\infty} y_n^\gamma(\alpha_1; \lambda; \delta; m)|a_n|z^n , \quad (11)$$

where

$$y_n^{\gamma}(\alpha_1;\lambda;\delta;m)=\left(\frac{(\alpha_1)_{n-1}\dots(\alpha_\delta)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}}\frac{[1+\lambda(n-1)]}{(n-1)!}\right)^{\gamma}, \tag{12}$$

$(n\in N\backslash\{1\},\gamma\in N_0\ )$.

Unless otherwise stated .We note that when $\gamma=1$and $\lambda=0$, the linear operator $\theta_{\lambda,\delta,m}^{\gamma,\alpha_1}$ would reduce to the familier Dziok-Srivastava linear operator given by(see[5]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [3], Owa [9] and Ruscheweyh [10].

For two analytic functions $f,g\in$K , we say that f is subordinate to g written f(z)$\prec g$(z) if there exists a schwarz function w(z), which (by definition) is analytic in U with w(0)=0 and |w(z)|<1 for all z$\in U$such that f(z)= g(w(z)),z$\in U$.

Furthermore, if the function g(z) is univalent in U, then we have the following equivalence (see [8]):

f(z) $\prec g$(z) $\leftrightarrow$ f(0)= g(0) and f(U) $\subset g$(U).

Definition (1): *For any function $f\in K$ and $\emptyset\geq 0$, the $\emptyset-neighborhood\ f$ is defined as,*

$$N_{n,\emptyset}(f)=\{g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n\in K;\sum_{n=2}^{\infty}n||a_n|-|b_n||\leq\emptyset\}. \tag{13}$$

In particular, for the function *e*(z)=z, we see that,

$$N_{n,\emptyset}(e)=\{g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n\in K;\sum_{n=2}^{\infty}n|\,b_n|\leq\emptyset\}. \tag{14}$$

The concept of neighborhoods was first introduced by Goodman [6] and then generalized by Ruscheweyh [11] , and studied by some authors, like Atshan [1] and Atshan and Kulkarni [2].

Definition (2): *For fixed parameters A and B, with -1$\leq B<A\leq 1$, we say that f$\in K$ is in class K($\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B$),if it satisfies the following subordination condition:*

$$1+\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1}f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1}f(z))'}\prec\frac{1+Az}{1+Bz}. \tag{15}$$

In view of the definition of subordination, (15) is equivalent to the following condition:

$$\left| \frac{\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'}}{B\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} + (B-A)} \right| < 1, \quad (z \in U).$$

For convenience, we write

$\mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,1-2\mu,-1) = \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,\mu)$,

where $\mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,\mu)$ denotes the class of functions in K satisfying the inequality:

$$\mathrm{Re}\left\{1 + \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{\left(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z)\right)'}\right\} > \mu, \quad 0 \le \mu < 1; z \in U).$$

2 Neighborhoods for the Class $\mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$

The following theorem gives a necessary and sufficient condition for a function *f* to be in the class $\mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$.

Theorem (1): *A function $f \in K$ belong to the class $K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$ if and only if*

$$\sum_{n=2}^{\infty} n(2n-1)\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)\left((1-B)y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B)\right)|a_n| \le (A-B)\,, \tag{16}$$

for $\gamma,\eta,m \in N_0, \delta \le m+1, \lambda \ge 0\ \ and -1 \le B < A \le 1$.

Proof: Let $f \in \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$. Then

$$1 + \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} \prec \frac{1+Az}{1+Bz}\ \ , \ z \in U\,. \tag{17}$$

Therefore, there exists an analytic function w such that

$$w(z) = \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{Bz(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))'' + (B-A)(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'}. \tag{18}$$

Hence,

$$|w(z)| = \left|\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{Bz(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))'' + (B-A)(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'}\right|$$

$$= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|z^n}{(B-A)z - \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)\left(B\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (B-A)\right)|a_n|z^n}\right| < 1.$$

Thus,

$$\left|\frac{\sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|z^n}{(A-B)z + \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)(B\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (B-A))|a_n|z^n}\right| < 1.$$

Therefore,

$$Re\left\{\frac{\sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|z^n}{(A-B)z + \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)(B\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (B-A))|a_n|z^n}\right\} < 1. \quad (19)$$

Taking |z|=r, for sufficiently small r with 0<r<1, since w(z) is analytic for |z|=1.Then , the inequality (19) yields

$$\sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|r^n < (A-B)r + \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)\left(B\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) - (A-B)\right)|a_n|r^n.$$

Equivalently,

$$\sum_{n=2}^{\infty} n(2n-1)\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))|a_n|r^n \leq (A-B)r$$

and (16) follows upon letting r→1.

Conversely, for |z|=r, 0<r<1, we have r^n <r. That is,

$$\sum_{n=2}^{\infty} n(2n-1)\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))|a_n|r^n$$

$$\leq \sum_{n=2}^{\infty} n(2n-1)\, y_n^{\gamma}\,(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))|a_n|r^n$$

$$\leq (A-B)r\ .$$

From (16), we have

$$\left|\sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|z^n\right| \leq \sum_{n=2}^{\infty} n(n-1)\, y_n^{\gamma+\eta}\,(\alpha_1;\lambda;\delta;m)|a_n|r^n$$

$$< (A-B)r + \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(B\ \ y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(B-A))|a_n|r^n$$

$$< |(A-B)z + \sum_{n=2}^{\infty} n^2\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(B\ \ y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(B-A))|a_n|z^n|.$$

This proves that

$$1+\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1}f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1}f(z))'} \prec \frac{1+Az}{1+Bz}\quad,\ \ z\in U,$$

and hence $f \in \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$.

Theorem (2): *If*

$$\emptyset = \frac{A-B}{3\left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\gamma}\left[(1-B)\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda))^{\eta}+(A-B)\right]}, \qquad (20)$$

then $\mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B) \subset \mathrm{N}_{n,\emptyset}(\mathrm{e})$.

Proof: It follows from (16), that if $f \in \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$, then

$$3\ y_2^{\gamma}(\alpha_1;\lambda;\delta;m)(\left(1-B\right)\, y_2^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B)\right)\sum_{n=2}^{\infty} n|a_n| \leq A-B\ .$$

Hence

$$3(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda))^{\gamma}\left[(1-B)\left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\eta}+\ (A-B)\right]$$

$$\times \sum_{n=2}^{\infty} n|a_n| \leq A - B, \tag{21}$$

which implies that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{A-B}{3(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda))^\gamma \left[(1-B)\left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^\eta + (A-B)\right]}$$

$$= \emptyset \tag{22}$$

Using (14), we get the result.

Definition (3): *The function g defined by g(z)=z* $- \sum_{n=2}^{\infty} |b_n| z^n$ *is said to be a member of the class* $K_c(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ *if there exist a function* $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ *such that*

$$\left|\frac{g(z)}{f(z)} - 1\right| \leq 1 - c\,, \qquad (z \in U, 0 \leq c < 1). \tag{23}$$

Theorem (3): *If* $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ *and*

$$c = 1 - \frac{3\emptyset\, y_2^\gamma(\alpha_1; \lambda; \delta; m)((1-B)\, y_2^\gamma(\alpha_1; \lambda; \delta; m) + (A-B))}{6\, y_2^\gamma(\alpha_1; \lambda; \delta; m)((1-B) y_2^\gamma(\alpha_1; \lambda; \delta; m) + (A-B)) - (A-B)}\,, \tag{24}$$

then $N_{n,\emptyset}(f) \subset K_c(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$.

Proof: Let $g \in N_{n,\emptyset}(f)$. Then we have from (13) that

$$\sum_{n=2}^{\infty} n||a_n| - |b_n|| \leq \emptyset,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} ||a_n| - |b_n|| \leq \frac{\emptyset}{2}.$$

Also, since $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, we have from (16)

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{A-B}{6\, y_2^\gamma(\alpha_1, \lambda, \delta, m)((1-B)\, y_2^\eta(\alpha_1; \lambda; \delta; m) + (A-B))},$$

where

$$y_2^{\gamma}(\alpha_1;\lambda;\delta;m) = (\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda))^{\gamma},$$

$$y_2^{\eta}(\alpha_1;\lambda;\delta;m) = (\frac{(\alpha_1)_1, \dots, (\alpha_\delta)_1}{(\beta_1)_1, \dots, (\beta_m)_1}(1+\lambda))^{\eta},$$

so that

$$\left|\frac{g(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}(|a_n| - |b_n|)z^n}{z - \sum_{n=2}^{\infty}|a_n| z^n}\right| < \frac{\sum_{n=2}^{\infty}||a_n| - |b_n||}{1 - \sum_{n=2}^{\infty}|a_n|}$$

$$\leq \emptyset. \frac{3\, y_2^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_2^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))}{6\, y_2^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_2^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B)) - (A-B)}$$

=1-c.

Thus by Definition (3), $g \in K_c(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$ for c given by (24). This completes the proof.

3 Convolution Properties

Theorem (4): *Let the functions f_j (j=1,2) defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}| z^n, (j = 1,2), \tag{25}$$

be in the class $K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$ .Then $f_1 * f_2 \in K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,\sigma)$, where

$$\sigma \leq \frac{An(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))^2 - (A-B)^2(A + y_n^{\eta}(\alpha_1;\lambda;\delta;m))}{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))^2 - (A-B)^2(1 + y_n^{\eta}(\alpha_1;\lambda;\delta;m))}$$

Proof: We must find the largest σ such that

$$\sum_{n=2}^{\infty} \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(\left(1-\sigma\right) y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-\sigma)\Big)}{A-\sigma} |a_{n,1}||a_{n,2}| \leq 1.$$

Since $f_j \in K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)(j = 1,2)$, then

$$\sum_{n=2}^{\infty} \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(\left(1-B\right) y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B)\Big)}{A-B} |a_{n,j}| \leq 1, (j = 1,2). \tag{26}$$

By Cauchy –Schwarz inequality, we get

$$\sum_{n=2}^{\infty}\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\sqrt{|a_{n,1}||a_{n,2}|}\leq 1\,. \quad (27)$$

We want only to show that

$$\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-\sigma)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\sigma))}{A-\sigma}|a_{n,1}||a_{n,2}|\leq$$

$$\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\sqrt{|a_{n,1}||a_{n,2}|}.$$

This equivalently to

$$\sqrt{|a_{n,1}||a_{n,2}|}\leq\frac{(A-\sigma)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{(A-B)((1-\sigma)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\sigma))}.$$

From (27), we have

$$\sqrt{|a_{n,1}||a_{n,2}|}\leq\frac{A-B}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}.$$

Thus, it is sufficient to show that

$$\frac{A-B}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}\leq$$

$$\frac{(A-\sigma)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{(A-B)((1-\sigma)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\sigma))},$$

which implies to

$$\sigma\leq\frac{An(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2-(A-B)^2(A+\,y_n^{\eta}(\alpha_1;\lambda;\delta;m))}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2-(A-B)^2(1+\,y_n^{\eta}(\alpha_1;\lambda;\delta;m))}$$

This completes the proof.

Theorem (5): *Let the functions f_j (j=1, 2) defined by (25) be in the class $K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$.Then the function h defined by*

$$h(z)=z-\sum_{n=2}^{\infty}(|a_{n,1}|^2+|a_{n,2}|^2)z^n \quad (28)$$

belong to the class $K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,\epsilon)$, where

$$\epsilon \leq \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2 - 2y_n^{\eta}(\alpha_1;\lambda;\delta;m)(A-B)^2 - 2(A-B)^2}{An(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2 + 2A(A-B)^2 + 2y_n^{\eta}(\alpha_1;\lambda;\delta;m)(A-B)^2}$$

Proof: We must find the largest ϵ such that

$$\sum_{n=2}^{\infty} \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)\left((1-\epsilon)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\epsilon)\right)}{A-\epsilon}\left(\left|a_{n,1}\right|^2+\left|a_{n,2}\right|^2\right) \leq 1.$$

Since $f_j \in \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$ $(j=1,2)$, we get

$$\sum_{n=2}^{\infty}\left(\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\right)^2 \left|a_{n,2}\right|^2$$

$$\leq \left(\sum_{n=2}^{\infty}\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\left|a_{n,1}\right|\right)^2 \leq 1\,, \qquad (29)$$

and

$$\sum_{n=2}^{\infty}\left(\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\right)^2 \left|a_{n,2}\right|^2$$

$$\leq \left(\sum_{n=2}^{\infty}\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\left|a_{n,2}\right|\right)^2 \leq 1. \qquad (30)$$

Combining the inequalities (29) and (30), gives

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))}{A-B}\right)^2 \left|a_{n,1}\right|^2+\left|a_{n,2}\right|^2 \leq 1\,. \qquad (31)$$

But $h \in \mathrm{K}(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,\epsilon)$ if and only if

$$\sum_{n=2}^{\infty}(\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-\epsilon)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\epsilon))}{A-\epsilon}\left(\left|a_{n,1}\right|^2+\left|a_{n,2}\right|^2\right) \leq 1\,. \qquad (32)$$

The inequality (32) will be satisfied if

$$\frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-\epsilon)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-\epsilon))}{A-\epsilon}$$

$$\leq \frac{(n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2}{2(A-B)^2},\ (n=2,3,\dots)\,, \qquad (33)$$

so that,

$$\epsilon \leq \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2-\ 2y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(A-B)^2-2(A-B)^2}{An(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))^2+2A(A-B)^2+\ 2y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(A-B)^2}$$

This completes the proof.

4 Integral Mean Inequalities for the Fractional Integral

Definition (4) [12]: *The fractional integral of order s (s>0) is defined for a function f by:*

$$D_z^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z-t)^{1-s}}\, dt\, ,$$

where the function f is an analytic in a simply- connected region of the complex z-plane containing the origin , and multiplicity of $(z-t)^{s-1}$ *is removed by requiring log(z-t) to be real, when (z-t)>0.*

In 1925, Littlewood [7] proved the following subordination theorem:-

Theorem (6) (Littlewood [7]): *If f and g are analytic in U with* $f \prec g$, *then for* $\mu > 0$ *and* $z = re^{i\theta}$ *(0<r<1)*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \leq \int_0^{2\pi} |g(z)|^{\mu}\, d\theta$$

Theorem (7): *Let* $f \in K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$ *and suppose that* f_n *is defined by*

$$f_n(z) = z - \frac{A-B}{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))} z^n, (n \geq 2). \quad (34)$$

Also, let

$$\sum_{n=2}^{\infty} (i-\tau)_{\tau+1}\, |a_i| \leq \frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+3)}{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(n+s+\tau+1)\Gamma(2-\tau)}, \quad (35)$$

for $0 \leq \tau \leq i, s > 0$, where $(i-\tau)_{\tau+1}$ denote the Pochhammer symbol defined by $(i-\tau)_{\tau+1} = (i-\tau)(i-\tau+1)\ldots i$.

If there exists an analytic function q defined by

$$\left(q(z)\right)^{n-1} = \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)(1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(n+s+\tau+1)}{(A-B)\Gamma(n+1)}$$

$$\times \sum_{i=2}^{\infty} (i-\tau)_{\tau+1} H(i)|a_i|z^{i-1} \quad , \tag{36}$$

where i$\geq \tau$ and

$$H(i) = \frac{\Gamma(i-\tau)}{\Gamma(i+s+\tau+1)}, \quad (s>0, i\geq 2), \tag{37}$$

then , for z $= re^{i\varphi}$ and $0<r<1$

$$\int_0^{2\pi} |D_z^{-s-\tau} f(z)|^{\mu}\, d\varphi \leq \int_0^{2\pi} |D_z^{-s-\tau} f_n(z)|^{\mu}\, d\varphi \quad (s>0, \mu>0)\,. \tag{38}$$

Proof: Let $f(z) = z - \sum_{i=2}^{\infty} |a_i|z^i$. For $\tau \geq 0$ and Definition (4), we get

$$D_z^{-s-\tau} f(z) = \frac{\Gamma(2) z^{s+\tau+1}}{\Gamma(s+\tau+2)}(1 - \sum_{i=2}^{\infty} \frac{\Gamma(i+1)\Gamma(s+\tau+2)}{\Gamma(2)\Gamma(i+s+\tau+1)} |a_i| z^{i-1})$$

$$= \frac{\Gamma(2) z^{s+\tau+1}}{\Gamma(s+\tau+2)}\left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\tau+2)}{\Gamma(2)} (i-\tau)_{\tau+1} H(i)|a_i|z^{i-1}\right),$$

where

$$H(i) = \frac{\Gamma(i-\tau)}{\Gamma(i+s+\tau+1)} \quad , \qquad (s>0, i\geq 2)$$

Since H is a decreasing function of i, we have

$$0 < H(i) \leq H(2) = \frac{\Gamma(2-\tau)}{\Gamma(s+\tau+3)}.$$

Similarly, from (34) and Definition (4), we get

$$D_z^{-s-\tau} f_n(z) = \frac{\Gamma(2) z^{s+\tau+1}}{\Gamma(s+\tau+2)}(1 -$$

$$\frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+2)}{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(n+s+\tau+1)} z^{n-1}).$$

For $\mu > 0$, and z $= re^{i\varphi}$ $(0<r<1)$, we must show that

$$\int_0^{2\pi}\left|1-\sum_{i=2}^{\infty}\frac{\Gamma(s+\tau+2)}{\Gamma(2)}(i-\tau)_{\tau+1}H(i)\,|a_i|z^{1-i}\right|d\varphi \le \int_0^{2\pi}|1-$$

$$\frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+2)}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(2)\Gamma(n+s+\tau+1)}z^{n-1}|^{\mu}d\varphi.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1-\sum_{i=2}^{\infty}\frac{\Gamma(s+\tau+2)}{\Gamma(2)}(i-\tau)_{\tau+1}H(i)\,|a_i|z^{i-1} \prec 1-$$

$$\frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+2)}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(2)\Gamma(n+s+\tau+1)}z^{n-1}.$$

By setting

$$1-\sum_{i=2}^{\infty}\frac{\Gamma(s+\tau+2)}{\Gamma(2)}(i-\tau)_{\tau+1}H(i)\,|a_i|z^{i-1}$$

$$=\frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+2)}{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(2)\Gamma(n+s+\tau+1)}\big(q(z)\big)^{n-1},$$

we find that

$$\big(q(z)\big)^{n-1}=\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B))\Gamma(n+s+\tau+1)}{(A-B)\Gamma(n+1)}$$

$$\times\sum_{i=2}^{\infty}(i-\tau)_{\tau+1}H(i)\,|a_i|z^{i-1},$$

which readily yields w(0)=0.For such a function q , we obtain

$$\big(q(z)\big)^{n-1}\le\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)\Big((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B)\Big)\Gamma(n+s+\tau+1)}{(A-B)\Gamma(n+1)}$$

$$\times\sum_{i=2}^{\infty}(i-\tau)_{\tau+1}H(i)\,|a_i||z|^{i-1}$$

$$\le\frac{n(2n-1)\,y_n^{\gamma}(\alpha_1;\lambda;\delta;m)\Big((1-B)\,y_n^{\eta}(\alpha_1;\lambda;\delta;m)+(A-B)\Big)\Gamma(n+s+\tau+1)}{(A-B)\Gamma(n+1)}$$

$$\times\ H(2)|z|\sum_{i=2}^{\infty}(i-\tau)_{\tau+1}\,|a_i|$$

$$= |z| \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))\Gamma(n+s+\tau+1)}{(\mathrm{A}-\mathrm{B})\Gamma(s+\tau+3)\Gamma(n+1)}$$

$$\times\ \Gamma(2-\tau)\sum_{i=2}^{\infty}(i-\tau)_{\tau+1}\,|a_i| \leq |z| < 1.$$

This completes the proof.

By taking $\tau = 0$, in the Theorem 7, we have the following corollary:

Corollary (1): *Let* $f \in K(\gamma,\eta,\alpha_1,\lambda,\delta,m,A,B)$, *and suppose that* f_n *is defined by (34). Also let*

$$\sum_{i=2}^{\infty} i|a_i| \leq \frac{(\mathrm{A}-\mathrm{B})\Gamma(n+1)\Gamma(s+3)}{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B))\Gamma(2)\Gamma(n+s+1)}$$

n≥ 2. If there exists an analytic function *q* defined by

$$\left(q(z)\right)^{n-1} = \frac{n(2n-1)\, y_n^{\gamma}(\alpha_1;\lambda;\delta;m)\left((1-B)\, y_n^{\eta}(\alpha_1;\lambda;\delta;m) + (A-B)\right)\Gamma(n+s+1)}{(\mathrm{A}-\mathrm{B})\Gamma(n+1)}$$

$$\times \sum_{i=2}^{\infty} iH(i)\,|a_i| z^{i-1} \quad ,$$

where

$$H(i) = \frac{\Gamma(i)}{\Gamma(i+s+1)}\ ,(\ s > 0, i \geq 2).$$

Then for $z = re^{i\varphi}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{-s} f(z)|^{\mu}\, d\varphi \leq \int_0^{2\pi} |D_z^{-s} f_n(z)|^{\mu}\, d\theta \quad (s > 0, \mu > 0)\,.$$

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