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On a Class of γ-b-Open Sets in a Topological Space

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Abstract

In this paper, we introduce some weak separation axioms by utilizing the notions of γ-b-open sets and the γ-b-closure operator.

Keywords: *γ-b-open, γ-b-closure, γD_b-set, γ-b-T_0, γ-b-T_1, γ-b-T_2, γ-b-R_0, γ-b-R_1, γ-b-continuous.*

1 Introduction

In [1] Andrijevi introduced b-open sets, Kasahara [3] defined an operation α on a topological space to introduce α-closed graphs. Following the same technique, Ogata [6] defined an operation γ on a topological space and introduced γ-open sets.

In this paper, we introduce the notion of γ-b-open sets, and γ-b-irresolute in topological spaces. By utilizing these notions we introduce some weak separation axioms. Also we show that some basic properties of γ-b-T_i, γ-b-D_i for $i = 0, 1, 2$ spaces and we ofer a new class of functions called γ-b-continuous functions and a new notion of the graph of a function called a γ-b-closed graph and investigate some of their fundamental properties.

2 Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subset A is said to be b-open [1] if $A \subseteq int(cl(A)) \cup cl(int(A))$. The complement of a b-open set is said to be b-closed.

An operation γ [3] on a topology τ is a mapping from τ in to power set $P(X)$ of X such that $V \subset \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [6] if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subset A$. Then, τ_γ denotes the set of all γ-open set in X. Clearly $\tau_\gamma \subset \tau$. Complements of γ-open sets are called γ-closed. The γ-closure [6] of a subset A of X with an operation γ on τ is denoted by τ_γ-$cl(A)$ and is defined to be the intersection of all γ-closed sets containing A, and the τ_γ-interior [4] of A is denoted by τ_γ-$int(A)$ and defined to be the union of all γ-open sets of X contained in A. A subset A of X with an operation γ on τ is called be γ-preopen set [5] if and only if $A \subseteq \tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A))$. A subset A of X with an operation γ on τ is called be γ-β-open set [2] if $A \subseteq \tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A)))$. A topological X with an operation γ on τ is said to be γ-regular [6] if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V. It is also to be noted that $\tau = \tau_\gamma$ if and only if X is a γ-regular space [6].

3 γ-b-Open Sets

Definition 3.1 *A subset A of a topological space (X, τ) is said to be γ-b-open if $A \subset \tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A)) \cup \tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(A))$.*

The complement of a γ-b-open set is said to be γ-b-closed. The family of all γ-b-open (resp. γ-b-closed) sets in a topological space (X, τ) is denoted by $\gamma bO(X, \tau)$ (resp. $\gamma bC(X, \tau)$).

Definition 3.2 *Let A be a subset of a topological space (X, τ). The intersection of all γ-b-closed sets containing A is called the γ-b-closure of A and is denoted by $\gamma cl_b(A)$.*

Definition 3.3 *Let (X, τ) be a topological space. A subset U of X is called a γ-b-neighbourhood of a point $x \in X$ if there exists a γ-b-open set V such that $x \in V \subset U$.*

Theorem 3.4 *For the γ-b-closure of subsets A, B in a topological space (X, τ), the following properties hold:*

1. *A is γ-b-closed in (X, τ) if and only if $A = \gamma cl_b(A)$.*
2. *If $A \subset B$ then $\gamma cl_b(A) \subset \gamma cl_b(B)$.*
3. *$\gamma cl_b(A)$ is γ-b-closed, that is $\gamma cl_b(A) = \gamma cl_b(\gamma cl_b(A))$.*
4. *$x \in \gamma cl_b(A)$ if and only if $A \cap V \neq \phi$ for every γ-b-open set V of X containing x.*

Proof. It is obvious.

Theorem 3.5 *For a family $\{A_\alpha : \alpha \in \Delta\}$ of subsets a topological space (X, τ), the following properties hold:*

1. *$\gamma cl_b(\cap_{\alpha\in\Delta} A_\alpha) \subset \cap_{\alpha\in\Delta} \gamma cl_b(A_\alpha)$.*
2. *$\gamma cl_b(\cup_{\alpha\in\Delta} A_\alpha) \supset \cup_{\alpha\in\Delta} \gamma cl_b(A_\alpha)$.*

Proof.

1. Since $\cap_{\alpha\in\Delta} A_\alpha \subset A_\alpha$ for each $\alpha \in \Delta$, by Theorem 3.4 we have $\gamma cl_b(\cap_{\alpha\in\Delta} A_\alpha) \subset \gamma cl_b(A_\alpha)$ for each $\alpha \in \Delta$ and hence $\gamma cl_b(\cap_{\alpha\in\Delta} A_\alpha) \subset \cap_{\alpha\in\Delta} \gamma cl_b(A_\alpha)$.
2. Since $A_\alpha \subset \cup_{\alpha\in\Delta} A_\alpha$ for each $\alpha \in \Delta$, by Theorem 3.4 we have $\gamma cl_b(A_\alpha) \subset \gamma cl_b(\cup_{\alpha\in\Delta} A_\alpha)$ for each $\alpha \in \Delta$ and hence $\cup_{\alpha\in\Delta} \gamma cl_b(A_\alpha) \subset \gamma cl_b(\cup_{\alpha\in\Delta} A_\alpha)$.

Theorem 3.6 *Every γ-preopen set is γ-b-open.*

Proof. It follows from the Definitions.

The converse of the above Theorem need not be true by the following Example.

Example 3.7 *Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\gamma(A) = A$ for all $A \in \tau$. Here $\{a, b\}$ is not γ-preopen however it is γ-b-open.*

Corollary 3.8 *Every γ-open set is γ-b-open.*

Proof. It follows from Theorem 3.6.

Theorem 3.9 *Every γ-b-open set is γ-β-open.*

Proof. It follows from the Definitions.

Remark 3.10 *The concepts of b-open and γ-b-open sets are independent, while in a γ-regular space these concepts are equivalent.*

Example 3.11 *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Define an operation γ on τ by*

$$\gamma(A) = \begin{cases} \{a\} & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, \{a\}, X\}$. Then $\{b\}$ is b-open but not γ-b-open. Again, if we define γ on τ by $\gamma(A) = X$, then $\{c\}$ is γ-b-open but not b-open.

Theorem 3.12 *An arbitrary union of γ-b-open sets is γ-b-open.*

Proof. Let $\{A_k : k \in \Delta\}$ be a family of γ-b-open sets. Then for each k, $A_k \subseteq \tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A_k)) \cup \tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(A_k))$ and so

$$\cup_{k\in\Delta} A_k \subseteq \cup_{k\in\Delta}[\tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A_k)) \cup \tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(A_k))]$$

$$\subseteq [\cup_{k\in\Delta}\tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A_k))] \cup [\cup_{k\in\Delta}\tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(A_k))]$$

$$\subseteq [\tau_\gamma\text{-}int(\cup_{k\in\Delta}\tau_\gamma\text{-}cl(A_k))] \cup [\tau_\gamma\text{-}cl(\cup_{k\in\Delta}\tau_\gamma\text{-}int(A_k))]$$

$$\subseteq [\tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(\cup_{k\in\Delta} A_k))] \cup [\tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(\cup_{k\in\Delta} A_k))].$$

Therefore, $\cup_{k\in\Delta} A_k$ is γ-b-open.

Remark 3.13

1. *An arbitrary intersection of γ-b-closed sets is γ-b-closed.*

2. *The intersection of even two γ-b-open sets may not be γ-b-open.*

Example 3.14 *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Define an operation γ on τ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \\ X & \text{otherwise} \end{cases}$$

Clearly, $\tau_\gamma = \{\phi, \{a, b\}, X\}$, take $A = \{a, c\}$ and $B = \{b, c\}$ are γ-b-open. Then $A \cap B = \{c\}$, which is not a γ-b-open set.

Definition 3.15 *A subset A of a topological space (X, τ) is called a γD_b-set if there are two $U, V \in \gamma bO(X, \tau)$ such that $U \neq X$ and $A = U \setminus V$.*

It is true that every γ-b-open set U different from X is a γD_b-set if $A = U$ and $V = \phi$. So, we can observe the following.

Remark 3.16 *Every proper γ-b-open set is a γD_b-set.*

Definition 3.17 *A topological space (X, τ) with an operation γ on τ is said to be*

1. γ-*b*-D_0 *if for any pair of distinct points x and y of X there exists a γD_b-set of X containing x but not y or a γD_b-set of X containing y but not x.*

2. γ-*b*-D_1 *if for any pair of distinct points x and y of X there exists a γD_b-set of X containing x but not y and a γD_b-set of X containing y but not x.*

3. γ-*b*-D_2 *if for any pair of distinct points x and y of X there exist disjoint γD_b-set G and E of X containing x and y, respectively.*

Definition 3.18 *A topological space (X, τ) with an operation γ on τ is said to be*

1. γ-*b*-T_0 *(resp. γ-pre T_0 [5] and γ-β T_0 [2]) if for any pair of distinct points x and y of X there exists a γ-b-open (resp. γ-preopen and γ-β-open) set U in X containing x but not y or a γ-b-open (resp. γ-preopen and γ-β-open) set V in X containing y but not x.*

2. γ-*b*-T_1 *(resp. γ-pre T_1 [5] and γ-β T_1 [2]) if for any pair of distinct points x and y of X there exists a γ-b-open (resp. γ-preopen and γ-β-open) set U in X containing x but not y and a γ-b-open (resp. γ-preopen and γ-β-open) set V in X containing y but not x.*

3. γ-*b*-T_2 *(resp. γ-pre T_2 [5] and γ-β T_2 [2]) if for any pair of distinct points x and y of X there exist disjoint γ-b-open (resp. γ-preopen and γ-β-open) sets U and V in X containing x and y, respectively.*

Remark 3.19 *For a topological space (X, τ), the following properties hold:*

1. *If (X, τ) is γ-b-T_i, then it is γ-b-T_{i-1}, for $i = 1, 2$.*

2. *If (X, τ) is γ-b-T_i, then it is γ-b-D_i, for $i = 0, 1, 2$.*

3. *If (X, τ) is γ-b-D_i, then it is γ-b-D_{i-1}, for $i = 1, 2$.*

4. *If (X, τ) is γ-pre T_i, then it is γ-b-T_i, for $i = 0, 1, 2$.*

5. *If (X, τ) is γ-b-T_i, then it is γ-β T_i, for $i = 0, 1, 2$.*

By Remark 3.19 we have the following diagram.

$$\begin{array}{ccccc}
\gamma\text{-pre } T_2 & \longrightarrow & \gamma\text{-pre } T_1 & \longrightarrow & \gamma\text{-pre } T_0 \\
\downarrow & & \downarrow & & \downarrow \\
\gamma\text{-b-}T_2 & \longrightarrow & \gamma\text{-b-}T_1 & \longrightarrow & \gamma\text{-b-}T_0 \\
\downarrow & & \downarrow & & \downarrow \\
\gamma\text{-}\beta\ T_2 & \longrightarrow & \gamma\text{-}\beta\ T_1 & \longrightarrow & \gamma\text{-}\beta\ T_0
\end{array}$$

Theorem 3.20 *A topological space (X, τ) is γ-b-D_1 if and only if it is γ-b-D_2.*

Proof. sufficiency. Follows from Remark 3.19.
Necessity. Let $x, y \in X$, $x \neq y$. Then there exist γD_b-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are γ-b-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.
(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
(a) $y \notin U_1$. From $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and by $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \phi$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, and $y \in U_2$. Therefore $(U_1 \setminus U_2) \cap U_2 = \phi$.
(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4$ and $x \in U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \phi$. Therefore X is γ-b-D_2.

Definition 3.21 *A point $x \in X$ which has only X as the γ-b-neighborhood is called a γ-b-neat point.*

Theorem 3.22 *If a topological space (X, τ) is γ-b-D_1, then it has no γ-b-neat point.*

Proof. Since (X, τ) is γ-b-D_1, so each point x of X is contained in a γD_b-set $A = U \setminus V$ and thus in U. By definition $U \neq X$. This implies that x is not a γ-b-neat point.

Theorem 3.23 *A topological space (X, τ) with an operation γ on τ is γ-b-T_0 if and only if for each pair of distinct points x, y of X, $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$.*

Theorem 3.24 *A topological space (X, τ) with an operation γ on τ is γ-b-T_1 if and only if the singletons are γ-b-closed sets.*

Proof. Let (X, τ) be γ-b-T_1 and x any point of X. Suppose $y \in X \setminus \{x\}$, then $x \neq y$ and so there exists a γ-b-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subset X \setminus \{x\}$ i.e., $X \setminus \{x\} = \cup\{U : y \in X \setminus \{x\}\}$ which is γ-b-open.

Conversely, suppose $\{p\}$ is γ-b-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a γ-b-open set contains y but not x. Similarly $X \setminus \{y\}$ is a γ-b-open set contains x but not y. Accordingly X is a γ-b-T_1 space.

Definition 3.25 *A topological space (X, τ) is γ-b-symmetric if for x and y in X, $x \in \gamma cl_b(\{y\})$ implies $y \in \gamma cl_b(\{x\})$.*

Theorem 3.26 *If a topological space (X, τ) with an operation γ on τ is a γ-b-T_1 space, then it is γ-b-symmetric.*

Proof. Suppose that $y \notin \gamma cl_b(\{x\})$. Then, since $x \neq y$, there exists a γ-b-open set U containing x such that $y \notin U$ and hence $x \notin \gamma cl_b(\{y\})$. This shows that $x \in \gamma cl_b(\{y\})$ implies $y \in \gamma cl_b(\{x\})$. Therefore, (X, τ) is γ-b-symmetric.

Definition 3.27 *Let (X, τ) and (Y, σ) be two topological spaces and γ, β operations on τ, σ, respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be γ-b-irresolute if for each $x \in X$ and each β-b-open set V containing $f(x)$, there is a γ-b-open set U in X containing x such that $f(U) \subset V$.*

Theorem 3.28 *If $f : (X, \tau) \to (Y, \sigma)$ is a γ-b-irresolute surjective function and E is a βD_b-set in Y, then the inverse image of E is a γD_b-set in X.*

Proof. Let E be a βD_b-set in Y. Then there are β-b-open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. By the γ-b-irresolute of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are γ-b-open in X. Since $U_1 \neq Y$ and f is surjective, we have $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)$ is a γD_b-set.

Theorem 3.29 *If (Y, σ) is β-b-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is γ-b-irresolute bijective, then (X, τ) is γ-b-D_1.*

Proof. Suppose that Y is a β-b-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is β-b-D_1, there exist βD_b-set G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(x) \notin G_y$ and $f(y) \notin G_x$. By Theorem 3.28, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are γD_b-set in X containing x and y, respectively, such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. This implies that X is a γ-b-D_1 space.

Theorem 3.30 *A topological space (X, τ) is γ-b-D_1 if for each pair of distinct points $x, y \in X$, there exists a γ-b-irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a β-b-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists a γ-b-irresolute, surjective function f of a space X onto a β-b-D_1 space Y such that $f(x) \neq f(y)$. By Theorem 3.20, there exist disjoint βD_b-set G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is γ-b-irresolute and surjective, by Theorem 3.28, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint γD_b-sets in X containing x and y, respectively. hence by Theorem 3.20, X is γ-b-D_1 space.

4 γ-b-R_0 and γ-b-R_1 Spaces

Definition 4.1 *Let A be a subset of a topological space (X, τ) with an operation γ on τ. The γ-b-kernel of A, denoted by $\gamma ker_b(A)$ is defined to be the set*

$$\gamma ker_b(A) = \cap\{U \in \gamma bO(X) \colon A \subset U\}.$$

Theorem 4.2 *Let (X, τ) be a topological space with an operation γ on τ and $x \in X$. Then $y \in \gamma ker_b(\{x\})$ if and only if $x \in \gamma cl_b(\{y\})$.*

Proof. Suppose that $y \notin \gamma ker_b(\{x\})$. Then there exists a γ-b-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \gamma cl_b(\{y\})$. The proof of the converse case can be done similarly.

Lemma 4.3 *Let (X, τ) be a topological space and A be a subset of X. Then, $\gamma ker_b(A) = \{x \in X \colon \gamma cl_b(\{x\}) \cap A \neq \phi\}$.*

Proof. Let $x \in \gamma ker_b(A)$ and suppose $\gamma cl_b(\{x\}) \cap A = \phi$. Hence $x \notin X \setminus \gamma cl_b(\{x\})$ which is a γ-b-open set containing A. This is impossible, since $x \in \gamma ker_b(A)$. Consequently, $\gamma cl_b(\{x\}) \cap A \neq \phi$. Next, let $x \in X$ such that $\gamma cl_b(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \gamma ker_b(A)$. Then, there exists a γ-b-open set V containing A and $x \notin V$. Let $y \in \gamma cl_b(\{x\}) \cap A$. Hence, V is a γ-b-neighborhood of y which does not contain x. By this contradiction $x \in \gamma ker_b(A)$ and the claim.

Remark 4.4 *The following properties hold for the subsets A, B of a topological space (X, τ) with an operation γ on τ:*

1. *$A \subset \gamma ker_b(A)$, if A is γ-b-open in (X, τ), then $A = \gamma ker_b(A)$.*
2. *If $A \subset B$, then $\gamma ker_b(A) \subset \gamma ker_b(B)$.*

Definition 4.5 *A topological space (X, τ) with an operation γ on τ is said to be γ-b-R_0 if every γ-b-open set U and $x \in U$ implies $\gamma cl_b(\{x\}) \subset U$.*

Theorem 4.6 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. (X,τ) *is* γ*-b-*R_0.

2. *For any* $F \in \gamma bC(X)$, $x \notin F$ *implies* $F \subset U$ *and* $x \notin U$ *for some* $U \in \gamma bO(X)$.

3. *For any* $F \in \gamma bC(X)$, $x \notin F$ *implies* $F \cap \gamma cl_b(\{x\}) = \phi$.

4. *For any distinct points* x *and* y *of* X, *either* $\gamma cl_b(\{x\}) = \gamma cl_b(\{y\})$ *or* $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$.

Proof. (1) $\Rightarrow$ (2). Let $F \in \gamma bC(X)$ and $x \notin F$. Then by (1) $\gamma cl_b(\{x\}) \subset X \setminus F$. Set $U = X \setminus \gamma cl_b(\{x\})$, then U is γ-b-open set such that $F \subset U$ and $x \notin U$.
(2) $\Rightarrow$ (3). Let $F \in \gamma bC(X)$ and $x \notin F$. There exists $U \in \gamma bO(X)$ such that $F \subset U$ and $x \notin U$. Since $U \in \gamma bO(X)$, $U \cap \gamma cl_b(\{x\}) = \phi$ and $F \cap \gamma cl_b(\{x\}) = \phi$.
(3) $\Rightarrow$ (4). Suppose that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$ for distinct points $x, y \in X$. There exists $z \in \gamma cl_b(\{x\})$ such that $z \notin \gamma cl_b(\{y\})$ (or $z \in \gamma cl_b(\{y\})$ such that $z \notin \gamma cl_b(\{x\})$). There exists $V \in \gamma bO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \gamma cl_b(\{y\})$. By (3), we obtain $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$. The proof for otherwise is similar.
(4) $\Rightarrow$ (1). let $V \in \gamma bO(X)$ and $x \in V$. For each $y \notin V$, $x \neq y$ and $x \notin \gamma cl_b(\{y\})$. This shows that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. By (4), $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$ for each $y \in X \setminus V$ and hence $\gamma cl_b(\{x\}) \cap (\bigcup_{y \in X \setminus V} \gamma cl_b(\{y\})) = \phi$. On other hand, since $V \in \gamma bO(X)$ and $y \in X \setminus V$, we have $\gamma cl_b(\{y\}) \subset X \setminus V$ and hence $X \setminus V = \bigcup_{y \in X \setminus V} \gamma cl_b(\{y\})$. Therefore, we obtain $(X \setminus V) \cap \gamma cl_b(\{x\}) = \phi$ and $\gamma cl_b(\{x\}) \subset V$. This shows that (X,τ) is a γ-b-R_0 space.

Theorem 4.7 *A topological space* (X,τ) *with an operation* γ *on* τ *is* γ*-b-*T_1 *if and only if* (X,τ) *is* γ*-b-*T_0 *and* γ*-b-*R_0 *space.*

Proof. Necessity. Let U be any γ-b-open set of (X,τ) and $x \in U$. Then by Theorem 3.24, we have $\gamma cl_b(\{x\}) \subset U$ and so by Remark 3.19, it is clear that X is γ-b-T_0 and γ-b-R_0 space.
Sufficiency. Let x and y be any distinct points of X. Since X is γ-b-T_0, there exists a γ-b-open set U such that $x \in U$ and $y \notin U$. As $x \in U$ implies that $\gamma cl_b(\{x\}) \subset U$. Since $y \notin U$, so $y \notin \gamma cl_b(\{x\})$. Hence $y \in V = X \setminus \gamma cl_b(\{x\})$ and it is clear that $x \notin V$. Hence it follows that there exist γ-b-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$. This implies that X is γ-b-T_1.

Theorem 4.8 *For a topological space* (X,τ) *with an operation* γ *on* τ, *the following properties are equivalent:*

1. (X,τ) *is* γ*-b-*R_0.

2. *$x \in \gamma cl_b(\{y\})$ if and only if $y \in \gamma cl_b(\{x\})$, for any points x and y in X.*

Proof. (1) $\Rightarrow$ (2). Assume that X is γ-b-R_0. Let $x \in \gamma cl_b(\{y\})$ and V be any γ-b-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every γ-b-open set which contain y contains x. Hence $y \in \gamma cl_b(\{x\})$.
(2) $\Rightarrow$ (1). Let U be a γ-b-open set and $x \in U$. If $y \notin U$, then $x \notin \gamma cl_b(\{y\})$ and hence $y \notin \gamma cl_b(\{x\})$. This implies that $\gamma cl_b(\{x\}) \subset U$. Hence (X, τ) is γ-b-R_0.

We observed that by Definition 3.25 and Theorem 4.8 the notions of γ-b-symmetric and γ-b-R_0 are equivalent.

Theorem 4.9 *The following statements are equivalent for any points x and y in a topological space (X, τ) with an operation γ on τ:*

1. *$\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$.*

2. *$\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$, then there exists a point z in X such that $z \in \gamma ker_b(\{x\})$ and $z \notin \gamma ker_b(\{y\})$. From $z \in \gamma ker_b(\{x\})$ it follows that $\{x\} \cap \gamma cl_b(\{z\}) \neq \phi$ which implies $x \in \gamma cl_b(\{z\})$. By $z \notin \gamma ker_b(\{y\})$, we have $\{y\} \cap \gamma cl_b(\{z\}) = \phi$. Since $x \in \gamma cl_b(\{z\})$, $\gamma cl_b(\{x\}) \subset \gamma cl_b(\{z\})$ and $\{y\} \cap \gamma cl_b(\{x\}) = \phi$. Therefore, it follows that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. Now $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$ implies that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$.
(2) $\Rightarrow$ (1). Suppose that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. Then there exists a point z in X such that $z \in \gamma cl_b(\{x\})$ and $z \notin \gamma cl_b(\{y\})$. Then, there exists a γ-b-open set containing z and therefore x but not y, namely, $y \notin \gamma ker_b(\{x\})$ and thus $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$.

Theorem 4.10 *Let (X, τ) be a topological space and γ be an operation on τ. Then $\cap\{\gamma cl_b(\{x\}) : x \in X\} = \phi$ if and only if $\gamma ker_b(\{x\}) \neq X$ for every $x \in X$.*

Proof. Necessity. Suppose that $\cap\{\gamma cl_b(\{x\}) : x \in X\} = \phi$. Assume that there is a point y in X such that $\gamma ker_b(\{y\}) = X$. Let x be any point of X. Then $x \in V$ for every γ-b-open set V containing y and hence $y \in \gamma cl_b(\{x\})$ for any $x \in X$. This implies that $y \in \cap\{\gamma cl_b(\{x\}) : x \in X\}$. But this is a contradiction.
Sufficiency. Assume that $\gamma ker_b(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \cap\{\gamma cl_b(\{x\}) : x \in X\}$, then every γ-b-open set containing y must contain every point of X. This implies that the space X is the unique γ-b-open set containing y. Hence $\gamma ker_b(\{y\}) = X$ which is a contradiction. Therefore, $\cap\{\gamma cl_b(\{x\}) : x \in X\} = \phi$.

Theorem 4.11 *A topological space (X, τ) with an operation γ on τ is γ-b-R_0 if and only if for every x and y in X, $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$ implies $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$.*

Proof. Necessity. Suppose that (X, τ) is γ-b-R_0 and $x, y \in X$ such that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. Then, there exists $z \in \gamma cl_b(\{x\})$ such that $z \notin \gamma cl_b(\{y\})$ (or $z \in \gamma cl_b(\{y\})$ such that $z \notin \gamma cl_b(\{x\})$). There exists $V \in \gamma bO(X)$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \gamma cl_b(\{y\})$. Thus $x \in [X \setminus \gamma cl_b(\{y\})] \in \gamma bO(X)$, which implies $\gamma cl_b(\{x\}) \subset [X \setminus \gamma cl_b(\{y\})]$ and $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$. The proof for otherwise is similar.
Sufficiency. Let $V \in \gamma bO(X)$ and let $x \in V$. We still show that $\gamma cl_b(\{x\}) \subset V$. Let $y \notin V$, i.e., $y \in X \setminus V$. Then $x \neq y$ and $x \notin \gamma cl_b(\{y\})$. This shows that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. By assumption, $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$. Hence $y \notin \gamma cl_b(\{x\})$ and therefore $\gamma cl_b(\{x\}) \subset V$.

Theorem 4.12 *A topological space (X, τ) with an operation γ on τ is γ-b-R_0 if and only if for any points x and y in X, $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$ implies $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\}) = \phi$.*

Proof. Suppose that (X, τ) is a γ-b-R_0 space. Thus by Theorem 4.9, for any points x and y in X if $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$ then $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. Now we prove that $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\}) = \phi$. Assume that $z \in \gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\})$. By $z \in \gamma ker_b(\{x\})$ and Theorem 4.2, it follows that $x \in \gamma cl_b(\{z\})$. Since $x \in \gamma cl_b(\{x\})$, by Theorem 4.6, $\gamma cl_b(\{x\}) = \gamma cl_b(\{z\})$. Similarly, we have $\gamma cl_b(\{y\}) = \gamma cl_b(\{z\}) = \gamma cl_b(\{x\})$. This is a contradiction. Therefore, we have $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\}) = \phi$.

Conversely, let (X, τ) be a topological space such that for any points x and y in X, $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$ implies $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\}) = \phi$. If $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$, then by Theorem 4.9, $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$. Hence, $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{y\}) = \phi$ which implies $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$. Because $z \in \gamma cl_b(\{x\})$ implies that $x \in \gamma ker_b(\{z\})$ and therefore $\gamma ker_b(\{x\}) \cap \gamma ker_b(\{z\}) \neq \phi$. By hypothesis, we have $\gamma ker_b(\{x\}) = \gamma ker_b(\{z\})$. Then $z \in \gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\})$ implies that $\gamma ker_b(\{x\}) = \gamma ker_b(\{z\}) = \gamma ker_b(\{y\})$. This is a contradiction. Therefore, $\gamma cl_b(\{x\}) \cap \gamma cl_b(\{y\}) = \phi$ and by Theorem 4.6 (X, τ) is a γ-b-R_0 space.

Theorem 4.13 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X, τ) is a γ-b-R_0 space.*

2. *For any nonempty set A and $G \in \gamma bO(X)$ such that $A \cap G \neq \phi$, there exists $F \in \gamma bC(X)$ such that $A \cap F \neq \phi$ and $F \subset G$.*

3. *Any $G \in \gamma bO(X)$, $G = \cup\{F \in \gamma bC(X)\colon F \subset G\}$.*

4. *Any $F \in \gamma bC(X)$, $F = \cap\{G \in \gamma bO(X)\colon F \subset G\}$.*

5. *For every $x \in X$, $\gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let A be a nonempty subset of X and $G \in \gamma bO(X)$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \gamma bO(X)$, $\gamma cl_b(\{x\}) \subset G$. Set $F = \gamma cl_b(\{x\})$, then $F \in \gamma bC(X)$, $F \subset G$ and $A \cap F \neq \phi$.
(2) $\Rightarrow$ (3). Let $G \in \gamma bO(X)$, then $G \supseteq \cup\{F \in \gamma bC(X)\colon F \subset G\}$. Let x be any point of G. There exists $F \in \gamma bC(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \gamma bC(X)\colon F \subset G\}$ and hence $G = \cup\{F \in \gamma bC(X)\colon F \subset G\}$.
(3) $\Rightarrow$ (4). This is obvious.
(4) $\Rightarrow$ (5). Let x be any point of X and $y \notin \gamma ker_b(\{x\})$. There exists $V \in \gamma bO(X)$ such that $x \in V$ and $y \notin V$, hence $\gamma cl_b(\{y\}) \cap V = \phi$. By (4) $(\cap\{G \in \gamma bO(X)\colon \gamma cl_b(\{y\}) \subset G\}) \cap V = \phi$ and there exists $G \in \gamma bO(X)$ such that $x \notin G$ and $\gamma cl_b(\{y\}) \subset G$. Therefore $\gamma cl_b(\{x\}) \cap G = \phi$ and $y \notin \gamma cl_b(\{x\})$. Consequently, we obtain $\gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\})$.
(5) $\Rightarrow$ (1). Let $G \in \gamma bO(X)$ and $x \in G$. Let $y \in \gamma ker_b(\{x\})$, then $x \in \gamma cl_b(\{y\})$ and $y \in G$. This implies that $\gamma ker_b(\{x\}) \subset G$. Therefore, we obtain $x \in \gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\}) \subset G$. This shows that (X, τ) is a γ-b-R_0 space.

Corollary 4.14 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X, τ) is a γ-b-R_0 space.*

2. *$\gamma cl_b(\{x\}) = \gamma ker_b(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Suppose that (X, τ) is a γ-b-R_0 space. By Theorem 4.13, $\gamma cl_b(\{x\}) \subset \gamma ker_b(\{x\})$ for each $x \in X$. Let $y \in \gamma ker_b(\{x\})$, then $x \in \gamma cl_b(\{y\})$ and by Theorem 4.6 $\gamma cl_b(\{x\}) = \gamma cl_b(\{y\})$. Therefore, $y \in \gamma cl_b(\{x\})$ and hence $\gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$. This shows that $\gamma cl_b(\{x\}) = \gamma ker_b(\{x\})$.
(2) $\Rightarrow$ (1). This is obvious by Theorem 4.13.

Theorem 4.15 *For a topological space (X, τ) with an operation γ on τ, the following properties are equivalent:*

1. *(X, τ) is a γ-b-R_0 space.*

2. *If F is γ-b-closed, then $F = \gamma ker_b(F)$.*

3. *If F is γ-b-closed and $x \in F$, then $\gamma ker_b(\{x\}) \subset F$.*

4. *If $x \in X$, then $\gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$.*

Proof. (1) $\Rightarrow$ (2). Let F be a γ-b-closed and $x \notin F$. Thus $(X \setminus F)$ is a γ-b-open set containing x. Since (X, τ) is γ-b-R_0. $\gamma cl_b(\{x\}) \subset (X \setminus F)$. Thus $\gamma cl_b(\{x\}) \cap F = \phi$ and by Lemma 4.3 $x \notin \gamma ker_b(F)$. Therefore $\gamma ker_b(F) = F$.
(2) $\Rightarrow$ (3). In general, $A \subset B$ implies $\gamma ker_b(A) \subset \gamma ker_b(B)$. Therefore, it follows from (2) that $\gamma ker_b(\{x\}) \subset \gamma ker_b(F) = F$.
(3) $\Rightarrow$ (4). Since $x \in \gamma cl_b(\{x\})$ and $\gamma cl_b(\{x\})$ is γ-b-closed, by (3), $\gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$.
(4) $\Rightarrow$ (1). We show the implication by using Theorem 4.8. Let $x \in \gamma cl_b(\{y\})$. Then by Theorem 4.2, $y \in \gamma ker_b(\{x\})$. Since $x \in \gamma cl_b(\{x\})$ and $\gamma cl_b(\{x\})$ is γ-b-closed, by (4) we obtain $y \in \gamma ker_b(\{x\}) \subset \gamma cl_b(\{x\})$. Therefore $x \in \gamma cl_b(\{y\})$ implies $y \in \gamma cl_b(\{x\})$. The converse is obvious and (X, τ) is γ-b-R_0.

Definition 4.16 *A topological space (X, τ) with an operation γ on τ, is said to be γ-b-R_1 if for x, y in X with $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$, there exist disjoint γ-b-open sets U and V such that $\gamma cl_b(\{x\}) \subset U$ and $\gamma cl_b(\{y\}) \subset V$.*

Theorem 4.17 *A topological space (X, τ) with an operation γ on τ is γ-b-R_1 if it is γ-b-T_2.*

Proof. Let x and y be any points of X such that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. By Remark 3.19, every γ-b-T_2 space is γ-b-T_1. Therefore, by Theorem 3.24, $\gamma cl_b(\{x\}) = \{x\}$, $\gamma cl_b(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since (X, τ) is γ-b-T_2, there exist disjoint γ-b-open sets U and V such that $\gamma cl_b(\{x\}) = \{x\} \subset U$ and $\gamma cl_b(\{y\}) = \{y\} \subset V$. This shows that (X, τ) is γ-b-R_1

Theorem 4.18 *For a topological space (X, τ) with an operation γ on τ, the following are equivalent:*

1. *(X, τ) is γ-b-T_2.*
2. *(X, τ) is γ-b-R_1 and γ-b-T_1.*
3. *(X, τ) is γ-b-R_1 and γ-b-T_0.*

Proof. Proof is easy and hence omitted.

Theorem 4.19 *For a topological space (X, τ) with an operation γ on τ, the following statements are equivalent:*

1. *(X, τ) is γ-b-R_1.*
2. *If $x, y \in X$ such that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$, then there exist γ-b-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.*

Proof. Proof is easy and hence omitted.

Theorem 4.20 *If (X,τ) is γ-b-R_1, then (X,τ) is γ-b-R_0.*

Proof. Let U be γ-b-open such that $x \in U$. If $y \notin U$, since $x \notin \gamma cl_b(\{y\})$, we have $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. So, there exists a γ-b-open set V such that $\gamma cl_b(\{y\}) \subset V$ and $x \notin V$, which implies $y \notin \gamma cl_b(\{x\})$. Hence $\gamma cl_b(\{x\}) \subset U$. Therefore, (X,τ) is γ-b-R_0.

The converse of the above Theorem need not be ture as shown in the following example.

Example 4.21 *Consider $X = \{a,b,c\}$ with the discrete topology on X. Define an operation γ on τ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Then X is a γ-b-R_0 space but not a γ-b-R_1 space.

Theorem 4.22 *A topological space (X,τ) with an operation γ on τ is γ-b-R_1 if and only if for $x,y \in X$, $\gamma ker_b(\{x\}) \neq \gamma ker_b(\{y\})$, there exist disjoint γ-b-open sets U and V such that $\gamma cl_b(\{x\}) \subset U$ and $\gamma cl_b(\{y\}) \subset V$.*

Proof. It follows from Theorem 4.9.

Theorem 4.23 *A topological space (X,τ) is γ-b-R_1 if and only if the inclusion $x \in X \setminus \gamma cl_b(\{y\})$ implies that x and y have disjoint γ-b-open neighborhoods.*

Proof. Necessity. Let $x \in X \setminus \gamma cl_b(\{y\})$. Then $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$ and x and y have disjoint γ-b-open neighborhoods.
Sufficiency. First, we show that (X,τ) is γ-b-R_0. Let U be a γ-b-open set and $x \in U$. Suppose that $y \notin U$. Then, $\gamma cl_b(\{y\}) \cap U = \phi$ and $x \notin \gamma cl_b(\{y\})$. There exist γ-b-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $\gamma cl_b(\{x\}) \subset \gamma cl_b(U_x)$ and $\gamma cl_b(\{x\}) \cap U_y \subset \gamma cl_b(U_x) \cap U_y = \phi$. Therefore, $y \notin \gamma cl_b(\{x\})$. Consequently, $\gamma cl_b(\{x\}) \subset U$ and (X,τ) is γ-b-R_0. Next, we show that (X,τ) is γ-b-R_1. Suppose that $\gamma cl_b(\{x\}) \neq \gamma cl_b(\{y\})$. Then, we can assume that there exists $z \in \gamma cl_b(\{x\})$ such that $z \notin \gamma cl_b(\{y\})$. There exist γ-b-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in \gamma cl_b(\{x\})$, $x \in V_z$. Since (X,τ) is γ-b-R_0, we obtain $\gamma cl_b(\{x\}) \subset V_z$, $\gamma cl_b(\{y\}) \subset V_y$ and $V_z \cap V_y = \phi$. This shows that (X,τ) is γ-b-R_1.

5 γ-b-Continuous Functions and γ-b-Closed Graphs

Definition 5.1 *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be γ-b-continuous if for every open set V of Y, $f^{-1}(V)$ is γ-b-open in X.*

Theorem 5.2 *The following are equivalent for a function $f : (X, \tau) \rightarrow (Y, \sigma)$:*

1. *f is γ-b-continuous.*
2. *The inverse image of every closed set in Y is γ-b-closed in X.*
3. *For each subset A of X, $f(\gamma cl_b(A)) \subset cl(f(A))$.*
4. *For each subset B of Y, $\gamma cl_b(f^{-1}(B)) \subset f^{-1}(cl(B))$.*

Proof. (1) $\Leftrightarrow$ (2). Obvious.
(3) $\Leftrightarrow$ (4). Let B be any subset of Y. Then by (3), we have $f(\gamma cl_b(f^{-1}(B))) \subset cl(f(f^{-1}(B))) \subset cl(B)$. This implies $\gamma cl_b(f^{-1}(B)) \subset f^{-1}(cl(B))$.

Conversely, let $B = f(A)$ where A is a subset of X. Then, by (4), we have, $\gamma cl_b(A) \subset \gamma cl_b(f^{-1}(f(A))) \subset f^{-1}(cl(f(A)))$. Thus, $f(\gamma cl_b(A)) \subset cl(f(A))$.
(2) $\Rightarrow$ (4). Let $B \subset Y$. Since $f^{-1}(cl(B))$ is γ-b-closed and $f^{-1}(B) \subset f^{-1}(cl(B))$, then $\gamma cl_b(f^{-1}(B)) \subset f^{-1}(cl(B))$.
(4) $\Rightarrow$ (2). Let $K \subset Y$ be a closed set. By (4), $\gamma cl_b(f^{-1}(K)) \subset f^{-1}(cl(K)) = f^{-1}(K)$. Thus, $f^{-1}(K)$ is γ-b-closed.

Definition 5.3 *For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to be γ-b-closed if for each $(x, y) \notin G(f)$, there exist a γ-b-open set U containing x and an open set V containing y such that $(U \times V) \cap G(f) = \phi$.*

Lemma 5.4 *The function $f : (X, \tau) \rightarrow (Y, \sigma)$ has an γ-b-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $y \neq f(x)$, there exist a γ-b-open set U and an open set V containing x and y respectively, such that $f(U) \cap V = \phi$.*

Proof. It follows readily from the above definition.

Theorem 5.5 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an injective function with the γ-b-closed graph, then X is γ-b-T_1.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Thus there exist a γ-b-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \phi$. Therefore $y \notin U$ and it follows that X is γ-b-T_1.

Theorem 5.6 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an injective γ-b-continuous with a γ-b-closed graph $G(f)$, then X is γ-b-T_2.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) \setminus G(f)$. Since the graph $G(f)$ is γ-b-closed, there exist a γ-b-open set U containing x_1 and open set V containing $f(x_2)$ such that $f(U) \cap V = \phi$. Since f is γ-b-continuous, $f^{-1}(V)$ is a γ-b-open set containing x_2 such that $U \cap f^{-1}(V) = \phi$. Hence X is γ-b-T_2.

Recall that a space X is said to be T_1 if for each pair of distinct points x and y of X, there exist an open set U containing x but not y and an open set V containing y but not x.

Theorem 5.7 *If $f : (X, \tau) \to (Y, \sigma)$ is an surjective function with the γ-b-closed graph, then Y is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists x in X such that $f(x) = y_2$. Therefore $(x, y_1) \notin G(f)$. By Lemma 5.4, there exist γ-b-open set U and an open set V containing x and y_1 respectively, such that $f(U) \cap V = \phi$. We obtain an open set V containing y_1 which does not contain y_2. It follows that $y_2 \notin V$. Hence, Y is T_1.

Definition 5.8 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be γ-b-W-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-b-open set U in X containing x such that $f(U) \subset cl(V)$.*

Theorem 5.9 *If $f : (X, \tau) \to (Y, \sigma)$ is γ-b-W-continuous and Y is Hausdorff, then $G(f)$ is γ-b-closed.*

Proof. Suppose that $(x, y) \notin G(f)$, then $f(x) \neq y$. By the fact that Y is Hausdorff, there exist open sets W and V such that $f(x) \in W$, $y \in V$ and $V \cap W = \phi$. It follows that $cl(W) \cap V = \phi$. Since f is γ-b-W-continuous, there exists a γ-b-open set U containing x such that $f(U) \subset cl(W)$. Hence, we have $f(U) \cap V = \phi$. This means that $G(f)$ is γ-b-closed.

Definition 5.10 *A subset A of a space X is said to be γ-b-compact relative to X if every cover of A by γ-b-open sets of X has a finite subcover.*

Theorem 5.11 *Let $f : (X, \tau) \to (Y, \sigma)$ have a γ-b-closed graph. If K is γ-b-compact relative to X, then $f(K)$ is closed in Y.*

Proof. Suppose that $y \notin f(K)$. For each $x \in K$, $f(x) \neq y$. By lemma 5.4, there exists a γ-b-open set U_x containing x and an open neighbourhood V_x of y such that $f(U_x) \cap V_x = \phi$. The family $\{U_x : x \in K\}$ is a cover of K by γ-b-open sets of X and there exists a fnite subset K_0 of K such that $K \subset \cup\{U_x : x \in K_0\}$. Put $V = \cap\{V_x : x \in K_0\}$. Then V is an open neighbourhood of y and $f(K) \cap V = \phi$. This means that $f(K)$ is closed in Y.

Theorem 5.12 *If $f : (X, \tau) \to (Y, \sigma)$ has a γ-b-closed graph $G(f)$, then for each $x \in X$. $\{f(x)\} = \cap\{cl(f(A) : A$ is γ-b-open set containing $x\}$.*

Proof. Suppose that $y \neq f(x)$ and $y \in \cap\{cl(f(A)) : A$ is γ-b-open set containing $x\}$. Then $y \in cl(f(A))$ for each γ-b-open set A containing x. This implies that for each open set B containing y, $B \cap f(A) \neq \phi$. Since $(x, y) \notin G(f)$ and $G(f)$ is a γ-b-closed graph, this is a contradiction.

Definition 5.13 *A function $f : (X, \tau) \to (Y, \sigma)$ is called a γ-b-open if the image of every γ-b-open set in X is open in Y.*

Theorem 5.14 *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective γ-b-open function with a γ-b-closed graph $G(f)$, then Y is T_2.*

Proof. Let y_1 and y_2 be any two distinct points of Y. Since f is surjective $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y)\backslash G(f)$. This implies that there exist a γ-b-open set A of X and an open set B of Y such that $(x, y_2) \in (A \times B)$ and $(A \times B) \cap G(f) = \phi$. We have $f(A) \cap B = \phi$. Since f is γ-b-open, then $f(A)$ is open such that $f(x) = y_1 \in f(A)$. Thus, Y is T_2.

Theorem 5.15 *If $f : (X, \tau) \to (Y, \sigma)$ is a γ-b-continuous injective function and Y is T_2, then X is γ-b-T_2.*

Proof. Let x and y in X be any pair of distinct points, then there exist disjoint open sets A and B in Y such that $f(x) \in A$ and $f(y) \in B$. Since f is γ-b-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are γ-b-open in X containing x and y respectively, we have $f^{-1}(A) \cap f^{-1}(B) = \phi$. Thus, X is γ-b-T_2.

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