Gen. Math. Notes, Vol. 4, No. 1, May 2011, pp. 59-66
ISSN 2219-7184; Copyright © ICSRS Publication, 2011
www.i-csrs.org
Available free online at http://www.geman.in

Biharmonic Curves in $H^2 \times R$

Talat Körpınar[1] and Essin Turhan[2]

[1]Fırat University, Department of Mathematics, 23119, Elazığ, Turkey
E-mail: talatkorpinar@gmail.com
[2]Fırat University, Department of Mathematics, 23119, Elazığ, Turkey
E-mail: essin.turhan@gmail.com

(Received: 14-5-11/ Accepted: 18-6-11)

Abstract

In this paper, we study biharmonic curves in the $H^2 \times R$. *We show that all of them are helices. By using the curvature and torsion of the curves, we give some characterizations biharmonic curves in the* $H^2 \times R$.

Keywords: *Biharmonic curve, curvature, helices, torsion.*

1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu. In the last decade there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

Let $f:(M,g)\to(N,h)$ be a smooth function between two Riemannian manifolds. The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))dv_g, \tag{1.1}$$

where $\tau(f)=\mathrm{trace}_g\nabla df$ is the tension field of f and dv_g is the volume form of M. Using the first variational formula one sees that f is a biharmonic function if and only if its bitension field vanishes identically, i.e.

$$\tau_2(f):=-\Delta^f(\tau(f))-\mathrm{trace}_g R^N(df,\tau(f))df=0, \tag{1.2}$$

where

$$\Delta^f=-\mathrm{trace}_g(\nabla^f)^2=-\mathrm{trace}_g\left(\nabla^f\nabla^f-\nabla^f_{\nabla^M}\right) \tag{1.3}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z.$$

In this paper we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in the $\mathbb{H}^2\times\mathbb{R}$ must satisfy. Then we prove that the non-geodesic biharmonic curves in the $\mathbb{H}^2\times\mathbb{R}$ are helices. Finally, we deduce the explicit parametric equations of the non-geodesic biharmonic curves in the $\mathbb{H}^2\times\mathbb{R}$.

2 Left Invariant Metric in $\mathbb{H}^2\times\mathbb{R}$

Let $\mathbb{H}^2$ be the upper half-plane model $\{(x,y)\in\mathbb{R}^2 : y>0\}$ of the hyperbolic plane endowed with the metric

$$g_{\mathbb{H}}=\frac{(dx^2+dy^2)}{y^2}$$

of constant Gauss curvature -1. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore the product $\mathbb{H}^2\times\mathbb{R}$ is a Lie group with the left invariant product metric

$$g=\frac{(dx^2+dy^2)}{y^2}+dz^2. \tag{2.1}$$

The left-invariant orthonormal orthonormal frame

$$e_1 = y\frac{\partial}{\partial x},\ e_2 = y\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}. \qquad (2.2)$$

Now let C_{ij}^k be the structure's constants of the Lie algebra g of G that is

$$[e_i, e_j] = C_{ij}^k e_k.$$

The corresponding Lie brackets are

$$[e_1, e_2] = -e_1, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

The Coshul formula for the Levi--Civita connection is:

$$2g(\nabla_{e_i} e_j, e_k) = C_{ij}^k - C_{jk}^i + C_{ki}^j := L_{ij}^k,$$

where the non zero L_{ij}^k 's are

$$L_{12}^1 = -2,\ L_{11}^2 = 2. \qquad (2.3)$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Moreover we put

$$R_{abc} = R(e_a, e_b)e_c,\ R_{abcd} = R(e_a, e_b, e_c, e_d),$$

where the indices a,b,c,d take the values $1,2,3$. Then,

$$R_{121} = e_2,\ R_{1212} = 1. \qquad (2.4)$$

3 Biharmonic Curves in $\mathrm{H}^2 \times \mathrm{R}$

Let $I \subset \mathrm{R}$ be an open interval and $\gamma : I \to \mathrm{H}^2 \times \mathrm{R}$ be a curve, parametrized by arc length, on a Riemannian manifold. Putting $T = \gamma'$, we can write the tension field of γ as $\tau(\gamma) = \nabla_{\gamma'} \gamma'$ and the biharmonic map equation (1.2) reduces to

$$\nabla_T T + R(T, \nabla_T T)T = 0. \qquad (3.1)$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve which is recalled in the following.
Let $\gamma : I \to \mathbb{H}^2 \times \mathbb{R}$ be a curve on $\mathbb{H}^2 \times \mathbb{R}$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to $\mathbb{H}^2 \times \mathbb{R}$ along γ defined as follows: T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T, N, B\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa T - \tau B \\ \nabla_T B &= \tau N, \end{aligned} \tag{3.2}$$

where κ is the curvature of γ and τ its torsion. With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned}$$

Theorem 3.1 $\gamma : I \to \mathbb{H}^2 \times \mathbb{R}$ *is a biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = B_3^2, \\ &\tau' = -N_3 B_3. \end{aligned} \tag{3.3}$$

Proof. From (3.1) we obtain

$$\begin{aligned} &\tau_2(\gamma) = \nabla_T T + R(T, \nabla_T T)T \\ &= (-3\kappa\kappa')T + (\kappa'' - \kappa^3 - \kappa\tau^2 + \kappa R(T, N, T, N))N \\ &+ (-2\kappa'\tau - \kappa\tau' + \kappa R(T, N, T, B))B \\ &= 0. \end{aligned}$$

We see that is a biharmonic curve if and only if

$$\begin{aligned} &\kappa\kappa' = 0, \\ &\kappa'' - \kappa^3 - \kappa\tau^2 + \kappa R(T, N, T, N) = 0, \\ &-2\kappa'\tau - \kappa\tau' + \kappa R(T, N, T, B) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned}&\kappa = \text{constant} \neq 0,\\&\kappa^2 + \tau^2 = R(T,N,T,N),\\&\tau' = R(T,N,T,B).\end{aligned} \tag{3.4}$$

A direct computation using (2.4) yields

$$R(T,N,T,N) = B_3^2, \;\; R(T,N,T,B) = -N_3B_3. \tag{3.5}$$

These, together with (3.4), complete the proof of the theorem.

Corollary 3.2. *Let* $\gamma : I \to \mathbb{H}^2 \times \mathbb{R}$ *be a curve with constant curvature and* $N_3B_3 \neq 0$. *Then,* γ *is not biharmonic.*

Proof. We use the covariant derivatives of the vector fields T, N and B, these, together with equations (3.2) we get

$$\begin{aligned}&T_3' = \kappa N_3,\\&N_3' = -\kappa T_3 - \tau B_3,\\&B_3' = \tau N_3.\end{aligned} \tag{3.6}$$

Assume now that γ is biharmonic. Then $\tau' = -N_3B_3 \neq 0$ and (3.3) we obtain

$$\tau\tau' = B_3B_3'.$$

Since $\tau' = -N_3B_3$, this is rewritten as

$$\tau = -\frac{B_3'}{N_3}. \tag{3.7}$$

From (3.6) we have

$$B_3' = \tau N_3. \tag{3.8}$$

If we substitue B_1' in equation (3.7)

$$\tau = 0.$$

Therefore also τ is constant and we have a contradiction. Complete the proof of the corollary.

By using Theorem 3.1 and Corollary 3.2, we have the following corollary:

Corollary 3.3. $\gamma : I \to \mathbb{H}^2 \times \mathbb{R}$ *is biharmonic if and only if*

$$\kappa = \text{constant} \neq 0,$$

$$\tau = \text{constant},$$
$$N_3 B_3 = 0,$$
$$\kappa^2 + \tau^2 = B_3^2.$$

Corollary 3.4.

(i) If $N_3 \neq 0$ then γ is not biharmonic.

(ii) If $N_3 = 0$, then

$$T(s) = \sin\alpha_0 \cos\beta(s) e_1 + \sin\alpha_0 \sin\beta(s) e_2 + \cos\alpha_0 e_3, \qquad (3.9)$$

where $\alpha_0 \in \mathbb{R}$.

Proof.

(i) Using the above Corollary 3.3 it is easy to see that γ is not biharmonic.

(ii) Since γ is s parametrized by arc length, we can write

$$T(s) = \sin\alpha \cos\beta(s) e_1 + \sin\alpha \sin\beta(s) e_2 + \cos\alpha e_3. \qquad (3.10)$$

From (3.6) we obtain

$$T_3' = \kappa N_3.$$

Since $N_3 = 0$

$$T_3' = 0.$$

Then T_3 is constant. We use (3.10), we have

$$T_3 = \cos\alpha_0 = \text{constant}.$$

Theorem 3.5. *The parametric equations of all biharmonic curves of $\mathbb{H}^2 \times \mathbb{R}$,*

$$x(s) = c_2 \sin\alpha_0 \int \sin\beta(s) e^{\sin\alpha_0 \int \cos\beta(s) ds} ds + c_1,$$
$$y(s) = c_2 e^{\sin\alpha_0 \int \cos\beta(s) ds}, \qquad (3.11)$$
$$z(s) = s\cos\alpha_0 + c_3,$$

where c_1, c_2, c_3 is arbitrary constants.

Proof. We note that

$$\frac{d\gamma}{ds} = T(s) = T_1 e_1 + T_2 e_2 + T_3 e_3,$$

and our left-invariant vector fields are

$$e_1 = y\frac{\partial}{\partial x},\ e_2 = y\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}$$

then

$$\frac{d\gamma}{ds} = (y(s)T_1, y(s)T_2, T_3).$$

By using equation (3.9), we have

$$\frac{d\gamma}{ds} = T(s) = (y(s)\sin\alpha_0 \cos\beta(s), y(s)\sin\alpha_0 \sin\beta(s), \cos\alpha_0).$$

In order to find the explicit equations for $\gamma(s) = (x(s), y(s), z(s))$, we must integrate the system $\frac{d\gamma}{ds} = T(s)$, that in our case is

$$\frac{dx}{ds} = y(s)\sin\alpha_0 \cos\beta(s),$$
$$\frac{dy}{ds} = y(s)\sin\alpha_0 \sin\beta(s),$$
$$\frac{dz}{ds} = \cos\alpha_0.$$

The integration is immediate and yields (3.11).

Corollary 3.7. *If $\beta(s) = s$, then the parametric equations of all biharmonic curves are*

$$\begin{aligned} x(s) &= e^{-\sin\alpha_0 \cos s}(\sin\alpha_0 \cos s + 1) + c_1, \\ y(s) &= c_2 e^{-\sin\alpha_0 \cos s}, \\ z(s) &= s\cos\alpha_0 + c_3, \end{aligned} \tag{3.12}$$

where c_1, c_2, c_3 is arbitrary constants.

The picture of biharmonic curve $\gamma(s)$ at $c_1 = c_2 = c_3 = 1$:

4 Conclusion

In this work, we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in the $\mathsf{H}^2 \times \mathsf{R}$ must satisfy. Then we prove that the non-geodesic biharmonic curves in the $\mathsf{H}^2 \times \mathsf{R}$ are helices. Finally, we deduce the explicit parametric equations of the non-geodesic biharmonic curves in the $\mathsf{H}^2 \times \mathsf{R}$.

References

[1] D.E. Blair, Contact manifolds in Riemannian geometry, *Lecture Notes in Mathematics*, Springer-Verlag 509, Berlin-New York, (1976).

[2] R. Caddeo and S. Montaldo, Biharmonic submanifolds of $\mathbb{S}^3$, *Internat. J. Math.*, 12(8) (2001), 867-876.

[3] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds of $\mathbb{S}^n$, *Israel J. Math.*, (To Appear).

[4] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, New Jersey (1976).

[5] B.Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.*, 17(1991), 169-188.

[6] J.P. Cleave, The form of the Tangent-developable at points of zero torsion on space curves, *Math. Proc. Camb. Phil. Soc*, (1980).

[7] J. Eells and L. Lemaire, A report on harmonic maps, *Bull. London Math. Soc.*, 10 (1978), 1-68.

[8] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, 86 (1964), 109-160.

[9] T. Hasanis and T. Vlachos, Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, *Math. Nachr.*, 172(1995), 145-169.

[10] G.Y. Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser. A*, 7(2) (1986), 130-144.

[11] T. Körpınar and E. Turhan, On horizontal biharmonic curves in the Heisenberg group $Heis^3$, Arab. *J. Sci. Eng. Sect. A Sci.*, 35(1) (2010), 79-85.

[12] T. Lamm, Biharmonic map heat flow into manifolds of nonpositive curvature, *Calc. Var.*, 22(2005), 421-445.

[13] E. Loubeau and C. Oniciuc, On the biharmonic and harmonic indices of the Hopf map, *Transactions of the American Mathematical Society*, 359(11) (2007), 5239--5256.

[14] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[15] I. Sato, On a structure similar to the almost contact structure, *Tensor, (N.S.)*, 30 (1976), 219-224.

[16] T. Takahashi, Sasakian ϕ-symmetric spaces, *Tohoku Math. J.*, 29(1977), 91-113.

[17] E. Turhan and T. Körpınar, Characterize on the Heisenberg group with left invariant Lorentzian metric, *Demonstratio Mathematica*, 42(2) (2009), 423-428.

[18] E. Turhan and T. Körpınar, On characterization of timelike horizontal biharmonic curves in the Lorentzian Heisenberg group Heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 65a(2010), 641-648.

[19] E. Turhan and T. Körpınar, The Weierstrass representation for minimal immersions in the Lie group $\mathrm{H}_3\times\mathbb{S}^1$, *J. Adv. Math. Studies*, (In Press).

[20] E. Turhan and T. Körpınar, Position vectors of spacelike biharmonic curves with spacelike bihormal in Lorentzian Heisenberg group Heis^3, *Int. J. Open Problems Compt. Math.*, 3(3) (2010), 413-422.