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Banach Type Fixed Point Theorem for Set Valued Maps on a Fuzzy Metric Space

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Abstract

In this paper, we introduce the notion of boundedness in a fuzzy metric space. This notion is a special case of the notion of F-boundedness due to [1]. Using this notion, we prove a fixed point theorem for maps from a fuzzy metric space to a sub class of the set of bounded subsets of that fuzzy metric space, under Hadzic type t −norm. The proof of our theorem is non-constructive.

Keywords: *Fuzzy metric space, F-boundedness, bounded subset of Fuzzy metric space, Hadzic type t- norm, set-valued M-maps.*

1 Introduction

The concept of fuzzy sets was introduced by L.A. Zadeh [8] in 1965 which became an active field of research for many researchers in fixed point theory. In 1975, Karmosil and Michalek [3] introduced the concept of a fuzzy metric space based on fuzzy sets; this notion was further modified by George and Veermani [1] with the help of t-norms. Many authors made use of the definition of a fuzzy metric space due to George and Veermani [1] in proving fixed point theorems in fuzzy metric spaces. In 1978, Hadzic [2] introduced a class $\mathcal{H}$ of t– norms later known as Hadzic type t-norms. K.P.R. Sastry, G.V.R. Babu, and M.L. Sandhya [5] extended the notion of weak contraction in metric spaces and obtained a fixed point theorem in Menger spaces with Hadzic type t −norm. K.P.R. Rao et al. [4] introduced the notion of M-maps with respect to single map and a pair of maps in fuzzy metric spaces and obtained common fixed point theorems for two pairs of sub compatible maps satisfying implicit relations.

In this paper we prove a fixed point theorem for maps from a fuzzy metric space X to $\mathfrak{B}(X)$, the set of all nonempty bounded subsets of the fuzzy metric space $(X, M,*)$ where $*$ is Hadzic type t −norm.

2 Preliminaries

We begin with some known definitions and results.

Definition 1.1: (L.A. Zadeh [8]) *A fuzzy set A in a nonempty set X is a function with domain X and values in* $[0,1]$.

Definition 1.2: (B. Schweizer and A. Sklar [7]) *A function* $*: [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous t-norm if* $*$ *satisfies the following conditions:*

For $a, b, c, d \in [0,1]$

(i) $*$ *is commutative and associative*
(ii) $*$ *is continuous*
(iii) $a * 1 = a \ \forall a \in [0,1]$
(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$

Examples of continuous t-norm are $a*b = ab$ *and* $a*b = \min\{a,b\}$.

Definition 1.3: (A. George and P. Veeramani [1]) *A triple* $(X, M,*)$ *is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set,* $*$ *is a continuous t-norm and M is a fuzzy set on* $X^2 \times [0,\infty)$ *satisfying the following conditions:*

For $x, y, z \in X$ *and* $s, t > 0$.

(i) $M(x,y,t) > 0, M(x,y,0) = 0$
(ii) $M(x,y,t) = 1 \; \forall \; t > 0$ *if and only if* $x = y$
(iii)$M(x,y,t) = M\,(y,x,t)$
(iv)$M(x,y,t) * M(y,z,s) \leq M(x,z,t + s)$
(v) $M(x,y,\ .): [0,\infty) \longrightarrow [0,1]$ *is continuous and* $lim_{t\to\infty} M(x,y,t) = 1.$

Then M *is called a fuzzy metric space on X.*

The function $M(x,y,t)$ *denotes the degree of nearness between* x *and* y *with respect to t.*

Definition 1.4: (A. George and P. Veeramani [1]) *Let* $(X, M,*)$ *be a fuzzy metric space. Then,*

(i) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *if* $lim_{n\to\infty} M(x_n\,, x, t) = 1 \;\forall t > 0.$
(ii) *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if* $lim_{n\to\infty} M\big(x_{n+p}, x_n, t\big) = 1 \;\forall t > 0$ *and* $p = 1,2,\ldots$
(iii)A FM –space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.5: (A. George and P. Veeramani [1]) *Let* $(X, M,*)$ *be a fuzzy metric space. Then, a subset A of X is said to be F-bounded if* $\exists\, t > 0$ and $r \in (0,1) \ni M(x,y,t) > 1 - r \;\; \forall x, y \in A$

Let $\mathfrak{B}(X)$ *be the set of F-nonempty bounded subsets of fuzzy metric space*$(X, M,*)$.

K.P.R. Rao et al. [4] Introduced the following, using the notion of F-boundedness due to [1].

For $A, B \in \mathfrak{B}(X)$ and for every $\mathrm{t} > 0$, define

$$\delta_M(A,B,t) = \inf\{M(a,b,t) / a \in A, b \in B\}$$

Using this notation, K.P.R. Rao et al. [4] Introduced the notion of M-map.

Definition 1.6: (K.P.R. Rao, K.R.K. Rao and V.C.C. Raju [4]) *Let* $(X, M,*)$ *be a fuzzy metric space and* $f : X \to X$ *and* $F : X \to \mathfrak{B}(X)$. *Then* (f, F) *is said to be a pair of M-maps with respect to* f *if there exists a sequence* $\{x_n\}$ *in* X *such that for every* $t > 0$, $M(fx_n, z, t) \to 1$ *and* $\delta_M(Fx_n, \{z\}, t) \to 1$ *as* $n \to \infty$ *for some* $z \in f(X)$.

Implicit Relation:

Let Φ denote the class of all continuous functions $\varphi:[0,1]^6 \to R_+$ satisfying $\varphi(u,1,1,v,v,1) \geq 0$ or $\varphi(u,1,v,1,1,v) \geq 0$ or $\varphi(u,v,1,1,v,v) \geq 0$ implies $u \geq v$.

Using the above implicit relation K.P.R.Rao et.al. [4] proved the following theorem.

Theorem 1.7: (K.P.R. Rao, K.R.K. Rao and V.C.C. Raju [4]) *Let $(X, M,*)$ be a fuzzy metric space and $f, g : X \to X$ and $F : X \to \mathfrak{B}(X)$ be maps satisfying*

$$\varphi\begin{pmatrix} \delta_M(Fx, Gy, kt), M(gx, gy, t), \delta_M(fx, Fx, t) \\ \delta_M(gy, Gy, t), \delta_M(fx, Gy, t), \delta_M(gy, Fx, t) \end{pmatrix} \geq 0$$

for all $x, y \in X$ and $k \in (0,1)$ where $\varphi \in \Phi$, the pairs (f, F) and (g, G) are compatible,

(a) (f, F) is a pair M-maps with respect to f and $Fx \subseteq g(X) \forall x \in X$

Or

(b) (g, G) is a pair M-maps with respect to g and $Gx \subseteq f(X) \forall x \in X$.

Then $f, g, F,$ and G have a unique common fixed point $z \in X$ such that $Fz = Gz = \{z\} = \{fz\} = \{gz\}$

Definition 1.8: (O. Hadzic[2]) *Let $*$ be a t- norm. For any $\in [0,1]$, write $*^0(x) = 1$ and $*^1(x) = *(*^0(x), x) = *(1, x) = x$.*

In general recursively define $*^{n+1}(x) = *(*^n(x), x)$, for $n = 0,1,2$ ...

Suppose that given ε in $(0,1)$ $\exists\ \delta \in (0,1)$ $\ni x > 1 - \delta \Rightarrow *^n(x) > 1 - \varepsilon\ \forall n \in N$

Then the sequence $\{*^n\}$ is said to be equicontinuous at 1.

If $\{*^n\}$ is equicontinuous at 1, then we say that $*$ is a Hadzic type t- norm.

Define t_{min} by $t_{min}(a, b) = min\{a, b\}$ for $a, b \in [0,1]$. Then we observe that t_{min} is a continuous t- norm of Hadzic type.

We make use of the following Lemma due to K.P.R. Sastry, G.A. Naidu and N. Umadevi [6]. See also [5].

Lemma 1.9: *Let*$(X, M,*)$ *be a fuzzy metric space with continuous Hadzic type* t *−norm* $*$. *Assume that. Suppose*$0 < a < 1$ *and* $\{x_n\}$ *is a sequence in* X *such that* $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{a})$ *for* $n = 1,2, \dots \dots \dots$

Then $\{x_n\}$ *is a Cauchy sequence in* X.

3 Main Result

In this section, we first introduce the notion of boundedness in a fuzzy metric space, which is a particular case of F-boundedness.

Definition 2.1: *Let A be a nonempty subset of fuzzy metric space*$(X, M,*)$. *Then A is said to be bounded if* $inf\{M(a, b, t)/a, b \in A\} > 0 \ \forall \ t > 0$.

Clearly this notion is a special case of F-boundedness defined by George and Veeramani [1].

Thus every bounded set is F- bounded.

However, a F-bounded set may not be bonded in view of the following example.

Example 2.2 (i): Let *H* be the Heaviside function defined by

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Let $X = [0,1]$

Define $M(x, y, t) = H(t - |x - y|) \forall x, y \in X$. Then X is F-bounded in$(X, M,*)$, $*$ being minimum *t*-norm. But *X* is not bounded since $\inf M(x, y, t) = 0$ if $t < 1$.

Example 2.2 (ii): Let $X = [0,1]$, $M(x, y, t) = \dfrac{t}{t + |x - y|} \forall x, y \in X$. Then it can be easily verified that *X* is bounded.
We observe the following:

Observation 2.3: Every finite set is bounded.
Observation 2.4: If *A* and *B* are bounded, then $A \cup B$ is bounded.
Observation 2.5: If *A* is bounded, $B \subseteq A$ then *B* is bounded.
Definition 2.6: Let $\mathfrak{B}(X)$ denote the set of all nonempty bounded subsets of the fuzzy metric space$(X, M,*)$.

For $A, B \in \mathfrak{B}(X)$ and for every $t > 0$, define

$$\delta_M\left(A,B,t\right)=\inf\left\{M\left(a,b,t\right)/\,a\in A,b\in B\right\}$$

We observe that

Observation 2.7: If $A=\{a\}$, then $\delta_M\left(A,B,t\right)=\delta_M\left(a,B,t\right)$

Observation 2.8: If $A=\{a\},B=\{b\}$, then $\delta_M\left(A,B,t\right)=M\left(a,b,t\right)$.

Observation 2.9: $\delta_M\left(A,B,t\right)=\delta_M\left(B,A,t\right)>0$

Observation 2.10: $\delta_M\left(A,B,t\right)=1\,\forall\, t>0\Leftrightarrow A=B=\left\{\text{singleton}\right\}$

Proof: Let $a\in A$ and $b\in B$. Then

$$M\left(a,b,t\right)\geq\delta_M\left(A,B,t\right)=1\,\forall\, t>0$$
$$\Rightarrow M\left(a,b,t\right)=1\,\forall\, t>0$$
$$\Rightarrow a=b$$
$$\therefore a\in A,b,c\in B\Rightarrow a=b \text{ and } a=c$$
$$\Rightarrow c=b$$

Thus, B is a singleton set.

Similarly A is a singleton set and $A=B$.

Conversely, $A=B=\text{singleton set}\left\{a\right\}$

$$\Rightarrow\quad M\left(a,a,t\right)=1\,\forall\, t>0$$
$$\Rightarrow\delta_M\left(A,B,t\right)=1\,\forall\, t>0$$

Observation 2.11: $\delta_M\left(A,B,t+s\right)\geq\delta_M\left(A,C,t\right)*\delta_M\left(C,B,s\right)\forall A,B,C\in\mathfrak{B}\left(X\right)$ and $s,t>0$.

Proof: Let $a\in A$, $b\in B$, and $c\in C$. Then we have

$$M\left(a,b,t+s\right)\geq M\left(a,c,t\right)*M\left(c,b,s\right)$$
$$\geq\delta_M\left(A,C,t\right)*\delta_M\left(C,B,s\right)>0$$
$$\inf\left\{M\left(a,b,t+s\right)/\,a\in A,b\in B\right\}\geq\delta_M\left(A,C,t\right)*\delta_M\left(C,B,s\right)>0$$
$$\delta_M\left(A,B,t+s\right)\geq\delta_M\left(A,C,t\right)*\delta_M\left(C,B,s\right)\forall A,B,C\in\mathfrak{B}\left(X\right) \text{ and } s,t>0.$$

Observation 2.12: $\delta_M\left(A,B,t\right)$ is an increasing function in *t*.

Proof: In order to prove $\delta_M\left(A,B,t\right)$ is increasing, we prove that

$$\delta_M(A,B,t) \geq \delta_M(A,B,s) \forall t \geq s$$

We have

$$M(a,b,t) \geq M(a,b,s) \geq \delta_M(A,B,s)$$
$$\Rightarrow M(a,b,t) \geq \delta_M(A,B,s) \;\; \forall a \in A, b \in B$$
$$\Rightarrow \inf\{M(a,b,t) / a \in A, b \in B\} \geq \delta_M(A,B,s)$$
$$\Rightarrow \delta_M(A,B,t) \geq \delta_M(A,B,s) \;\forall\; t \geq s$$

$\therefore \delta_M(A,B,t)$ is increasing in *t*.

Definition 2.13: *A sequence* $\{A_n\}$ *in* $\mathscr{B}(X)$ *is said to converge to* $a \in X$ *if* $\delta_M(A_n,a,t) \to 1 \, as \;\; n \to \infty \;\forall\; t > 0$

Lemma 2.14: *If* $\{A_n\} \to a$ *and* $\{A_n\} \to b$, *then* $a = b$.

Proof: $\{A_n\} \to a$ and $\{A_n\} \to b$

$\Rightarrow \delta_M(A_n,a,t) \to 1$ and $\delta_M(A_n,b,t) \to 1$ as $n \to \infty \;\forall t > 0$ (2.14.1)

We have, from observation (2.11)

$$\delta_M(a,b,t) \geq \delta_M\left(a, A_n, \frac{t}{2}\right) * \delta_M\left(A_n, b, \frac{t}{2}\right)$$

$\therefore$From (2.14.1), $\delta_M(a,b,t) \to 1$ as $n \to \infty \;\forall t > 0$

$\therefore a = b$

Lemma 2.15: *Let* $\{A_n\}$ *and* $\{S_n\}$ *be sequences in* $\mathscr{B}(X)$ *converging to* $z \in X$. *Then* $\lim_{n\to\infty} \delta_M(A_n,S_n,t) = 1$.

Proof: $\{A_n\} \to z$ and $\{S_n\} \to z$

$\Rightarrow \delta_M(A_n,z,t) \to 1$ and $\delta_M(S_n,z,t) \to 1$ as $n \to \infty \;\forall t > 0$ (2.15.1)

We have, from observation (2.11)

$$\delta_M(A_n,S_n,t) \geq \delta_M\left(A_n, z, \frac{t}{2}\right) * \delta_M\left(z, S_n, \frac{t}{2}\right)$$

$\therefore$ From (2.15.1), $\delta_M\left(A_n, S_n, t\right) \to 1$ as $n \to \infty$ $\forall t > 0$

$\therefore \lim\limits_{n\to\infty} \delta_M\left(A_n, S_n, t\right) = 1$

Definition 2.16: *Let* $A, B \in \mathscr{B}(X)$. *We say that* $\delta_M\left(A, B, t\right)$ *is attained if there exists* $a \in A, b \in B$ *such that* $M\left(a, b, t\right) = \delta_M\left(A, B, t\right)$.

In example 2.2(ii), $\delta_M\left(X, X, t\right)$ is attained (Take $a = 0$ and $b = 1$). Also $\delta_M\left(A, A, t\right)$ is not attained where $A = [0,1)$.

Lemma 2.17: *Suppose*

(2.17.1) $A, B \in \mathfrak{B}(X)$

(2.17.2) $\delta_M\left(A, B, t\right)$ *is attained, i.e., there exist* $a \in A, b \in B$ *depending on t such that*

$$\delta_M\left(A, B, t\right) = M\left(a, b, t\right).$$

(2.17.3) $\exists k \in (0,1) \ni \delta_M\left(A, B, kt\right) \geq \delta_M\left(A, B, t\right)$

Then $A = B = \text{singleton set}$.

Proof: From (2.17.3), we have

$\delta_M\left(A, B, kt\right) \geq \delta_M\left(A, B, t\right) \geq \delta_M\left(A, B, kt\right)$ (Since $\delta_M\left(A, B, t\right)$ is increasing in *t*, by (2.12))

$\Rightarrow \delta_M\left(A, B, kt\right) = \delta_M\left(A, B, t\right)$
$= \lambda > 0$, a constant, say

Let $a \in A, b \in B$ be such that $M\left(a, b, t\right) = \lambda$
Let $x \in A, y \in B$ be such that $M\left(x, y, kt\right) = \delta_M\left(A, B, kt\right)$
Then, by (2.17.3), $M\left(x, y, kt\right) = M\left(a, b, t\right)$

Now,

$$M\left(a, b, t\right) = M\left(x, y, kt\right) = \delta_M\left(A, B, kt\right) \leq M\left(a, b, kt\right)$$
$$\Rightarrow M\left(a, b, kt\right) \geq M\left(a, b, t\right)$$

Since *M* is increasing in *t*, so

$$M(a,b,t) \geq M(a,b,kt) \geq M(a,b,t)$$
$$\Rightarrow M(a,b,kt) = M(a,b,t) = \lambda$$
$$\therefore M(a,b,kt) = \lambda = \delta_M(A,B,kt)$$

In a similar way, we can prove that

$$M\left(a,b,\frac{t}{\lambda}\right) = M(a,b,t)$$

Hence,

$$\lambda = M(a,b,t) = M(a,b,kt) \; \forall\; t > 0$$

But from (v) of the definition of fuzzy metric space, $M(a,b,t) \to 1$ as $n \to \infty$

$\therefore \lambda = 1$

$\therefore \delta_M(A,B,t) = \lambda = 1 \; \forall \; t > 0$

$\therefore A = B = \{\text{singleton}\}$ (from Observation 2.10)

Before we state and prove our main result, first we introduce a subclass of $\mathcal{B}(X)$ in analogy with compact subsets of a metric space.

Definition 2.18: *Let $\mathcal{C}(X)$ be a subclass of $\mathcal{B}(X)$ such that*

(2.18.1) $\{x\} \in \mathcal{C}(X) \; \forall x \in X$

(2.18.2) $A, B \in \mathcal{C}(X) \Rightarrow \delta_M(A,B,t)$ *is attained*

Now, we state and prove our main result. This theorem may be regarded as an extension of Banach contraction principle to set valued maps on a fuzzy metric space. The proof we have given is a non-constructive proof.

Theorem 2.19: *Let $(X, M, *)$ be any complete fuzzy metric space where $*$ is continuous t-norm of Hadzic type and $\mathcal{B}(X)$ be the set of all nonempty bounded subsets of X. Let $\mathcal{C}(X)$ be a subclass of $\mathcal{B}(X)$ such that*

(2.19.1) $\{x\} \in \mathcal{C}(X) \; \forall x \in X$

(2.19.2) $A, B \in \mathcal{C}(X) \Rightarrow \delta_M(A,B,t)$ is attained.

Let $S : X \to \mathcal{C}(X)$ such that

(2.19.3) $\exists \; k \in (0,1) \ni \delta_M(Sx, Sy, kt) \geq \delta_M(x, y, t) \forall x, y \in X \;, x \neq y$

Then, S has unique fixed point in X.

Proof: Suppose S has no fixed point, that is, $a \notin Sa \, \forall \, a \in X$ (2.19.4)
Let $x_0 \in X$, then $x_0 \notin Sx_0$.

Since $x_0 \notin Sx_0$ and $\{x_0\} \in \mathcal{C}(X)$ (by 2.19.1)
$\exists x_1 \in Sx_0 \ni \delta_M\left(x_0, Sx_0, kt\right) = M\left(x_0, x_1, kt\right)$ (by 2.19.2)

Clearly $x_1 \neq x_0$ (by 2.19.4)
$\therefore \ \delta_M\left(Sx_0, Sx_0, kt\right) \geq \delta_M\left(x_0, x_1, t\right)$ (by 2.19.3)

Also, since $x_1 \notin Sx_1$ and $\{x_1\} \in \mathcal{C}(X)$ (by 2.19.1)
$\exists x_2 \in Sx_1 \ni \delta_M\left(x_1, Sx_1, kt\right) = M\left(x_1, x_2, kt\right)$ (by 2.19.2)

Since $x_1 \neq x_2$ (by 2.19.4)
$\delta_M\left(Sx_1, Sx_2, kt\right) \geq \delta_M\left(x_1, x_2, t\right)$ (by 2.19.3)

Therefore

$$\begin{aligned} M\left(x_1, x_2, kt\right) &= \delta_M\left(x_1, x_2, kt\right) \\ &\geq \delta_M\left(x_1, Sx_1, kt\right) \quad \left(\because x_2 \in Sx_1\right) \\ &\geq \delta_M\left(Sx_0, Sx_1, kt\right) \left(\because x_1 \in Sx_0\right) \\ &\geq M\left(x_0, x_1, t\right) \qquad \left(\because x_1 \neq x_2, \text{by}(2.19.3)\right) \end{aligned}$$

$\therefore M\left(x_1, x_2, kt\right) \geq M\left(x_0, x_1, t\right)$

Similarly we can prove that, $\exists x_3 \in Sx_2$, (hence $x_3 \neq x_2$) such that

$$M\left(x_2, x_3, kt\right) \geq M\left(x_2, x_3, t\right)$$

In this way we can construct a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1}, x_n \in Sx_{n+1}$ and
$M\left(x_{n+1}, x_{n+2}, kt\right) \geq M\left(x_n, x_{n+1}, t\right)$, for $n = 1,2, ...$ (2.19.5)

From (2.19.5), we have

$$M\left(x_{n+1}, x_{n+2}, t\right) \geq M\left(x_n, x_{n+1}, \frac{t}{k}\right)$$

Now using Lemma 1.9, $\{x_n\}$ is a Cauchy sequence-

Since X is complete, $\{x_n\} \to x \in X$

Let N be any positive integer.

Now, suppose $x \neq x_n \; \forall n \geq N$. (2.19.6)

Then

$$\delta_M(Sx, x_{n+1}, kt) \geq \delta_M(Sx, Sx_n, kt) \quad (\because x_{n+1} \in Sx_n)$$
$$\geq M(x, x_n, t) \quad (\because x \neq x_n, \text{by}(2.19.3))$$

Letting $n \to \infty$, we get

$$\delta_M(Sx, x_{n+1}, kt) \to 1 \tag{2.19.7}$$

Let $y \in Sx$. Then, for $n \geq N$,

$$\delta_M(y, x_{n+1}, kt) \geq \delta_M(Sx, x_{n+1}, kt) \quad (\because y \in Sx)$$
$$\to 1 \text{ as } n \to \infty \text{ (by 2.19.7)}$$

$\therefore \delta_M(y, x_{n+1}, kt) \to 1$
$\therefore x_{n+1} \to y$
$\therefore x = y$ (by Lemma 2.14)
$\therefore x \in Sx(\because x = y \text{ and } y \in Sx)$, which is a contradiction to (2.19.4).

This contradiction arose due to our supposition (2.19.6). Hence (2.19.6) is not valid.

Hence $\exists n \geq N \ni x = x_n$.
Consequently there exists a strictly increasing sequence $\{n_l\} \ni x = x_{n_l}, l = 1, 2, 3, \ldots$

$$\delta_M(Sx, x, kt) = \delta_M(Sx_{n_1}, x_{n_l}, kt) \quad (\because x = x_{n_1}, x = x_{n_l})$$
$$\geq \delta_M(Sx_{n_1}, Sx_{n_l-1}, kt) \quad (\because x_{n_l} \in Sx_{n_l-1})$$
$$\geq M(x_{n_1}, x_{n_l-1}, t) \quad (\because x_{n_1} \neq x_{n_l-1}, \text{by}\,(2.19.3))$$
$$\therefore \delta_M(Sx, x, kt) \geq M(x_{n_1}, x_{n_l-1}, t)$$

On letting $l \to \infty$

$$M(x_{n_1}, x_{n_l-1}, t) \to M(x_{n_1}, x, t) = 1(\because x = x_{n_1})$$

Therefore

$$\delta_M\left(Sx, x, kt\right) = 1 \qquad (2.19.8)$$

Let $y \in Sx$. Then,

$$\delta_M\left(y, x, kt\right) \geq \delta_M\left(Sx, x, kt\right) \quad \left(\because y \in Sx\right)$$
$$= 1 \quad \left(\text{by } 2.19.8\right)$$

$\Rightarrow \delta_M\left(y, x, kt\right) = 1$
$\therefore x = y$

$\therefore x \in Sx\left(\text{since } x = y \text{ and } y \in Sx\right)$, which is a contradiction to (2.19.4).

Hence, $a \notin Sa \,\forall a \in X$ is false.
Therefore there exists $a \in X \ni a \in Sa$.
Hence, *S* has a fixed point in *X*.

Uniqueness: Suppose *x, y* is fixed points of *S*.
Then $x \in Sx$ and $y \in Sy$

Suppose, $x \neq y$. Then

$$M\left(x, y, kt\right) = \delta_M\left(Sx, Sy, kt\right) \geq \delta_M\left(x, y, t\right)\left(\text{by } 2.19.3\right)$$
$$= M\left(x, y, t\right) \forall t > 0$$

$\therefore M\left(x, y, kt\right) = M\left(x, y, t\right) \;\forall\; t > 0$

$\therefore$ By definition 1.3(v), $x = y$.

Hence, fixed point of *S* is unique.

Note: The proof we have given to Theorem 2.19 is non-constructive in nature, since we assumed (2.19.4) and arrived at a contradiction.

Now, we give an example to support our result.

Example 2.20: Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$.

Define $M\left(x, y, t\right) = \dfrac{t}{t + \left|x - y\right|}$ and $*$ be the minimum *t*-norm.

Define $S : X \to \mathcal{C}\left(X\right)$ by

$$S\left(\frac{1}{2^n}\right)=\left\{\frac{1}{2^{n+1}},\frac{1}{2^{n+2}}\right\}, n=0,1,2,\ldots \text{ and } S0=0$$

Then it can be easily verified that (2.19.3) holds, for $k=\frac{3}{4}$.

Further, 0is the unique fixed point of *S*.

Observation 2.21: It may be observed that in the above example, (2.19.3) does not hold if $x=y$.

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