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The Strongly Summable Generalized Difference Double Sequence Spaces in 2-Normed Spaces Defined by an Orlicz Functions

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Abstract

The main aim of this paper is to introduce a new class of sequence spaces namely $_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, p, \|.,.\|]_\sigma$ *where* $\sigma = 0, 1, \infty$, *using the concept of 2-norm and the notion of de la Valee-Pousin means when* $A = (a_{m,n,j,k}), j, k = 0, 1, ...$ *is a doubly infinite matrix of real numbers for all* $m, n = 0, 1, ...$ *To construct these spaces we use an Orlicz function, a bounded sequence of positive real numbers and a generalized difference operator which was introduced by Dutta[2]. We obtain various inclusion relations involving these sequence spaces.*

Keywords: *Double sequence Spaces, Difference operator, de la Valee-Pousin mean, 2-normed space, Orlicz Function.*
[1]

1 Introduction

The concept of 2-normed spaces was initially introduced by G̈ahler[5] in the mid of 1960's. Since then, many researchers have studied this concept and

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obtained various results, see for instance [6,7,8].
Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies the following four conditions (see[9,13]):

(i) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent;

(ii) $\|x_1, x_2\| = \|x_2, x_1\|$:

(iii) $\|\alpha x_1, x_2\| = |\alpha|\|x_1, x_2\|$, for any $\alpha \in R$:

(iv) $\|x + x', x_2\| \leq \|x, x_2\| + \|x', x_2\|$
The pair $(X, \|.,.\|)$ is then called a 2-normed space.

Example 1.1. A standard example of a 2-normed space is R^2 equipped with the following 2-norm
$\|x, y\| :=$ the area of the triangle having vertices $0, x, y$.

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively normed by

$$\|x\| = \sup_k |x_k|.$$

Kizmaz [14], defined the difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.
The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k \|\Delta x_k\|.$$

The notion of difference sequence spaces was generalized by Et. and Colak[4] as follows:

$$Z(\Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $n \in \mathbb{N}, (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and so that

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

In 2005, Tripathy and Esi [19], introduced the following new type of difference sequence spaces:

$$Z(\Delta_m) = \{x = (x_k) \in w : \Delta_m x \in Z\}, \text{ for } Z = l_\infty, c \text{ and } c_0$$

where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$, for all $k \in \mathbb{N}$.

Later on Tripathy, Esi and Tripathy[20], generalized the above notions and unified these as follows:
Let m, n be non negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

where

$$\Delta_m^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+mi}$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et an Colak[4]. Taking $n = 1$, we get the spaces $l_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi[18]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz[14].

Let $v = v_k$ be sequence of non-zero scalars. Also let $Z = \{l_\infty, c, c_o\}$. Recently Dutta[2] defined the following sequence spaces

$$Z(\Delta_{(mv)}^n x_k) = \{x = (x_k) \in w : (\Delta_{mv}^n x_k) \in Z\},$$

where $(\Delta_{(mv)}^n x_k) = (\Delta_{(mv)}^{n-1} x_k - \Delta_{(mv)}^{n-1} x_{k-m})$ and $\Delta_{(mv)}^o x_k = v_k x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_{(mv)}^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} v_{k-mi} x_{k-mi}$$

Let $\lambda = (\lambda_r)$ be a non decreasing sequences of positive real numbers both of which tending to ∞, and $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 0$. The generalized de la Valee-Pousin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k$$

where
$I_r = [r - \lambda_r + 1, r]$ (see[3]).

A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L if $t_r(x) \to L$ as $r \to \infty$. If $\lambda_r = r$, then $(V, \lambda)-$summability reduced to

$(C,1)-$summability. We write

$$[V,\lambda]=\{x=(x_k):\lim_{r\to\infty}\frac{1}{\lambda_r}\sum_{k\in I_r}|x_k-L|=0 \ \text{ for some } L\}$$

for sets of sequences $x=(x_k)$ which are strongly $(V,\lambda)-$ summable to L.

Subsequently strongly $(V,\lambda)-$ summable as well as generalized kind of summable sequence spaces have been studied by various authors[1,18].

An *Orlicz Function* is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, nondecreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$, as $x\to\infty$.

An Orlicz function M satisfies the Δ_2- *condition* $(M\in\Delta_2$ for short $)$ if there exist constant $k\geq 2$ and $u_0>0$ such that

$$M(2u)\leq KM(u)$$

whenever $|u|\leq u_0$.

An Orlicz function M can always be represented in the integral form $M(x)=\int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t\geq 0, q(t)>0$ for $t>0, q$ is non-decreasing and $q(t)\to\infty$ as $t\to\infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x)\leq\lambda M(x) \ \text{ for all } \ \lambda \text{ with } 0<\lambda<1,$$

since M is convex and $M(0)=0$.
Lindesstrauss and Tzafriri [15] used the idea of Orlicz sequence space;

$$l_M:=\left\{x\in w:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty, \ \text{ for some } \ \rho>0\right\}$$

which is Banach space with the norm

$$\|x\|_M=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\leq 1\right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x)=x^p$ for $1\leq p<\infty$.

2 Preliminaries

Throughout $x = (x_{jk})$ is a double sequence that is a double infinite array of elements x_{jk}, for $j, k \in \mathbb{N}$. By the convergence of a double sequence we mean the convergence on the Pringsheim sence that is, a double sequence $x = (x_{jk})$ is said to be Pringsheim convergent (or P-convergent) if for $\epsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \epsilon$ whenever $j, k > N$ (see[16]).

We shall write this as
$\lim_{j,k\to\infty} x_{jk} = L$, where j, k tends to infinity independent of each other.

A double sequence $x = (x_{jk})$ is bounded if

$$\|x\| = \sup_{j,k\geq 0} |x_{jk}| < \infty \ (see[16]).$$

Let $A = (a_{m,n,j,k}), j, k = 0, 1, ...$ be a doubly infinite matrix of real numbers for all $m, n = 0, 1, ...$ Forming the sums

$$y_{mn} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n,j,k}x_{jk},$$

called the $A-$means of the double sequence x, yeilds a method of summability. More exactly, we say that a sequence is $A-$summable to the limit L if the A-means exist for all $m, n = 0, 1, ...$ in the sense of Pringsheim's convergence:

$$\lim_{p,q\to\infty}\sum_{j=0}^{p}\sum_{k=0}^{q} a_{m,n,j,k}x_{jk} = y_{mn} \text{ and } \lim_{m,n\to\infty} y_{mn} = L.$$

Double sequence have been studied by Vakeel.A.Khan[9] and Vakeel.A.Khan and Sabiha Tabassum[10,11,12,13] and many others.

A double sequence space E is said to be solid if $(\alpha_{i,j}x_{i,j}) \in E$, whenever $(x_{i,j}) \in E$, for all double sequences $(\alpha_{i,j})$ of scalars with $|\alpha_{i,j}| \leq 1$, for all $i, j \in \mathbb{N}$.

Let $K = \{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 < .... \text{ and } k_1 < k_2 < k_3 < ...\} \subseteq N \otimes N$ and E be a double sequence space. A K-step space of E is a sequence space

$$\lambda_K^E = \{(\alpha_{i,j}x_{i,j}) : (x_{i,j}) \in E\}.$$

A *canonical pre-image* of a sequence $(x_{n_i,k_j}) \in E$ is a sequence $(b_{n,k}) \in E$

defined as follows:

$$b_{nk} = \begin{cases} a_{nk} & \text{if } (n,k) \in K, \\ 0 & \text{otherwise .} \end{cases}$$

A canonical pre-image of step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A double sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

A double sequence space E is said to be symmetric if $(x_{i,j}) \in E$ implies $(x_{\pi(i),\pi(j)}) \in E$, where π is a permutation of $\mathbb{N}$.

Lemma 2.1 *A sequence space E is solid implies E is monotone.*

The following inequality will be used throughout the paper. Let $q = q_{jk}$ be a double sequence of positive real numbers with $0 < q_{jk} \leq \sup q_{jk} = H$ and let $C = \max\{1, 2^{H-1}\}$. Then for the factorable sequences (a_{jk}) and (b_{jk}) in the complex plane, we have

$$|a_{jk} + b_{jk}|^{q_{jk}} \leq C(|a_{jk}|^{q_{jk}} + |b_{jk}|^{q_{jk}})$$

3 Main Results

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non decreasing sequences of positive real numbers both of which tends to ∞ as r, s approach ∞, respectively. Also let $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 0$ and $\mu_{s+1} \leq \mu_s + 1, \mu_1 = 0$. The generalized double de la Valee-Pousin mean was defined by M.Mursaleen, C. Çakan, S.A.Mohiuddine and E. Savas [17] as:

$$t_{r,s}(x) = \frac{1}{\lambda_r \mu_s} \sum_{j \in I_r} \sum_{k \in I_s} x_{j,k}$$

where $I_r = [r - \lambda_r + 1, r]$ and $I_s = [s - \mu_s + 1, s]$.
Throughout this paper we shall denote $\lambda_r \mu_s$ by $\bar{\lambda}_{rs}$ and $(j \in I_r, k \in I_s)$ by $(j,k) \in \bar{I}_{r,s}$.

Let M be an Orlicz function, $x = (x_{jk})$ be double sequence space and $q = (q_{jk})$ be any factorable double sequence of strictly positive real numbers($0 < h = \inf q_{jk} \leq q_{jk} \leq \sup q_{jk} < \infty$). Let $A = (a_{m,n,j,k})$ be an infinite four dimensional matrix of complex numbers and $(X, \|., .\|)$ be 2-normed space. We define

$$ {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o = \left\{ x=(x_{jk}) : P-\lim_{r,s}\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\right)\right]^{q_{jk}}=0, \right. $$

$$ \left. \text{for some } \rho>0 \text{ and } \text{ for every } z\in X \right\} $$

$$ {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|] = \left\{ x=(x_{jk}) : P-\lim_{r,s}\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(x)-L,z\|}{\rho}\right)\right]^{q_{jk}}=0, \right. $$

$$ \left. \text{for some } \rho>0, L>0 \text{ and } \text{ for every } z\in X \right\} $$

$$ {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty = \left\{ x=(x_{jk}) : \sup_{r,s}\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\right)\right]^{q_{jk}}<\infty, \right. $$

$$ \left. \text{for some } \rho>0 \text{ and } \text{ for every } z\in X \right\} $$

Where

$$ \Delta^n_{(mv)}a_{m,n,j,k} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}v_{j-mi,k-mi}a_{m,n,j-mi,k-mi} $$

Theorem 3.1 *Let $q=(q_{jk})$ be bounded. Then ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o$, ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]$ and ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x_{jk},y_{jk}\in_2 V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,p,\|.,.\|]_o$ and $\alpha,\beta\in\mathbb{C}$. Then there exist some $\rho_1,\rho_2>0$ such that

$$ P-\lim_{r,s}\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho_1}\right)\right]^{q_{jk}}=0, $$

$$ P-\lim_{r,s}\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(y),z\|}{\rho_2}\right)\right]^{q_{jk}}=0, $$

Let $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Then we have

$$ \frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(\alpha x+\beta y),z\|}{\rho_3}\right)\right]^{q_{jk}} $$

$$ \leq \frac{1}{\bar{\lambda} rs}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(\alpha x),z\|}{\rho_3}+\frac{\|\Delta^n_{(mv)}A_{jk}(\beta y),z\|}{\rho_3}\right)\right]^{q_{jk}} $$

$$\leq C\Big\{\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho_1}\Big)\Big]^{q_{jk}}$$

$$+\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(y),z\|}{\rho_2}\Big)\Big]^{q_{jk}}\Big\}.$$

This implies that

$$\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(\alpha x+\beta y),z\|}{\rho_3}\Big)\Big]^{q_{jk}}=0.$$

This proves that ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o$. is a linear space.
Similarly we can prove that ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]$ and ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.

Theorem 3.2 *Let M be any Orlicz function, then ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o\subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]\subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$ hold.*

Proof. The inclusion ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o\subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]$ is obvious.
Let $x_{jk}\in {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]$ then there exists some $\rho>0$ and $L>0$ such that

$$\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x)-L,z\|}{\rho}\Big)\Big]^{q_{jk}}=0.$$

Taking $\rho_1=2\rho$, we have
$\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\Big)\Big]^{q_{jk}}$

$$=\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x)-L+L,z\|}{\rho}\Big)\Big]^{q_{jk}}$$

$$\leq C\Big\{\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[\frac{1}{2}M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x)-L,z\|}{\rho}\Big)\Big]^{q_{jk}}$$

$$+\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[\frac{1}{2}M\Big(\frac{\|L,z\|}{\rho}\Big)\Big]^{q_{jk}}\Big)\Big\}.$$

$$\leq C\Big\{\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[\frac{1}{2}M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x)-L,z\|}{\rho}\Big)\Big]^{q_{jk}}$$

$$+\max\Big(\Big[\frac{1}{2}M\Big(\frac{\|L,z\|}{\rho}\Big)\Big]^{H}\Big)\Big\}.$$

Hence $x_{jk}\in {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$.

As a consequence of above theorem we state the following corollary.

Corollary 3.3 ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o$ *and* ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]$ *are nowhere dense subsets of* ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$.

Theorem 3.4 *The sequence spaces* ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o$ *and* ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty$ *are solid and hence monotone.*

Proof. Let $\alpha=(\alpha_{jk})$ be double sequence of scalars such that $|\alpha_{jk}|\leq 1$, for all $j,k\in\mathbb{N}$. Since M is monotone, we get for some $\rho>0$

$$\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(\alpha x),z\|}{\rho}\Big)\Big]^{q_{jk}}\leq\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\sup|\alpha_{jk}|\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\Big)\Big]^{q_{jk}}$$

$$\leq\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}\Big[M\Big(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\Big)\Big]^{q_{jk}}$$

Hence the result.$\diamond$

Theorem 3.5 *Let M_1 and M_2 be two Orlicz functions. Then we have*

(i) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1,q,\|.,.\|]_o\cap{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]_o\subseteq{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1+M_2,q,\|.,.\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1,q,\|.,.\|]\cap{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]\subseteq{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1+M_2,q,\|.,.\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1,q,\|.,.\|]_\infty\cap{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]_\infty\subseteq{}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1+M_2,q,\|.,.\|]_\infty$.

Theorem 3.6 *Let the Orlicz functions M_1 and M_2 satisfy the Δ_2-condition. Then*

(i) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1, q, \|., .\|]_o \subseteq {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1 \circ M_2, q, \|., .\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1, q, \|., .\|] \subseteq {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1 \circ M_2, q, \|., .\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1, q, \|., .\|]_\infty \subseteq {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1 \circ M_2, q, \|., .\|]_\infty$.

Proof(i). We consider only case ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1, q, \|., .\|]_o \subseteq {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1 \circ M_2, q, \|., .\|]_o$.Let $x_{jk} \in {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M_1, q, \|., .\|]_o$ and $\epsilon > 0$. Now using the continuity of M choose $0 < \delta < 1$ such that $0 < t < \delta \Rightarrow M(t) < \epsilon$. Write $y_{jk} = M_1\left(\frac{\|\Delta^n_{(mv)} A_{jk}(x), z\|}{\rho}\right)$.
Now consider

$$\frac{1}{\bar{\lambda}_{rs}} \sum_{(j,k) \in \bar{I}_{r,s}} [M_2(y_{jk})]^{q_{jk}} = \frac{1}{\bar{\lambda} rs} \sum_1 [M_2(y_{jk})]^{q_{jk}} + \frac{1}{\bar{\lambda}_{rs}} \sum_2 [M_2(y_{jk})]^{q_{jk}}$$

Where the first summation is over $y_{jk} \leq \delta$ and the second summation is over $y_{jk} > \delta$. Since M_2 is continuous, we have

$$\frac{1}{\bar{\lambda}_{rs}} \sum_1 [M_2(y_{jk})]^{q_{jk}} < \frac{1}{\bar{\lambda}_{rs}} \sum_{(j,k) \in \bar{I}_{r,s}} \max(1, \epsilon^H).$$

For $y_{jk} > \delta$, we use the fact that

$$y_{jk} < \frac{y_{jk}}{\delta} \leq 1 + \left(\frac{y_{jk}}{\delta}\right).$$

Since M_2 is non decreasing and convex, it follows that

$$M(y_{jk}) < M(1 + \delta^{-1} y_{jk}) = M\left(\frac{2}{2} + \frac{2}{2}\delta^{-1} y_{jk}\right)$$

$$< \frac{1}{2} M(2) + \frac{1}{2} M(2\delta^{-1} y_{jk})$$

Since M satisfies Δ_2-condition, there is a constant $K > 2$ such that

$$M(2\delta^{-1} y_{jk}) \leq \frac{1}{2} K \delta^{-1} y_{jk} M(2)$$

Hence

$$\frac{1}{\bar{\lambda}_{rs}} \sum_2 [M_2(y_{jk})]^{q_{jk}} < \max(1, (K\delta^{-1} M(2))) \frac{1}{\bar{\lambda}_{rs}} \sum_{(j,k) \in \bar{I}_{r,s}} (y_{jk})^{q_{jk}}.$$

Thus we have

$$\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}[M_2(y_{jk})]^{q_{jk}} < \max(1,\epsilon^H)+\max(1,(K\delta^{-1}M(2)))\frac{1}{\bar{\lambda}_{rs}}\sum_{(j,k)\in\bar{I}_{r,s}}(y_{jk})^{q_{jk}}.$$

Therefore $x \in {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_1\circ M_2,p,\|.,.\|]_o$

Taking $M_1(x)=x$, for all $x=x_{jk}\in[0,\infty)$, we have the following result

Proof of other two cases follow similarly.

Theorem 3.7 *Let the Orlicz function M_2 satisfy the Δ_2-condition. Then*

(i) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},q,\|.,.\|]_o \subseteq {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},q,\|.,.\|] \subseteq {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},q,\|.,.\|]_\infty \subseteq {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M_2,q,\|.,.\|]_\infty$.

Theorem 3.8 *Let $0<p_{jk}\le q_{jk}$ for all j,k and (q_{jk}/p_{jk}) be bounded. Then*

(i) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o \subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,p,\|.,.\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|] \subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,p,\|.,.\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_\infty \subset {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,p,\|.,.\|]_\infty$.

Proof. Let $x \in {}_2V^{\bar{\lambda}}[A,\Delta^n_{(mv)},M,q,\|.,.\|]_o$
Write $t_{j,k}=\left[M\left(\frac{\|\Delta^n_{(mv)}A_{jk}(x),z\|}{\rho}\right)\right]^{p_{jk}}$ for all j,k and $\lambda_{j,k}=\frac{p_{j,k}}{q_{j,k}}$.
Since $0<p_{j,k}\le q_{j,k}$, therefore $0<\lambda_{j,k}\le 1$.
Take $0<\lambda\le\lambda_{j,k}$.
Define

$$u_{j.k}=\begin{cases} t_{j,k} & t_{j,k}\ge 1\\ 0 & t_{j,k}<1.\end{cases}$$

$$v_{j,k}=\begin{cases} 0 & t_{j,k}\ge 1\\ t_{j,k} & t_{j,k}<1.\end{cases}$$

So $t_{j,k}=u_{j,k}+v_{j,k}$ and

$$t_{j,k}^{\lambda_{j,k}}=u_{j,k}^{\lambda_{j,k}}+v_{j,k}^{\lambda_{j,k}}$$

Now it follows that

$u_{j,k}^{\lambda_{j,k}} \leq u_{j,k} \leq t_{j,k}$ and $v_{j,k}^{\lambda_{j,k}} \leq v_{j,k}^{\lambda}$
Therfore

$$\frac{1}{\bar{\lambda}_{rs}} \sum_{(j,k)\in \bar{I}_{r,s}} t_{j,k}^{\lambda_{j,k}} \leq \frac{1}{\bar{\lambda}_{rs}} \sum_{(j,k)\in \bar{I}_{r,s}} t_{j,k} + \left[\frac{1}{\bar{\lambda} rs} \sum_{(j,k)\in \bar{I}_{r,s}} v_{j,k}\right]^{\lambda}$$

Hence $x \in {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, p, \|., .\|]_o$
By using above theorem it is easy to prove the following result.

Corollary 3.9 (a). *If* $0 < \inf q_{k,l} \leq q_{j,k} \leq 1$ *for all* $j, k \in \mathbb{N}$ *then,*

(i) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|]_o \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|] \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|]_\infty \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|]_\infty$.

(b). *If* $0 \leq q_{j,k} \leq \sup q_{j,k} = H < \infty$ *for all* $j, k \in \mathbb{N}$ *then,*

(i) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|]_o \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|]_o$,

(ii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|] \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|]$ *and*

(iii) ${}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, q, \|., .\|]_\infty \subset {}_2V^{\bar{\lambda}}[A, \Delta^n_{(mv)}, M, \|., .\|]_\infty$.

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