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Linearized Comparison Criteria for a First Order Nonlinear Neutral Delay Difference Equation

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Abstract

In this paper, we consider a class of first order nonlinear neutral delay difference equations with variable coefficients of the form

$$\Delta\left[x(n) - P(n)x(n-\tau(n))\right] + \sum_{j=1}^{m} Q_j(n) f_j\left(x(n-\sigma_j(n))\right) = 0, \quad n \geq n_0. (*)$$

We establish sufficient conditions for oscillation of all solutions of (*) and a linearized comparison theorem is derived which establishes a connection between our nonlinear equations and a class of linear neutral equations with constant coefficients.

Keywords: *Nonlinear, neutral, delay difference equations.*

1 Introduction

In this paper, we are concerned with a functional difference equation of the form

$$\Delta\left[x(n) - P(n)x(n-\tau(n))\right] + \sum_{j=1}^{m} Q_j(n) f_j\left(x(n-\sigma_j(n))\right) = 0, \quad n \geq n_0, \quad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$. The following conditions are assumed to be hold:

(H_1) $\{P(n)\}$, $\{Q_j(n)\}$, $j = 1, 2, 3, ..., m$ are sequences of positive real numbers, and $0 \leq P(n) \leq 1$.
(H_2) $\{\tau(n)\}$, $\{\sigma_j(n)\}$, $j = 1, 2, 3, ..., m$, are sequences of positive integers such that $n - \tau(n) > 0$ and $n - \sigma_j(n) > 0$,
(H_3) $0 < \tau_* \leq \tau(n) \leq \tau^*$, $0 < \sigma_* \leq \sigma_j(n) \leq \sigma^*$ for $j = 1, 2, 3, ..., m$,
and
(H_4) $\{f_j\}$, $j = 1, 2, 3, ..., m$ are real valued functions such that $uf_j(u) > 0$ for $u \neq 0$ and $f_j(-u) = -f_j(u)$.

For comparison purposes, we will consider a linear equation of the form

$$\Delta\left[x(n) - px(n-\tau)\right] + \sum_{j=1}^{m} q_j x(n-\sigma_j) = 0, \quad n \geq n_0, \tag{2}$$

where $p \in [0, 1)$, q_1, q_2,...,q_m are positive real numbers, and τ, σ_1, σ_2,...,σ_m are positive integers.

By a solution of difference equation (1), we mean a real sequence $\{x(n)\}$ which satisfies the equation (1) for all $n \geq n_0$. A solution $\{x(n)\}$ is said to be oscillatory if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

In recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions for difference classes of difference equations, we refer to the books [1,2,5] and the papers [3,4, 6-10]. Also, the oscillatory behavior of neutral functional difference equations has been the subject of intensive study.

Lalli [7] established oscillation criteria for the first order neutral difference equations

$$\Delta\left[x(n) + Px(n+\delta k)\right] + q(n)f(x(\tau(n))) = F(n), \quad n = 0, 1, 2, ...,$$

where $\delta = \pm 1$, P is a nonnegative real number, k is a positive integer $\{\tau(n)\}$ is a sequence of nonnegative integer, and $\{q(n)\}$ and $\{F(n)\}$ are sequences of real numbers. Later Peng et al. [10] established oscillation criteria for the equation

$$\Delta\left[x(n) - Px(n-\tau)\right] + \sum_{i=1}^{m} q_i f_i\left(x(n-\sigma_i)\right) = 0; \quad n \geq n_0,$$

where $P \in [0, 1)$, $f_i \in C(R, R)$, $q_i \in (0, \infty)$, and τ, $\sigma_i \in \{0, 1, 2, ...\}$, $i = 0, 1, 2, ..., m$.

In the next section we establish sufficient condition for oscillation of all solution of (1). In the third section, the desired linearized comparison theorem is established.

2 Necessary Conditions

Let

$$y(n) = x(n) - P(n)x(n - \tau(n)), \tag{3}$$

where $\{x(n)\}$ is an eventually positive solution of (1).

Lemma 2.1 *Suppose $\{x(n)\}$ is an eventually positive solution of (1). Then $y(n) > 0$, $\Delta y(n) < 0$ for all large n.*

Proof. In view of (1), we see that

$$\Delta y(n) = -\sum_{j=1}^{m} Q_j(n) f_j\left(x(n - \sigma_j(n))\right) < 0$$

for all large n. Thus $\{y(n)\}$ is eventually positive or negative. Assume to the contrary that $y(n) < 0$ and $\Delta y(n) < 0$ for all large n. Then $y(n) \le -\alpha < 0$ for n greater than or equal to some integer N_1, so that

$$x(n) \le -\alpha + P(n)x(n - \tau(n)), \quad n \ge N_1.$$

We have two cases to consider. First, assume that $\{x(n)\}$ is unbounded. Then there is a real sequence $\{m_k\}$ of integers which tends to infinity and

$$x(m_k) = \max_{N_1 \le n \le m_k} x(n). \tag{4}$$

However, in view of the assumption that $0 \le P(n) \le 1$, we see that

$$\begin{aligned} x(m_k) &\le -\alpha + P(m_k)x(m_k - \tau(m_k)) \\ &\le -\alpha + x(m_k - \tau(m_k)) \\ &\le -\alpha + x(m_k), \end{aligned}$$

which is a contradiction.

Next, assume that $\{x(n)\}$ is bounded. Then there is a sequence $\{v_k\}$ of integers which tends to infinity and

$$\limsup_{k\to\infty} x(v_k) = L < \infty.$$

Let $\{\xi_k\}$ be the sequence of integers defined by

$$x\left(\xi_k\right) = max\left\{x(n)/v_k - \tau(v_k) \le n \le v_k\right\}.$$

Then $\xi_k \to \infty$ and $\limsup_{k\to\infty} x(\xi_k) = L$. Furthermore, we have

$$x(v_k) \le -\alpha + P(v_k)x(\xi_k) \le -\alpha + x(\xi_k)$$

for all large n. Taking superior limits on both sides of the inequality, we see that $L \le -\alpha + L$, which is also a contradiction. The proof is complete.

By means of Lemma 2.1, we now derive one of our main results related to the existence of oscillatory solution of (1).

Theorem 2.2 *Assume that $f_j(u)/u \geq 1$ for $u > 0$ and $j = 1, 2, ..., m$, and suppose there is a sufficiently large integer N such that*

$$\inf_{n\geq N,\lambda>1}\left\{\frac{1}{\lambda}\sum_{j=1}^{m}Q_j(n)e^{\lambda\sigma_j(n)}+\frac{1}{\lambda}\sum_{j=1}^{m}Q_j(n)P(n-\sigma_j(n))e^{\lambda\tau(n-\sigma_j(n))}\right\}>1. \quad (5)$$

Then every solution of (1) is oscillatory.

Proof. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1). Then by means of Lemma 2.1, we see that, $x(n) > 0$, $y(n) > 0$ and $\Delta y(n) < 0$ for n greater than or equal to some N. Furthermore, we have

$$x(n-\sigma_j(n)) \leq f_j(x(n-\sigma_j(n))),$$

and

$$0 < y(n-\tau(n-\sigma_j(n))) < y(n-\sigma_j(n)-\tau(n-\sigma_j(n))) \leq x(n-\sigma_j(n)-\tau(n-\sigma_j(n)))$$

for $1 \leq j \leq m$. Define

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)}, \quad n \geq N.$$

Then $\lambda(n) > 0$ for $n \geq N$, and

$$\frac{y(s)}{y(n)} \geq \exp\left(\sum_{k=s}^{n-1}\lambda(k)\right), \quad s, n \geq N.$$

In view of (1), we see further that

$$\begin{aligned}\lambda(n) &= -\frac{\Delta y(n)}{y(n)}\\ &= \sum_{j=1}^{m}\frac{Q_j(n)f_j(x(n-\sigma_j(n)))}{y(n)}\\ &\geq \sum_{j=1}^{m}\frac{Q_j(n)x(n-\sigma_j(n))}{y(n)}\end{aligned}$$

$$\geq \frac{\sum_{j=1}^{m}Q_j(n)\left\{y(n-\sigma_j(n))+P(n-\sigma_j(n))x(n-\sigma_j(n)-\tau(n-\sigma_j(n)))\right\}}{y(n)}$$

$$\geq \sum_{j=1}^{m}Q_j(n)\exp\left\{\sum_{s=n-\sigma_j(n)}^{n-1}\lambda(s)\right\}+\sum_{j=1}^{m}Q_j(n)P(n-\sigma_j(n))\frac{y(n-\tau(n-\sigma_j(n)))}{y(n)}$$

$$\geq \sum_{j=1}^{m} Q_j(n) \exp\left\{\sum_{s=n-\sigma_j(n)}^{n-1} \lambda(s)\right\}$$
$$+\sum_{j=1}^{m} Q_j(n) P(n-\sigma_j(n)) \exp\left\{\sum_{s=n-\tau(n-\sigma_j(n))}^{n-1} \lambda(s)\right\}. \qquad (6)$$

Next, we assert that $\liminf_{n\to\infty} \lambda(n) > 0$. Assume to the contrary that $\liminf_{n\to\infty} \lambda(n) = 0$. Choose a sequence $\{s_k\}$ of integers which tends to infinity and

$$\lambda(s_k) = \min_{N\leq n\leq s_k} \lambda(n).$$

Then we see from (6) that

$$\begin{aligned}
\lambda(s_k) \ \geq\ & \sum_{j=1}^{m} Q_j(s_k) \exp\left\{\sum_{s=s_k-\sigma_j(s_k)}^{s_k-1} \lambda(s)\right\} \\
& +\sum_{j=1}^{m} Q_j(s_k) P(s_k-\sigma_j(s_k)) \exp\left\{\sum_{s=s_k-\tau(s_k-\sigma_j(s_k))}^{s_k-1} \lambda(s)\right\} \\
\geq\ & \sum_{j=1}^{m} Q_j(s_k) \exp\left\{\lambda(s_k)\sigma_j(s_k)\right\} \\
& +\sum_{j=1}^{m} Q_j(s_k) P(s_k-\sigma_j(s_k)) \exp\left\{\lambda(s_k)\tau(s_k-\sigma_j(s_k))\right\},
\end{aligned}$$

so that

$$1 \geq \inf_{k\geq 1}\left\{\frac{1}{\lambda(s_k)}\sum_{j=1}^{m} Q_j(s_k)\exp(\lambda(s_k)\sigma_j(s_k)) + \frac{1}{\lambda(s_k)}\sum_{j=1}^{m} Q_j(s_k) P(s_k-\sigma_j(s_k))\exp(\lambda(s_k)\tau(s_k-\sigma_j(s_k)))\right\},$$

contrary to our assumption (5).

Since $0 < \lambda(n) < 1$, we have $\liminf_{n\to\infty} \lambda(n) < \infty$. To complete our proof, let us denote $\liminf_{\to\infty} \lambda(n)$ by λ_*. Also let $\eta > 1$ be an arbitrary number such that

$$\inf_{n\geq N, \lambda>1}\left\{\frac{1}{\lambda}\sum_{j=1}^{m} Q_j(n) e^{\lambda\sigma_j(n)} + \frac{1}{\lambda}\sum_{j=1}^{m} Q_j(n) P(n-\sigma_j(n)) e^{\lambda\tau(n-\sigma_j(n))}\right\} > \eta. \qquad (7)$$

For sufficiently large n, since $\eta\lambda(n-\tau(n-\sigma_j(n))) > \lambda_*$ and $\eta\lambda(n-\sigma_j(n)) > \lambda_*$ for $1 \le j \le m$, we see from (6) that

$$\lambda_* \ge \inf_{s\ge N}\left\{\sum_{j=1}^m Q_j(s)\exp\left(\frac{\lambda_*}{\eta}\sigma_j(s)\right)\right.$$
$$\left.+\sum_{j=1}^m Q_j(s)P(s-\sigma_j(s))\exp\left(\frac{\lambda_*}{\eta}\tau(s-\sigma_j(s))\right)\right\}.$$

After rewriting this inequality, we see that

$$\eta \ge \inf_{s\ge N}\left\{\frac{\eta}{\lambda_*}\sum_{j=1}^m Q_j(s)\exp\left(\frac{\lambda_*}{\eta}\sigma_j(s)\right)\right.$$
$$\left.+\frac{\eta}{\lambda_*}\sum_{j=1}^m Q_j(s)P(s-\sigma_j(s))\exp\left(\frac{\lambda_*}{\eta}\tau(s-\sigma_j(s))\right)\right\},$$

contrary to our assumption (7). The proof is complete.

There are two variants of the above theorem. The first one assumes the additional condition that $\tau(n)\equiv\tau$, $\sigma_j(n)\equiv\sigma_j$ and $f_j(u)\le\delta_j u$ for $u>0$.

Theorem 2.3 *Assume that $\tau(u)\equiv\tau$, $\sigma_j(n)\equiv\sigma_j$ for $1\le j\le m$, that*

$$1 \le \frac{f_j(u)}{u} \le \delta_j \quad \textit{for} \quad u>0 \quad \textit{and} \quad 1\le j\le m, \tag{8}$$

and that there is an integer N such that

$$\inf_{n\ge N,\lambda>1}\left\{\frac{1}{\lambda}\sum_{j=1}^m Q_j(n)e^{\lambda\sigma_j} + \frac{R(n)}{\delta}e^{\lambda\tau}\right\} > 1, \tag{9}$$

where $\delta = \max_{1\le j\le m}\delta_j$ and

$$R(n) = \min_{1\le j\le m}\frac{Q_j(n)P(n-\sigma_j)}{Q_j(n-\tau)}.$$

Then every solution of (1) is oscillatory.

Proof. We only need to note that (6) now changes to

$$\lambda(n) \ge \sum_{j=1}^m Q_j(n)\exp\left\{\sum_{s=n-\sigma_j}^{n-1}\lambda(s)\right\} + \sum_{j=1}^m\frac{Q_j(n)P(n-\sigma_j)x(n-\sigma_j-\tau)}{y(n)},$$

and the second sum $S(n)$ in the above inequality is equal to

$$\sum_{j=1}^{m}\left\{\frac{Q_j(n)P(n-\sigma_j)}{Q_j(n-\tau)}.\frac{Q_j(n-\tau)x(n-\sigma_j-\tau)}{y(n)}\right\},$$

so that

$$\begin{aligned}
S(n) &\geq R(n)\sum_{j=1}^{m}\frac{Q_j(n-\tau)f_j(x(n-\sigma_j-\tau))}{y(n)\delta_j}\\
&\geq \frac{R(n)}{\delta y(n)}\sum_{j=1}^{m}Q_j(n-\tau)f_j(x(n-\sigma_j-\tau))\\
&\geq \frac{R(n)}{\delta y(n)}\{-\Delta y(n-\tau)\}\\
&= \frac{R(n)}{\delta}\left\{\frac{\lambda(n-\tau)y(n-\tau)}{y(n)}\right\}\\
&\geq \frac{R(n)}{\delta}\lambda(n-\tau)\exp\left\{\sum_{s=n-\tau}^{n-1}\lambda(s)\right\}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.2 and is thus omitted.

Theorem 2.4 *Assume that $f_j(u)/u \geq 1$, for $u > 0$ and $j = 1, 2, ..., m$,*

$$\lim_{n\to\infty} P(n) = p \in [0, 1) \tag{10}$$

and that one of the sequences $\{Q_j(n)\}$, $j = 1, 2, ..., m$, say $\{Q_{j*}(n)\}$ satisfies

$$\sum_{s=n_0}^{\infty} Q_{j*}(s) = \infty. \tag{11}$$

Then every nonoscillatory solution of (1) converges to zero.

Proof. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solutions of (1). Then in view of Lemma 2.1, the sequence $\{y(n)\}$ defined by (3) satisfies $y(n) > 0$ and $\Delta y(n) < 0$ for all large n. Thus we have

$$0 < y(n) \leq y(n-\sigma_{j*}(n)) \leq x(n-\sigma_{j*}(n)) \leq f_{j*}(x(n-\sigma_{j*}(n)))$$

for n greater than or any equal to some positive integer N. Without loss of any generality, we may assume that $P(n) < p'$ for $n \geq N$, where $p' \in (p, 1)$. Employing these facts, we then deduce from (1) that

$$\Delta y(n) + Q_{j*}(n)y(n) \leq 0, \quad n \geq N.$$

We can easily show that

$$y(n)\exp\left(\sum_{s=N}^{n-1} Q_{j*}(s)\right) \le y(N), \quad n \ge N,$$

which implies

$$y(n) \le \exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right) y(N),$$

or

$$x(n) \le P(n)x(n-\tau(n)) + y(N)\exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right)$$

$$< p'x(n-\tau(n)) + y(N)\exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right).$$

If $\{x(n)\}$ is not bounded, then there is a sequence $\{m_k\}$ of integers which tends to infinity and (4) holds. Thus,

$$x(m_k) < p'x(m_k-\tau(m_k)) + y(N)\exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right)$$

$$\le p'x(m_k) + y(N)\exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right),$$

which implies

$$x(m_k) < \frac{y(N)}{1-p'}\exp\left(-\sum_{s=N}^{n-1} Q_{j*}(s)\right)$$

for all large n. This is impossible as can be seen by taking limits on both sides. We have thus shown that $\{x(n)\}$ is bounded.

Next we show that $\{x(n)\}$ has a limit. Indeed, let $\{r_k\}$ be a divergent sequence such that

$$\limsup_{n\to\infty} x(n) = \lim_{k\to\infty} x(r_k).$$

Then in view of (3), we see that

$$\limsup_{n\to\infty} x(n) = \lim_{n\to\infty} y(n) + p\lim_{k\to\infty} x(r_k - \tau r_1)) \le \lim_{n\to\infty} y(n) + p\limsup_{n\to\infty} x(n),$$

which implies

$$\limsup_{n\to\infty} x(n) \le \frac{\lim_{n\to\infty} y(n)}{1-p}.$$

Similarly, we have

$$\frac{\lim_{n\to\infty} y(n)}{1-p} \le \liminf_{n\to\infty} x(n).$$

From this, we have $\limsup_{n\to\infty} x(n) = \liminf_{n\to\infty} x(n)$ and hence $\lim_{n\to\infty} x(n)$ exists.

Finally, if $\lim_{n\to\infty} x(n) = \alpha > 0$, then $0 < \alpha/2 \le x(n)$ for n greater than or equal to some integer n_1. Since $f(n) \ge n$ for $n \ge n_0$, we see that

$$f_{j*}(x(n-\sigma_j(n))) \ge x(n-\sigma_j(n)) \ge \alpha/2$$

for n greater than or equal to some integer $n_2 \ge n_1$. Thus by means of (1), we have

$$\Delta y(n) = -\sum_{j=1}^{m} Q_j(n) f_j(x(n-\sigma_j(n))) \le -Q_{j*}(n)\alpha/2, \quad n \ge n_2.$$

By summing the above inequality from n_2 to ∞, we conclude from (11) that $\lim_{n\to\infty} y(n) = -\infty$. This contradicts the conclusion of Lemma 2.1.

3 Linearized Comparison Theorem

In this section, we will exhibit a connection between equation (1) and an appropriate linear equation of the form (2). Recall that the assumptions that $p \in [0,1)$, and $q_1, q_2, ..., q_m,\ \sigma_1, \sigma_2, ..., \sigma_m > 0$ have been made. Next, we establish two properties of (2) which are needed for our linearized comparison theorem.

Theorem 3.1 *If the condition*

$$F(\lambda) = \left(\frac{1}{\lambda} - 1\right)(1 - p\lambda^{\tau}) + \sum_j q_j \lambda^{\sigma_j} > 0 \tag{12}$$

holds for all $\lambda > 1$. Then every solution of equation (2) is oscillatory. The converse also holds.

Proof. The first statement follows from Theorem 2.3 by taking $f_j(x) = x$ and $\delta_j = 1$ for $1 \le j \le m$. To see that the converse holds, suppose there is a positive number λ^* such that $F(\lambda^*) \le 0$. Note that $F(+\infty) = +\infty$, thus there is a number $\xi \in [\lambda^*, \infty)$ such that $F(\xi) = 0$. It is then easily verified that the sequence $x(n) = \{\xi^{-n}\}$ is an eventually positive solution of (12).

Next, we establish a theorem on continuous dependence on parameters for linear equations of the form (2).

Theorem 3.2 *Suppose $p > 0$ and that every solution of (2) is oscillatory. Then there is a positive number $\mu < \min\{p, q_1, q_2, ..., q_n\}$ such that for every $\varepsilon \in [0, \mu]$, the equation*

$$\Delta[x(n) - (p-\varepsilon)x(n-\tau)] + \sum_{j=1}^{m} (q_j - \varepsilon) x(n-\sigma_j) = 0 \tag{13}$$

is oscillatory.

Proof. By means of Theorem 3.1, we see that the function $F = F(\lambda)$ defined by (12) satisfies $F(\lambda) > 0$ for $\lambda > 1$. Furthermore, it is easily verified that

$$\lim_{\lambda\to 1} F(\lambda) > 0, \quad F(+\infty) = +\infty$$

and $F''(\lambda) > 0$ for $\lambda > 1$. Thus $F(\lambda) \geq c > 0$ for $\lambda > 1$.
Define

$$F(\lambda,\theta) = \left(\frac{1}{\lambda} - 1\right)(1 - (p-\theta)\lambda^{\tau}) + \sum_{j=1}^{m}(q_j - \theta)\lambda^{\sigma_j}, \quad \lambda > 1, -\infty < \theta < \infty.$$

Note that $F(\lambda, 0) = F(\lambda) \geq c > 0$ for $\lambda > 1$, that $F_\theta(\lambda,\theta) < 0$ and that $F(+\infty,\theta) = +\infty$ for $0 < \theta < \min\{p, q_1, q_2, ..., q_m\}$. Therefore, since $F(\lambda,\theta)$ is continuous in λ and θ, it is not difficult to find a positive number $\mu < \min\{p, q_1, q_2, ..., q_n\}$ such that $F(\lambda,\mu) > 0$ for $\lambda > 1$. Next, since $F(\lambda,\theta)$ is decreasing in θ for each fixed λ, we see that $F(\lambda,\theta) > 0$ for each $\lambda > 1$ and $\theta \in [0,\mu]$. The proof is complete.

The same idea can be employed to show the following variant of Theorem 3.2: Suppose $p = 0$ and that every solution of (2) is oscillatory. Then there is a positive number $\mu < \min\{q_1, q_2, ..., q_m\}$ such that for every $\varepsilon \in [0,\mu]$, every solution of

$$\Delta x(n) + \sum_{j=1}^{m}(q_j - \varepsilon)x(n - \sigma_j) = 0$$

is oscillatory.

We now state and prove our final linearized comparison theorem, we need the assumptions that

$$\lim_{n\to\infty} P(n) = p \in [0,1), \tag{14}$$

$$\lim_{n\to\infty} Q_j(n) = q_j, \quad 1 \leq j \leq m, \tag{15}$$

and

$$\lim_{u\to 0} \frac{f_j(u)}{u} = 1, \quad 1 \leq j \leq m. \tag{16}$$

Theorem 3.3 *Assume that (14)-(16) hold. Assume further that* $\tau(n) \equiv \tau$ *and* $\sigma_j(n) \geq \sigma_j$ *for* $1 \leq j \leq m$ *for* $n \geq n_1$. *If every solution of (2) is oscillatory then every solution of (1) is oscillatory.*

Proof. We assume that $0 < p < 1$. Since (2) does not have any eventually positive solutions, by Theorem 3.2, there is a positive number $\mu <$

$\min\{p, q_1, q_2, ..., q_m\}$ such that for every $\varepsilon \in [0, \mu]$, the equation (13) cannot have any eventually positive solutions either. Thus, by Theorem 3.1,

$$\left(\frac{1}{\lambda} - 1\right)[1 - (p - \varepsilon)\lambda^{\tau}] + \sum_{j=1}^{m}(q_j - \varepsilon)\lambda^{\sigma_j} > 0, \quad \lambda > 1, \varepsilon \in [0, \mu].$$

That is,

$$\frac{\lambda}{\lambda - 1}\sum_{j=1}^{m}(q_j - \varepsilon)\lambda^{\sigma_j} + (p - \varepsilon)\lambda^{\tau} > 1, \quad \lambda > 1, \varepsilon \in [0, \mu].$$

Note further that (14) and $\lim_{n\to\infty} R(n) = p$ imply respectively that $Q_j(n) \geq q_j - \varepsilon$ and $R(n) \geq p - \varepsilon$ for all large n. Thus,

$$\frac{\lambda}{\lambda - 1}\sum_{j=1}^{m} Q_j(n)\lambda^{\sigma_j} + R(n)\lambda^{\tau}$$

$$\geq \frac{\lambda}{\lambda - 1}\sum_{j=1}^{m}(q_j - \varepsilon\lambda^{\sigma_j} + (p - \varepsilon)\lambda^{\tau} > 1,$$

or

$$\frac{\lambda}{\lambda - 1}\sum_{j=1}^{m} Q_j(n)\lambda^{\sigma_j} + R(n)\lambda^{\tau} > 1$$

for all $\lambda > 1$ and all large n. By Theorem 2.3, (1) cannot have any eventually positive solutions. The case where $p = 0$ is similarly proved.

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