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Analytical Solution of Differential Equation Associated with Simple Pendulum

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Abstract

In the present work, we provide the exact equation of motion of a simple pendulum of arbitrary amplitude. For first time, a new and exact expression is obtained for the time "t" of swinging of a simple pendulum from the vertical position to an arbitrary angular position "θ". The time period "T" of such a pendulum is also exactly expressible in terms of hypergeometric functions.

Keywords: *Pochhammer's symbol; Gauss ordinary hypergeometric function; Kampé de Fériet's double hypergeometric function.*

1 Introduction and Preliminaries

In our investigations we shall apply the following results.

The Pochhammer's Symbol

The Pochhammer's symbol or shifted factorial or generalized factorial function is defined by

$$(h)_r = \frac{\Gamma(h+r)}{\Gamma(h)} = \begin{cases} 1 & , \text{ if } \ r=0 \\ h(h+1)(h+2)\cdots(h+r-1), & \text{ if } \ r=1,\ 2,\ 3,\cdots \end{cases}$$

where $h \neq 0, -1, -2, -3, \cdots$ and the notation Γ denotes the gamma function.

Reduction Formula

We know that

$$I_n = \int \sin^n(x)\ \mathrm{d}x = \frac{-\sin^{(n-1)}(x)\cos(x)}{n} + \frac{n-1}{n}\ I_{n-2},\ n \geq 2 \tag{1}$$

$$I_0 = x, I_1 = -\cos x$$

By the successive application of above reduction formula, we can find

$$\int \sin^{2m}(x)\ \mathrm{d}x = \frac{-(\frac{1}{2})_m\ \sin(x)\cos(x)}{(1)_m} \sum_{r=0}^{m-1} \frac{(1)_r(\sin(x))^{2r}}{(\frac{3}{2})_r} + \frac{x\ (\frac{1}{2})_m}{(1)_m} + \text{Arbitrary constant} \tag{2}$$

for $m = 0, 1, 2, 3, \ldots$.

Series Identities

$$\text{The empty sum} \sum_{r=0}^{-1} F(r) \text{ is treated as zero.} \tag{3}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m} F(m,r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} F(m+r,r) \tag{4}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m-1} F(m,r) = \sum_{r=0}^{-1} F(0,r) + \sum_{m=1}^{\infty}\sum_{r=0}^{m-1} F(m,r) =$$

$$= 0 + \sum_{m=0}^{\infty}\sum_{r=0}^{m} F(m+1,r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} F(m+r+1,r) \tag{5}$$

provided that involved multiple power series, are absolutely convergent.

Gauss Ordinary Hypergeometric Function

In 1812, C. F. Gauss defined the following function

$${}_2F_1\left[\begin{array}{r} a,\ b\ ; \\ c\ ; \end{array} z\right] = \sum_{m=0}^{\infty} \frac{(a)_m\ (b)_m\ (z)^m}{(c)_m\ m!} = 1 + \frac{ab\ z}{c} + \frac{a(a+1)\ b(b+1)\ z^2}{c(c+1)\ 2!} +$$

$$+ \frac{a(a+1)(a+2)\ b(b+1)(b+2)\ z^3}{c(c+1)(c+2)\ 3!} + \cdots \text{ad inf.} \tag{6}$$

It is always convergent for $|z| < 1$ and the denominator parameter $c \neq 0, -1, -2, -3, \ldots$
Note:

$$ {}_2F_1\left[\begin{array}{c} a,\ b\ ;\\ c\ ; \end{array} 0\right] = {}_2F_1\left[\begin{array}{c} 0,\ b\ ;\\ c\ ; \end{array} z\right] = {}_2F_1\left[\begin{array}{c} a,\ 0\ ;\\ c\ ; \end{array} z\right] = 1 \tag{7} $$

Binomial theorem in hypergeometric notation, is given by

$$ (1-z)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r\ z^r}{r!} = {}_1F_0\left[\begin{array}{c} a\ \ ;\\ \text{—}\ ; \end{array} z\right]; |z| < 1 \tag{8} $$

Kampé de Fériet's Double Hypergeometric Function

In 1921, P. Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and P. Humbert's seven confluent double hypergeometric functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by J. Kampé de Fériet.
We recall the definition of general double hypergeometric functions of Kampé de Fériet in the slightly modified notation of Srivastava and Panda [8,pp.423-424 (26,27); see also 9,p.23(1.2,1.3)]

$$ F^{A:B;D}_{E:G;H}\left[\begin{array}{ll} (a_A) : (b_B); (d_D)\ ; & \\ & x\ , y \\ (e_E) : (g_G); (h_H)\ ; & \end{array}\right] $$

$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a_A)]_{m+n}\ [(b_B)]_m\ [(d_D)]_n\ x^m\ y^n}{[(e_E)]_{m+n}\ [(g_G)]_m\ [(h_H)]_n\ m!\ n!} \tag{9} $$

$$ = 1 + \sum_{m=1}^{\infty} \frac{[(a_A)]_m\ [(b_B)]_m\ x^m}{[(e_E)]_m\ [(g_G)]_m\ m!} + \sum_{n=1}^{\infty} \frac{[(a_A)]_n\ [(d_D)]_n\ y^n}{[(e_E)]_n\ [(h_H)]_n\ n!} + $$

$$ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{[(a_A)]_{m+n}\ [(b_B)]_m\ [(d_D)]_n\ x^m\ y^n}{[(e_E)]_{m+n}\ [(g_G)]_m\ [(h_H)]_n\ m!\ n!} \tag{10} $$

where (a_A) denotes the array of A parameters $a_1, a_2, \ldots, a_A$, $[(a_A)]_m = \prod_{j=1}^{A} (a_j)_m$ with similar interpretation for others and denominator parameters are neither zero nor negative integers. For convergence conditions of double series (9), we have
(i) $A + B < E + G + 1,\ A + D < E + H + 1$, for $|x| < \infty$, $|y| < \infty$
or, (ii) $A + B = E + G + 1,\ A + D = E + H + 1$ and

$$ \begin{cases} |x|^{\frac{1}{A-E}} + |y|^{\frac{1}{A-E}} < 1 & \text{if } A > E \\ \max\{|x|, |y|\} < 1 & \text{if } A \leq E \end{cases} $$

2 Incomplete Elliptic Integral Related with Simple Pendulum

Figure 1 shows a simple pendulum, where its motion along an arc is considered. The motion may be described in terms of the angle θ. We intend to find an exact expression for θ as a function of time t; subject to a given set of initial conditions.
Suppose θ_0 is the maximum angular displacement of pendulum bob (towards right or left) from the vertical position(PA).

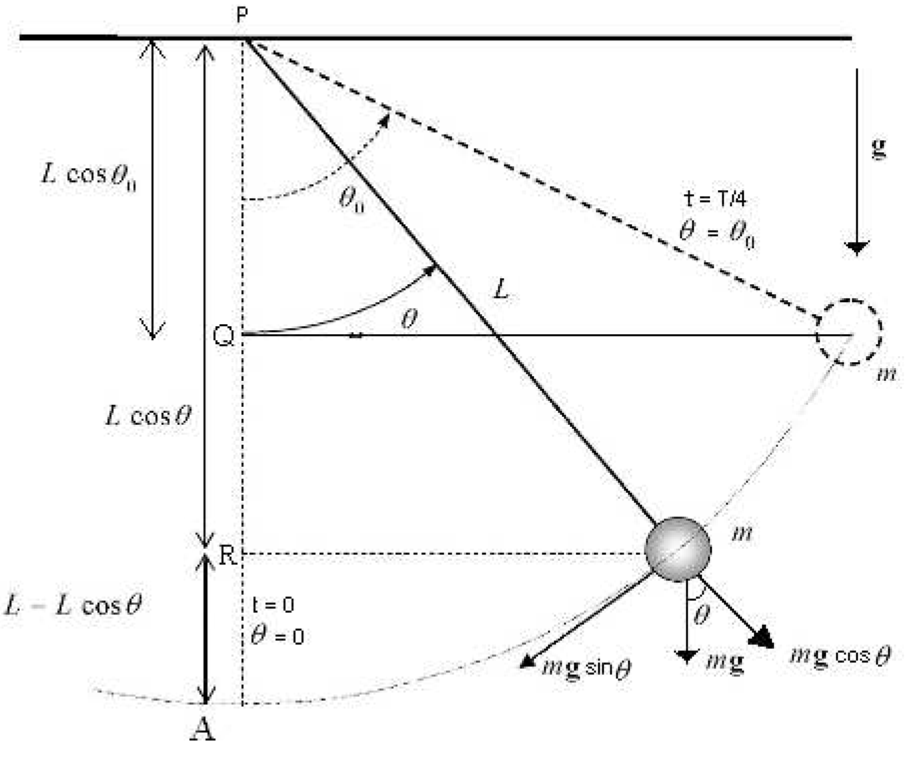

Figure 1

Let $t = 0$ be identified with the vertical position(i.e., with $\theta = 0$)

$$\text{At any arbitrary time } t, \text{the energy consists of Kinetic Energy} = \frac{\mathrm{mL}^2\dot{\theta}^2}{2} \tag{11}$$

$$\text{and Potential Energy} = \mathrm{mg}(\mathrm{L} - \mathrm{L}\cos\theta) \tag{12}$$

By the law of conservation of energy, the total of the two terms is constant.

$$\text{Total Energy} = \text{Kinetic Energy} + \text{Potential Energy} = \text{a constant}$$

Since the kinetic energy is zero, when the angular displacement is maximum (i.e., θ_0), we may write,

$$\frac{\mathrm{mL}^2\dot{\theta}^2}{2} + \mathrm{mgL}(1-\cos\theta) = \mathrm{mgL}(1-\cos\theta_0) \tag{13}$$

or

$$\frac{\mathrm{mL}^2\dot{\theta}^2}{2} + \mathrm{mgL}(\cos\theta_0-\cos\theta) = 0 \tag{14}$$

therefore,

$$\dot{\theta}^2 = \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \frac{2\mathrm{g}}{\mathrm{L}}(\cos\theta-\cos\theta_0)$$

i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2\mathrm{g}}{\mathrm{L}}(\cos\theta-\cos\theta_0)} \tag{15}$$

which is the well known differential equation of motion of a simple pendulum.

Let " t " be the time of swinging of simple pendulum from vertical position to an arbitrary angular position θ. It has been assumed that at $t=0$, the angular displacement θ is zero. Then the above differential equation may be integrated to yield

$$\int_{y=0}^{y=\theta} \frac{\mathrm{d}y}{\sqrt{\cos y-\cos\theta_0}} = \sqrt{\frac{2\mathrm{g}}{\mathrm{L}}}\int_{t=0}^{t}\mathrm{d}t = \left(\sqrt{\frac{2\mathrm{g}}{\mathrm{L}}}\right)t \tag{16}$$

$$\text{Hence } t = \sqrt{\frac{\mathrm{L}}{2\mathrm{g}}}\int_{y=0}^{y=\theta}\frac{\mathrm{d}y}{\sqrt{\cos y-\cos\theta_0}} = \sqrt{\frac{\mathrm{L}}{2\mathrm{g}}}\int_{y=0}^{y=\theta}\frac{\mathrm{d}y}{\sqrt{2\sin^2(\frac{\theta_0}{2})-2\sin^2(\frac{y}{2})}}$$

$$t = \frac{1}{2}\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\int_{y=0}^{y=\theta}\frac{\mathrm{d}y}{\sqrt{\sin^2(\frac{\theta_0}{2})-\sin^2(\frac{y}{2})}} \tag{17}$$

The integral involved in equation (17) is the incomplete elliptic integral of the first kind.

Here we are interested in the exact solution of equation (17), that can be obtained by using the hypergeometric approach as follows:

3 Hypergeometric Approach

Using the substitution $\sin(\frac{y}{2}) = \sin(\frac{\theta_0}{2}).\sin z$, we get

$$\mathrm{d}y = \frac{2\,\sin\left(\frac{\theta_0}{2}\right)\cos z\;\mathrm{d}z}{\sqrt{1-\sin^2(\frac{\theta_0}{2})\sin^2 z}}$$

It may now be noted that: when $y = 0 \Rightarrow \; z = 0$

$$\text{and when } y = \theta \Rightarrow \; z = \sin^{-1}\left[\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}\right] = z_1 \text{ (say)} \tag{18}$$

Therefore, (17) reduces to

$$t = \frac{1}{2}\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\int_0^{z_1} \frac{2\,\sin\left(\frac{\theta_0}{2}\right)\cos z\;\mathrm{d}z}{\sqrt{1-\sin^2(\frac{\theta_0}{2})\sin^2 z}\sqrt{\sin^2(\frac{\theta_0}{2})-\sin^2(\frac{\theta_0}{2})\sin^2 z}} \tag{19}$$

$$t = \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\int_0^{z_1} \frac{\mathrm{d}z}{\sqrt{1-\sin^2(\frac{\theta_0}{2})\sin^2 z}} \tag{20}$$

or

$$t = \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\int_{z=0}^{z=z_1}\left[1-\sin^2\left(\frac{\theta_0}{2}\right)\sin^2 z\right]^{\frac{-1}{2}}\mathrm{d}z \tag{21}$$

In hypergeometric notation (8), the integrand of incomplete elliptic integral of first kind involved in (20) or (21) can be written as

$$t = \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\int_{z=0}^{z=z_1} {}_1F_0\left[\begin{array}{c}\frac{1}{2}\\ \rule{1.5em}{0.4pt}\end{array};\;\sin^2\left(\frac{\theta_0}{2}\right)\sin^2 z\right]\mathrm{d}z$$

Using power series form of (8), interchanging the order of summation and integration, we get

$$t = \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m\left[\sin^2\left(\frac{\theta_0}{2}\right)\right]^m}{m!}\int_{z=0}^{z=z_1}\sin^{2m} z\;\mathrm{d}z \tag{22}$$

Now using the integral (2) in (22), we get

$$t = \left[\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(\sin\frac{\theta_0}{2})^{2m}}{m!}\right]\times$$

$$\times\left[\left\{\frac{-(\frac{1}{2})_m\sin(z_1)\cos(z_1)}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r(\sin(z_1))^{2r}}{(\frac{3}{2})_r}\right\}+\left\{\frac{z_1\;(\frac{1}{2})_m}{(1)_m}\right\}\right] \tag{23}$$

$$= z_1\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m\;(\frac{1}{2})_m\;(\sin\frac{\theta_0}{2})^{2m}}{m!\;(1)_m} - \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sin(z_1)\cos(z_1)\times$$

$$\times\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\frac{(\frac{1}{2})_m\;(\frac{1}{2})_m\;(1)_r\;(\sin^2(\frac{\theta_0}{2}))^m\;(\sin(z_1))^{2r}}{m!\;(1)_m(\frac{3}{2})_r} \tag{24}$$

Further using double series identity (5) and hypergeometric notation (6), in (24) we get

$$t = z_1\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\,{}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{array};\ \begin{array}{c};\\;\end{array} \sin^2\left(\frac{\theta_0}{2}\right)\right] - \sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sin(z_1)\cos(z_1)\times$$

$$\times\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\frac{1}{2})_{m+r+1}\ (\frac{1}{2})_{m+r+1}\ (1)_r\ (\sin^2(\frac{\theta_0}{2}))^{m+r+1}\ (\sin(z_1))^{2r}}{(1)_{m+r+1}\ (1)_{m+r+1}\ (\frac{3}{2})_r} \tag{25}$$

$$= z_1\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\,{}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{array};\ \begin{array}{c};\\;\end{array} \sin^2\left(\frac{\theta_0}{2}\right)\right] - \frac{1}{4}\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\sin(z_1)\cos(z_1)\ \sin^2\left(\frac{\theta_0}{2}\right)\times$$

$$\times\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\frac{3}{2})_{m+r}\ (\frac{3}{2})_{m+r}\ (1)_r\ (1)_m\ (1)_r\ (\sin^2(\frac{\theta_0}{2}))^m\ (\sin^2(\frac{\theta_0}{2}))^r\ (\sin^2(z_1))^r}{(2)_{m+r}\ (2)_{m+r}\ (\frac{3}{2})_r\ m!\ r!} \tag{26}$$

Using $z_1 = \sin^{-1}\left[\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}\right]$ from (18) and writing double power series of (26) in hypergeometric notation (9), after simplification we get

$$t = \frac{T(0)z_1}{2\pi}\,{}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2} \\ 1 \end{array}\begin{array}{c};\\;\end{array}\ {\alpha_0}^2\right] - \frac{T(0)\alpha\sqrt{{\alpha_0}^2-\alpha^2}}{8\pi}F^{2:1;2}_{2:0;1}\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2} \\ 2,\ 2 \end{array}\begin{array}{c}:\\:\end{array}\begin{array}{c} 1 \\ \_\_\_ \end{array}\begin{array}{c};\\;\end{array}\begin{array}{c} 1,\ 1 \\ \frac{3}{2} \end{array}\begin{array}{c};\\;\end{array}\ {\alpha_0}^2,\ \alpha^2\right] \tag{27}$$

where

$$\left.\begin{array}{l} \alpha = \sin(\frac{\theta}{2}) \\ \alpha_0 = \sin(\frac{\theta_0}{2}) \\ T(0) = 2\pi\sqrt{\frac{\mathrm{L}}{\mathrm{g}}} \\ \text{and } z_1 = \sin^{-1}(\frac{\alpha}{\alpha_0}) \end{array}\right\} \tag{28}$$

where $0 \le \theta \le \theta_0 \le \frac{\pi}{2}$ and $\theta = 0$ at $t = 0$.

The equation (27) is the exact solution of equation of motion (17) of a simple pendulum yielding an implicit function $\theta(t)$. The above result for arbitrary time t is always convergent [see convergence conditions of Kampé de Fériet's double hypergeometric function (9) and Gauss ordinary hypergeometric function ${}_2F_1$ (6)].

The result (27) is not available in literature, though many approximate expressions for time period "T" are available[1-7].

4 Some Deductions

(a) Exact Time Period for Arbitrary Amplitude

To find the time period, we use equation (27). In (27), when $\theta = \theta_0$ then $z_1 = \sin^{-1}(1) = \frac{\pi}{2}$ and $t = \frac{T}{4}$ (The bob has completed one fourth of an oscillation).
The exact time period T of simple pendulum of given length L, is therefore given by

$$T(\theta_0) = \left(2\pi\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}\right) {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} & ; \\ 1 & ; \end{matrix} \sin^2\left(\frac{\theta_0}{2}\right)\right] = T(0) {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} & ; \\ 1 & ; \end{matrix} \sin^2\left(\frac{\theta_0}{2}\right)\right] \tag{29}$$

where "g" is the acceleration due to gravity and ${}_2F_1$ is the Gauss ordinary hypergeometric function given by (6). Here θ_0 is the maximum angular displacement of the pendulum from vertical position (and corresponds to $t = \frac{T}{4}$). Obviously $0 \leq \theta_0 \leq \frac{\pi}{2}$.

(b) Simple Harmonic Approximation

When $\theta_0 \approx 0$ then (29) reduces to

$$\lim_{\theta_0 \to 0} T(\theta_0) \approx 2\pi\sqrt{\frac{\mathrm{L}}{\mathrm{g}}} \approx T(0) \tag{30}$$

which is the well known approximate formula for the time period of a simple pendulum.

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