Gen. Math. Notes, Vol. 17, No. 1, July, 2013, pp.32-44

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On Ideal Version of Lacunary Statistical Convergence of Double Sequences

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(Received: 13-2-13 / Accepted: 24-4-13)

Abstract

For any double lacunary sequence $\theta_{rs} = \{(k_r, l_s)\}$ and an admissible ideal $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$, the aim of present work is to define the concepts of $N_{\theta_{rs}}(\mathcal{I}_2)-$ and $S_{\theta_{rs}}(\mathcal{I}_2)-$convergence for double sequence of numbers. We also present some inclusion relations between these notions and prove that $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty$ and $S_2(\mathcal{I}_2) \cap \ell^2_\infty$ are closed subsets of ℓ^2_∞, the space of all bounded double sequences of numbers.

Keywords: *Double sequences and multiple sequences, $\mathcal{I}-$convergence, Lacunary sequences, Statistical convergence.*

1 Introduction and Background

Fast[4] presented an interesting generalization of the usual sequential limit which he called statistical convergence for number sequences. This idea turns out very useful functional tool to resolve many convergence problems arising in Fourier Analysis, Ergodic Theory, Number Theory and Analysis. In past few years, statistical convergence is further investigated from the sequence space point of view and linked with summability theory by Connor [2], Fridy [5], Maddox [10], Šalát [12] and many others.

Definition 2.1[4] *A number sequence $x = (x_k)$ is said to be statistically convergent to a number L (denoted by $S - \lim_{k\to\infty} x_k = L$) provided that for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bars denote the cardinality of the enclosed set. Let S denotes the set of all statistically convergent sequences of numbers.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, where the ratio $\frac{k_r}{k_{r-1}}$ is denoted by q_r.

Using lacunary sequence, Fridy and Orhan [6] generalized statistical convergence as follows:

Definition 2.2[6] *Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent to a number L (denoted by $S_\theta - \lim_{k\to\infty} x_k = L$) if for each $\epsilon > 0$,*

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0.$$

Let S_θ denotes the set of all lacunary statistically convergent sequences of numbers.

For any non-empty set X, $\mathcal{P}(X)$ denotes the power set of X.

A family of sets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal in X if and only if (i) $\emptyset \in \mathcal{I}$; (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$; (iii) For $A \in \mathcal{I}$ and $B \subseteq A$ we have $B \in \mathcal{I}$.

A non-empty family of sets $\mathcal{F} \subset \mathcal{P}(X)$ is called a filter on X if and only if (i) $\emptyset \notin \mathcal{F}$; (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$; (iii) For $A \in \mathcal{F}$ and $B \supseteq A$ we have $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.

It immediately implies that $\mathcal{I} \subset \mathcal{P}(X)$ is a non-trivial ideal if and only if the class $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on X. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

A non-trivial ideal $\mathcal{I} \subset \mathcal{P}(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e. if it contains $\{\{x\} : x \in X\}$. Throughout the paper, $\mathcal{I}$ is considered as a non-trivial admissible ideal.

An admissible ideal $\mathcal{I} \subset \mathcal{P}(X)$ is said to be satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2...\}$ belonging to $\mathcal{I}$ there exists a countable family $\{B_1, B_2...\}$ in $\mathcal{I}$ such that $A_i \triangle B_i$ is a finite set for each $i \in N$ and $B = \cup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Using the above terminology, Kostyrko et.al. [9] defined $\mathcal{I}-$convergence in a metric space as follows:

Definition 2.3[9] *Let $\mathcal{I} \subset \mathcal{P}(N)$ be a non-trivial ideal in N and (X, ρ) be a metric space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}-$convergent to ξ if for each $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geq \epsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim_{k\to\infty} x_k = \xi$.*

Recently, Das et.al. [3] unified the ideas of statistical convergence and ideal convergence to introduce new concepts of $\mathcal{I}-$statistical convergence and $\mathcal{I}-$lacunary statistical convergence as follows:

Definition 2.4[3] *A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$statistical convergent or $S(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$ and $\delta > 0$*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S(\mathcal{I}))$ or $S(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. Let $S(\mathcal{I})$ denotes the set of all $\mathcal{I}-$statistically convergent sequences of numbers.

Definition 2.5[3] *Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}-$lacunay statistical convergent or $S_\theta(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$ and $\delta > 0$*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L(S_\theta(\mathcal{I}))$ or $S_\theta(\mathcal{I}) - \lim_{k\to\infty} x_k = L$. The set of all $\mathcal{I}-$lacunary statistically convergent sequences will be denoted by $S_\theta(\mathcal{I})$.

Definition 2.6[3] *Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be $N_\theta(\mathcal{I})-$convergent to L, if for every $\epsilon > 0$ we have*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \epsilon \right\} \in \mathcal{I}.$$

It is denoted by $x_k \to L(N_\theta(\mathcal{I}))$ and the class of such sequences will be denoted by simply $N_\theta(\mathcal{I})$.

In recent years, the above ideas of statistical convergence, lacunary statistical convergence and $\mathcal{I}-$convergence have been respectively extended from single to double sequences in [8], [11], [13], [14] and [15]. We now quote the following definitions, which will be needed in the sequel.

Definition 2.7[11] *A double sequence $x = (x_{ij})$ of numbers is said to be statistically convergent to a number L in the Pringsheim sense (denoted by $S_2 - \lim_{i,j\to\infty} x_{ij} = L$) provided that for every $\epsilon > 0$,*

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{i \leq m, j \leq n : |x_{ij} - L| \geq \epsilon\}| = 0.$$

In this case we write, $S_2 - P - \lim_{i,j\to\infty} x_{ij} = L$. Let S_2 denotes the set of all double sequences, which are statistically convergent.

By a double lacunary sequence $\theta_{rs} = \{(k_r, l_s)\}$, we mean there exists two lacunary sequences $\theta_r = (k_r)$ and $\theta_s = (l_s)$. Let $h_r = k_r - k_{r-1}$, $q_r = \frac{k_r}{k_{r-1}}$, $I_r = (k_{r-1}, k_r]$, $\overline{h}_s = l_s - l_{s-1}$, $\overline{q}_s = \frac{l_s}{l_{s-1}}$, $\overline{I}_s = (l_{s-1}, l_s]$, $k_{rs} = k_r l_s$, $h_{rs} = h_r \overline{h}_s$, $q_{rs} = q_r \overline{q}_s$ and the interval determined by θ_{rs} is denoted by $I_{rs} = \{(i,j) : k_{r-1} < i \leq k_r, l_{s-1} < j \leq l_s\}$.

Definition 2.8[13] *Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be lacunary statistically convergent to a number L in the Pringsheim sense (denoted by $S_{\theta_{rs}} - P - \lim_{i,j\to\infty} x_{ij} = L$) if for each $\epsilon > 0$,*

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| = 0.$$

Let $S_{\theta_{rs}}$ denotes the set of all lacunary statistically convergent double sequences.

Definition 2.9[13] *A double sequence $x = (x_{ij})$ is said to be strongly Cesàro summable to a number L if*

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0.$$

$$\text{Let} \quad |\sigma_{11}| = \left\{ x = (x_{ij}) : \exists \text{ some} L, P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1,j=1}^{m,n} |x_{ij} - L| = 0 \right\}$$

whereas $|\sigma_1|$ denotes the space of all strongly Cesàro summable single sequences.

Definition 2.10[13] *Let θ_{rs} be a double lacunary sequence. A double sequence $x = (x_{ij})$ of numbers is said to be $N_{\theta_{rs}} - P-$convergent to a number L* if

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = 0.$$

$$\text{Let} \quad N_{\theta_{rs}} = \left\{ x = (x_{ij}) : \exists \text{some } L,\ P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = 0 \right\}.$$

Definition 2.11[8] *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$ convergent in the Pringsheim sense to a number L, if for every $\epsilon > 0$*

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - P - \lim_{i,j\to\infty} x_{ij} = L$.

Definition 2.12[1] *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$statistical convergent or $S_2(\mathcal{I}_2)-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(S_2(\mathcal{I}_2))$ or $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. Let $S_2(\mathcal{I}_2)$ denotes the set of all $\mathcal{I}_2-$statistically convergent double sequences of numbers.

We now consider the lacunary statistical ideal convergence of double sequences, which we call $S_{\theta_{rs}}(\mathcal{I}_2)-$convergence of double sequences.

2 Main Results

Definition 3.1 *Let θ_{rs} be a double lacunary sequence and $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ be a non-trivial ideal. A double sequence $x = (x_{ij})$ of numbers is said to be $\mathcal{I}_2-$lacunary statistical convergent or $S_{\theta_{rs}}(\mathcal{I}_2)-$convergent to L, if for each $\epsilon > 0$ and $\delta > 0$*

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$ *or* $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. *Let* $S_{\theta_{rs}}(\mathcal{I}_2)$ *denotes the set of all* $\mathcal{I}_2-$*lacunary statistically convergent double sequences of numbers.*

Definition 3.2 *Let* θ_{rs} *be a double lacunary sequence and* $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ *be a non-trivial ideal. A double sequence* $x = (x_{ij})$ *of numbers is said to be* $N_{\theta_{rs}}(\mathcal{I}_2)-$*convergent to* L*, if for every* $\epsilon > 0$ *we have*

$$\left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \epsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$ *or* $N_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$. *Let* $N_{\theta_{rs}}(\mathcal{I}_2)$ *denotes the set of all* $N_{\theta_{rs}}(\mathcal{I}_2)-$*convergent double sequences of numbers.*

Example 3.1 If we take $\mathcal{I}_2 = \{E \subset \mathbb{N}\times\mathbb{N} : E = (\mathbb{N}\times H)\cup(H\times\mathbb{N})$ for some finite subset H of $\mathbb{N}\}$ and $\theta_r = (2^r)$, $\theta_s = (3^s)$ be two lacunary sequences. We take a special set $A \in \mathcal{I}_2$ and define a sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} \sqrt{ij}, & \text{for } (r,s) \notin A,\ 2^{r-1}+1 \leq i \leq 2^{r-1} + [\sqrt{h_r}] \text{ and } 3^{s-1}+1 \leq j \leq 3^{s-1} + [\sqrt{\overline{h}_s}], \\ \sqrt{ij}, & \text{for } (r,s) \in A,\ 2^{r-1} < i \leq 2^{r-1} + h_r \text{ and } 3^{s-1} < j \leq 3^{s-1} + \overline{h}_s, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_r = (2^{r-1}, 2^r]$ and $I_s = (3^{s-1}, 3^s]$.
Then for each $\epsilon > 0$, we have

$$P - \lim_{r,s\to\infty} \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \leq \frac{[\sqrt{h_r}].[\sqrt{\overline{h}_s}]}{h_{rs}} \to 0$$

for $(r,s) \notin A$.
For $\delta > 0$, there exists a positive integer r_0 such that

$$\frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| < \delta$$

for every $(r,s) \notin A$ and $r,s \geq r_0$.
Let $B = \{1, 2, \cdots r_0 - 1\}$ and $K = \{(r,s) \notin A : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\}$. Then clearly $K \subseteq (\mathbb{N}\times B)\cup(B\times\mathbb{N})$ and $K \in \mathcal{I}_2$ by structure of the ideal $\mathcal{I}_2$. Hence

$$\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\} \subset A \cup K.$$

It follows that $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = 0$. But similarly $P - \lim_{r,s\to\infty} \frac{1}{h_r \overline{h}_s} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \nrightarrow 0$. This example shows that $S_{\theta_{rs}}(\mathcal{I}_2)-$convergence is a generalization of $S_{\theta_{rs}}-$convergence for the double sequences.

Theorem 3.1 *For an admissible ideal $\mathcal{I}_2$, $S_{\theta_{rs}}(\mathcal{I}_2)-$limit of any double sequence if exists is unique.*

Theorem 3.2 *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be an admissible ideal and θ_{rs} be a double lacunary sequence. Then, the set $S_{\theta_{rs}}(\mathcal{I}_2)$ is closed under the operations of additions and scalar multiplication.*

Proof The proof of this theorem is obvious.

The following theorem is a multidimensional analogue of Fridy and Orhan's theorem presented in [6].

Theorem 3.3 *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be an admissible ideal and θ_{rs} be a double lacunary sequence. Then we have the following:*
(i) $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$ implies $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$;
(ii)$N_{\theta_{rs}}(\mathcal{I}_2)$ is a proper subset of $S_{\theta_{rs}}(\mathcal{I}_2)$;
(iii) If $x=(x_{ij}) \in \ell^2_\infty$ and $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$ then $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$;
(iv) $S_{\theta_{rs}}(\mathcal{I}_2)\cap \ell^2_\infty = N_{\theta_{rs}}(\mathcal{I}_2)\cap \ell^2_\infty$;
where ℓ^2_∞ denotes the space of all bounded double sequences.

Proof. (i) Suppose $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$. For $\epsilon>0$, we can write

$$\begin{aligned}\sum_{(i,j)\in I_{rs}} |x_{ij}-L| &\geq \sum_{(i,j)\in I_{rs}\&|x_{ij}-L|\geq\epsilon} |x_{ij}-L| \\ &\geq \epsilon|\{(i,j)\in I_{rs} : |x_{ij}-L|\geq\epsilon\}|;\end{aligned}$$

which implies

$$\frac{1}{\epsilon.h_{rs}}\sum_{(i,j)\in I_{rs}} |x_{ij}-L| \geq \frac{1}{h_{rs}}|\{(i,j)\in I_{rs} : |x_{ij}-L|\geq\epsilon\}|.$$

Thus for any $\delta>0$, we have the containment

$$\begin{aligned}&\left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j)\in I_{rs} : |x_{ij}-L|\geq\epsilon\}|\geq\delta\right\} \\ &\subseteq \left\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}\sum_{(i,j)\in I_{rs}}|x_{ij}-L|\geq\epsilon\,\delta\right\}.\end{aligned}$$

Since $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$, it follows that the later set belongs to $\mathcal{I}_2$ and hence $\{(r,s)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j)\in I_{rs} : |x_{ij}-L|\geq\epsilon\}|\geq\delta\}\in\mathcal{I}_2$. This shows that $x_{ij}\to L(S_{\theta_{r,s}}(\mathcal{I}_2))$.

(ii) Let $x = (x_{ij})$ be defined as follows:

$$x_{ij} = \begin{pmatrix} 1 & 2 & 3 & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ 2 & 2 & 3 & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & [\sqrt[3]{h_{rs}}] & \dots & \dots & [\sqrt[3]{h_{rs}}] & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is clear that (x_{ij}) is an unbounded double sequence. Moreover, for each $\epsilon > 0$

$$\frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \leq \frac{[\sqrt[3]{h_{rs}}]}{h_{rs}}$$

which immediately implies for any $\delta > 0$, the containment

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{[\sqrt[3]{h_{rs}}]}{h_{rs}} \geq \delta\right\}.$$

Since $P - \lim \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0$, it follows that the set on the right side is finite and therefore belongs to $\mathcal{I}_2$. This shows that

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - 0| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}_2$$

and therefore we have $x_{ij} \to 0(S_{\theta_{rs}}(\mathcal{I}_2))$. On the other hand

$$\frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - 0| = \frac{1}{h_{rs}} \frac{[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}] + 1))}{2} \to \frac{1}{2} \; as \; r,s \to \infty$$

implies that the sequence $(\frac{1}{h_{rs}}[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}] + 1))) \to 1 \; as \; r,s \to \infty$, which gives for $\epsilon = \frac{1}{4}$

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j) \in I_{rs}} |x_{ij} - 0| \geq \frac{1}{4}\right\}$$
$$= \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}[\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}]([\sqrt[3]{h_{rs}}] + 1)) \geq \frac{1}{2}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

This shows that $x_{ij} \to 0(N_{\theta_{rs}}(\mathcal{I}_2))$ does not hold.
(iii) Suppose that $x = (x_{ij}) \in \ell^2_\infty$ such that $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$. Then there

exists a $M > 0$ such that $|x_{ij} - L| \leq M$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Also for each $\epsilon > 0$, we can write

$$\frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| = \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}, |x_{ij}-L|\geq \frac{\epsilon}{2}} |x_{ij} - L| + \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}, |x_{ij}-L|< \frac{\epsilon}{2}} |x_{ij} - L|$$
$$\leq \frac{M}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \frac{\epsilon}{2}\}| + \frac{\epsilon}{2}.$$

Consequently, we get

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \epsilon\right\}$$
$$\subseteq \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} |\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \frac{\epsilon}{2}\}| \geq \frac{\epsilon}{2M}\right\}.$$

Since $x_{ij} \to L(S_{\theta_{rs}}(\mathcal{I}_2))$, it follows that the later set belongs to $\mathcal{I}_2$, which immediately implies $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(i,j)\in I_{rs}} |x_{ij} - L| \geq \epsilon\right\} \in \mathcal{I}_2$. This shows that $x_{ij} \to L(N_{\theta_{rs}}(\mathcal{I}_2))$.
(iv) This is an immediate consequence of (i) and (iii). ■

Theorem 3.4 *Let $\mathcal{I}_2 \subseteq \mathcal{P}(\mathbb{N}\times\mathbb{N})$ be an admissible ideal satisfying the condition (AP) and θ_{rs} be a double lacunary sequence such that $\theta_{rs} \in \mathcal{F}(\mathcal{I}_2)$. If $x = (x_{ij}) \in S_2(\mathcal{I}_2) \cap S_{\theta_{rs}}(\mathcal{I}_2)$ then $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij}$.*

Proof. Suppose $S_2(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L$ and $S_{\theta_{r,s}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L'$ where $L \neq L'$. Select $0 < \epsilon < \frac{|L-L'|}{2}$. Since $\mathcal{I}_2$ satisfies the condition (AP), so there is a set $M = \{(m_p, n_q) : p, q = 1, 2, \ldots\} \subseteq \mathbb{N}\times\mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I}_2)$ and

$$P - \lim_{p,q\to\infty} \frac{1}{m_p n_q} |\{i \leq m_p, j \leq n_q : |x_{ij} - L| \geq \epsilon\}| = 0.$$

Let

$$A = |\{i \leq m_p, j \leq n_q : |x_{ij} - L| \geq \epsilon\}| \quad \text{and}$$
$$B = |\{i \leq m_p, j \leq n_q : |x_{ij} - L'| \geq \epsilon\}|.$$

Then $m_p\, n_q = |A \cup B| \leq |A| + |B|$, which implies that $1 \leq \frac{|A|}{m_p\, n_q} + \frac{|B|}{m_p\, n_q}$. Since $\lim_{p,q\to\infty} \frac{|B|}{m_p\, n_q} \leq 1$ and $\lim_{p,q\to\infty} \frac{|A|}{m_p\, n_q} = 0$, so we must have $\lim_{p,q\to\infty} \frac{|B|}{m_p . n_q} = 1$. Let $M^\star = M \cap \theta_{rs}$, then $M^\star \in F(\mathcal{I}_2)$ and therefore an infinite set. Let $M^\star = \{(k_{\alpha_t}, l_{\beta_{t'}}) : t, t' = 1, 2, \ldots\}$. Consider the $(k_{\alpha_t}\, l_{\beta_{t'}})^{th}$ term of statistical limit expression $\frac{1}{m_p n_q}|\{i \leq m_p, j \leq n_q : |x_{ij} - L'| \geq \epsilon\}|$;

$$\begin{aligned}
&\frac{1}{k_{\alpha_t} l_{\beta_{t'}}}|\{i \leq k_{\alpha_t}, j \leq l_{\beta_{t'}} : |x_{ij} - L'| \geq \epsilon\}| \\
&= \frac{1}{k_{\alpha_t} l_{\beta_{t'}}}|\{(i,j) \in \bigcup_{u=1,v=1}^{\alpha_t,\beta_{t'}} I_{uv} : |x_{ij} - L'| \geq \epsilon\}| \\
&= \frac{1}{k_{\alpha_t} l_{\beta_{t'}}} \sum_{u=1,v=1}^{\alpha_t,\beta_{t'}} |\{(i,j) \in I_{uv} : |x_{ij} - L'| \geq \epsilon\}| \\
&\leq (\frac{1}{\sum_{u=1,v=1}^{\alpha_t,\beta_{t'}} h_{uv}}) \sum_{u=1,v=1}^{\alpha_t,\beta_{t'}} h_{uv}\, z_{uv} \quad (*)
\end{aligned}$$

where $z_{uv} = \frac{1}{h_{uv}}|\{(i,j) \in I_{uv} : |x_{ij} - L'| \geq \epsilon\}| \to 0(\mathcal{I}_2)$ as $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim_{i,j\to\infty} x_{ij} = L'$. Since θ_{rs} be a double lacunary sequence and $(*)$ satisfies all the conditions for a four dimensional matrix transformation to map pringsheim null sequence into pringsheim null sequence [7] and therefore it is also $\mathcal{I}_2-$ convergent to zero as $t, t' \to \infty$ and so it has a subsequence which is convergent to zero since $\mathcal{I}_2$ satisfies the (AP) condition. But since this is also a subsequence of $(\frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L'| \geq \epsilon\}|)_{(m,n)\in\mathbb{M}}$, we infer that $\{\frac{1}{mn}|\{1 \leq i \leq m, 1 \leq j \leq n : |x_{ij} - L'| \geq \epsilon\}|\}$ does not converge to 1. Which is a contradiction. Hence $L = L'$.

Next we give two results on the closed-ness of the sets $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty$ and $S_2(\mathcal{I}_2) \cap \ell^2_\infty$ out of which first is proved and the proof for the later can be obtained similarly.

Theorem 3.5 *The set $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty$ is closed subset of ℓ^2_∞, the space of all bounded double sequences endowed with the superior norm.*

Proof Let $x^{mn} = (x^{mn}_{ij})$ be a convergent sequence in $S_{\theta_{rs}}(\mathcal{I}_2) \cap \ell^2_\infty$. Suppose $x^{(mn)}$ converges to x. It is clear $x \in \ell^2_\infty$. Since $x^{(mn)} \in S_{\theta_{rs}}(\mathcal{I}_2)$, therefore there exists L_{mn} such that $S_{\theta_{rs}}(\mathcal{I}_2) - P - \lim x^{(mn)}_{ij} = L_{mn}$ $(m, n = 1, 2, 3, \ldots)$. As $x^{(mn)} \to x$ implies $x^{(mn)}$ is a Cauchy sequence. So for each $\epsilon > 0$, there exists a positive integer n_0 such that for every $p \geq m \geq n_0, q \geq n \geq n_0$, we have

$$|x^{(pq)} - x^{(mn)}| < \frac{\epsilon}{3}. \tag{1}$$

Since $x^{(mn)}_{ij} \to L_{mn}(S_{\theta_{rs}}(\mathcal{I}_2))$, therefore for every $\epsilon > 0$ and $\delta > 0$, if we denote the sets

$$K_1 = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x^{mn}_{ij} - L_{mn}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}\right\} \quad \text{and}$$

$$K_2 = \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x^{pq}_{ij} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}\right\},$$

then $\phi \neq K_1 \cap K_2 \in F(\mathcal{I}_2)$. Let $(r,s) \in K_1 \cap K_2$, then we have $\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{mn} - L_{mn}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}$ and $\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{pq} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \frac{\delta}{3}$, which implies that

$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{mn} - L_{mn}| \geq \frac{\epsilon}{3} \vee |x_{ij}^{pq} - L_{pq}| \geq \frac{\epsilon}{3}\}| < \delta < 1.$$

This shows that there exists a pair $(i_0, j_0) \in I_{r,s}$ for which $|x_{i_0j_0}^{mn} - L_{mn}| < \frac{\epsilon}{3}$ and $|x_{i_0j_0}^{pq} - L_{pq}| < \frac{\epsilon}{3}$. Moreover, for $p \geq m \geq n_0$ and $q \geq n \geq n_0$, we have $|L_{pq} - L_{mn}| = |L_{pq} - x_{i_0j_0}^{pq}| + |x_{i_0j_0}^{pq} - x_{i_0j_0}^{mn}| + |x_{i_0j_0}^{mn} - L_{mn}| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Thus (L_{mn}) is a Cauchy double sequence in $\mathbb{R}(or\mathbb{C})$ and consequently there is a number L such that $L_{mn} \longrightarrow L$ as $m,n \longrightarrow \infty$. Now to prove the theorem it is sufficient to show that the sequence $x = (x_{ij}) \longrightarrow L(S_{\theta_{rs}}(\mathcal{I}_2))$. Since x^{mn} is convergent to $x \in \ell_\infty^2$ so by the structure of ℓ_∞^2, it is coordinate-wise convergent. Therefore for each $\epsilon > 0$, there exists a positive integer $n_1(\epsilon)$ such that

$$|x_{ij}^{mn} - x_{ij}| < \frac{\epsilon}{3}, \forall\ m,n \geq n_1(\epsilon). \tag{2}$$

Also $L_{mn} \to L$, so for each $\epsilon > 0$, we can find another positive integer $n_2(\epsilon)$ such that

$$|L_{mn} - L| < \frac{\epsilon}{3}, \forall\ m,n \geq n_2(\epsilon). \tag{3}$$

Choose $n_3(\epsilon) = \max\ \{n_1(\epsilon), n_2(\epsilon)\}$ and $m_0, n_0 \geq n_3(\epsilon)$. Then for any $(i,j) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned} |x_{ij} - L| &\leq |x_{ij} - x_{ij}^{(m_0n_0)}| + |x_{ij}^{(m_0n_0)} - L_{m_0n_0}| + |L_{m_0n_0} - L| \\ &< \frac{\epsilon}{3} + |x_{ij}^{(m_0n_0)} - L_{m_0n_0}| + \frac{\epsilon}{3}(by\ (2) \text{ and } (3)) \end{aligned}$$

and therefore the containment

$$\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\} \subseteq \{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}\text{implies}$$
$$\frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \leq \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}|.$$

Further, for any $\delta > 0$ we have

$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}| < \delta\right\} \subseteq$$
$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| < \delta\right\}.$$

Since $\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij}^{m_0n_0} - L_{m_0n_0}| \geq \frac{\epsilon}{3}\}| < \delta\right\} \in F(\mathcal{I}_2)$ so

$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{r,s} : |x_{ij} - L| \geq \epsilon\}| < \delta\right\} \in F(\mathcal{I}_2)$. Hence $\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}}|\{(i,j) \in I_{rs} : |x_{ij} - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}_2$. This shows that $x = (x_{ij}) \to L(S_{\theta_{rs}}(\mathcal{I}_2)$. Which completes the proof of the theorem.

Theorem 3.6 *The set $S_2(\mathcal{I}_2) \cap \ell^2_\infty$ is closed subset of ℓ^2_∞, the space of all bounded double sequences endowed with the superior norm.*

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