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Strong Lacunary Statistical Limit and Cluster Points on Probabilistic Normed Spaces

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Abstract

For any lacunary sequence $\theta = (k_r)$, the aim of the present work is to introduce strong θ-statistical limit and strong θ-statistical cluster points of sequences on probabilistic normed spaces (briefly PN-spaces). Some relations among the sets of ordinary limit points, strong θ-statistical limit and strong θ-statistical cluster points of sequences on PN-spaces are obtained.

Keywords: *Lacunary sequence, PN-space, statistical convergence, statistical limit and cluster point.*

1 Introduction

The idea of statistical convergence of a number sequence was introduced by Fast [5], later developed in [3], [6], [16], [17] and many others. Fridy [7] used statistical convergence to introduce the set Λ_x of all statistical limit points and the set Γ_x of all statistical cluster points of a sequence $x = (x_k)$ of real numbers and discussed some interesting relations. These issues have been further explored in different directions by many authors (see [14], [2], [8] and [4]).

Menger [13] introduced probabilistic metric space (PM-space) to resolve the interpretative issue of quantum mechanics. He replaced the distance between points p and q by a distribution function F_{pq} whose value $F_{pq}(x)$ at the real number x is interpreted as the probability that the distance between p and q is less than x.

An important family of PM-spaces are PN-spaces. PN-spaces were first introduced by Šerstnev [19] by means of a definition that was closely molded to the definition of normed space. In 1993, Alsina *et al.* [1] presented a new definition of a PN-space which includes the definition of Šerstnev as a special case. In recent years, statistical convergence and related notions are found useful to handle many convergence problems arising on PN-spaces. For instance [8], [9], [10], [11], [12], [15] and [18].

In this paper, we use lacunary sequence $\theta = (k_r)$ to define strong θ-statistical limit and strong θ-statistical cluster points of sequences on PN-spaces. For the sake of convenience we recall some definitions. Let $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of reals, $\mathbb{R}^+ = [0, \infty]$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

Definition 1.1 *A distribution function is a non decreasing function F defined on $\overline{\mathbb{R}}$ with $F(-\infty) = 0$ and $F(\infty) = 1$.*

Let Δ denotes the set of all distribution functions that are left continuous on $(-\infty, \infty)$. The elements of Δ are partially ordered via $F \leq G$ if and only if $F(x) \leq G(x)\ \forall x \in \mathbb{R}$. For any $a \in \mathbb{R}$, ε_a, the unit step at a, is the function in Δ given by

$$\epsilon_a(x) = \begin{cases} 0, & \text{if } -\infty \leq x \leq a, \\ 1, & \text{if } a \leq x \leq \infty \end{cases}$$

and

$$\epsilon_\infty(x) = \begin{cases} 0, & \text{if } -\infty \leq x \leq \infty, \\ 1, & \text{if } x = \infty \end{cases}$$

The distance $d_L(F, G)$ between two functions $F, G \in \Delta$ is defined as the infimum of all numbers $h \in (0, 1]$ such that the inequalities

$$F(x-h) - h \leq G(x) \leq F(x+h) + h, G(x-h) - h \leq F(x) \leq G(x+h) + h$$

hold for every $x \in (-\frac{1}{h}, \frac{1}{h})$. It is known that d_L is a metric on Δ.

Definition 1.2 *A distance distribution function is a non decreasing function F defined on $\mathbb{R}^+ = [0, \infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0, \infty)$.*

Let $\triangle^+$ denotes the set of all distance distribution functions.

Definition 1.3 *A triangular norm, briefly, a t-norm is a function* $T : [0,1] \times [0,1] \longrightarrow [0,1]$ *that satisfies the following conditions:*
(i) T is commutative, *i.e.*, $T(s,t) = T(t,s)$ for all s and t in $[0,1]$;
(ii) T is associative, *i.e.*, $T(T(s,t),u) = T(s, T(t,u))$ for all s, t and u in $[0,1]$;
(iii) T is nondecreasing, *i.e.*, $T(s,t) < T(s^{'},t)$ for all $t, s, s^{'} \in [0,1]$ whenever $s < s^{'}$;
(iv) T satisfies the boundary condition $T(1,t) = t$ for every $t \in [0,1]$.
The most important $t-$norms are M and $\prod$ respectively given by $M(x,y) = \min\{x,y\}$ and $\prod(x,y) = xy$. Given a t-norm T, its t-conorm T^* is defined on $[0,1] \times [0,1]$ by $T^*(x,y) = 1 - T(1-s, 1-t)$.

Definition 1.4 *A triangle function is a binary operation on* $\triangle^+$ *namely a function* $\tau : \triangle^+ \times \triangle^+ \to \triangle^+$ *such that for all* F, G *and* H *in* $\triangle^+$*, we have*
(i) $\tau(\tau(F,G), H) = \tau(F, \tau(G,H))$;
(ii) $\tau(F,G) = \tau(G,F)$;
(iii) $F \leq G \Rightarrow \tau(F,H) \leq \tau(G,H)$ and
(iv) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Definition 1.5 *A* PN*-space is a quadruple* $(V, \vartheta, \tau, \tau^*)$*, where* V *is a real linear space,* τ *and* τ^* *are continuous triangle functions with* $\tau \leq \tau^*$ *and* ϑ *is a mapping (the probabilistic norm) from* V *into* $\triangle^+$ *such that for all* p, q *in* V*, the following conditions hold:*
$(PN1)$ $\vartheta_p = \epsilon_0$ if and only if, $p = \theta$ (θ is the null vector in V);
$(PN2)$ $\vartheta_{-p} = \vartheta_p$;
$(PN3)$ $\vartheta_{p+q} \geq \tau(\vartheta_p, \vartheta_q)$ and
$(PN4)$ $\vartheta_p \leq \tau^*(\vartheta_{\lambda p}, \vartheta_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

A PN-space is called a Šerstnev space if it satisfies $(PN1)$, $(PN3)$ and the following condition: For all $p \in V$, $\alpha \in \mathbb{R} - \{0\}$ and $x > 0$ one has

$$\vartheta_{\alpha p}(x) = \vartheta_p\left(\frac{x}{|\alpha|}\right).$$

which clearly implies $(PN2)$ and also $(PN4)$ in the strengthened form for all $\lambda \in [0,1]$, $\vartheta_p = \tau_M\left(\vartheta_{\lambda p}, \vartheta_{(1-\lambda)p}\right)$.

A PN-space in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for a suitable continuous t-norm T and its t-conorm T^*, is called a Menger PN-space where
$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t))$ and $\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t))$.

Definition 1.6 *Let* $(V, \vartheta, \tau, \tau^*)$ *be a* PN*-space. For* $p \in V$ *and* $t > 0$*, the strong* t*-neighborhood of* p *is the set*

$$N_p(t) = \{q \in V : \vartheta_{q-p}(t) > 1 - t\},$$

and the strong neighborhood system for V *is the union* $\cup_{p\in V} N_p$ *where* $N_p = \{N_p(t) : t > 0\}$.

There is a natural topology define on a PN-space $(V, \vartheta, \tau, \tau^*)$ called the strong topology in terms of strong neighborhood system. In the sequel, when we consider a PN-space $(V, \vartheta, \tau, \tau^*)$ we mean it is endowed with the strong topology.

Definition 1.7 *A sequence* $p = (p_k)$ *in a* PN*-space* $(V, \vartheta, \tau, \tau^*)$ *is said to be strongly convergent to a point* p_0 *in* V*, symbolically,* $\lim_k p_k = p_0$*, if for any* $t > 0$ *there exists a positive integer* m *such that* p_k *is in* $N_{p_0}(t)$ *whenever* $k \geq m$.

For any set $K \subseteq \mathbb{N}$, let K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density $\delta(K)$ of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$. The natural density may not exist for each set K. But the upper density $\overline{\delta}$ defined by $\overline{\delta}(K) = \limsup_n n^{-1} |K_n|$ always exists for any set $K \subseteq \mathbb{N}$. Also $\delta(K)$ different from zero we mean $\overline{\delta}(K) > 0$. Moreover, $\delta\left(K^C\right) = 1 - \delta(K)$; and for $A \subseteq B$ then $\overline{\delta}(A) \leq \overline{\delta}(B)$. Using natural density, statistical convergence on a PN-space is defined as follows.

Definition 1.8 *Let* $(V, \vartheta, \tau, \tau^*)$ *be a* PN*-space. A sequence* $p = (p_k)$ *in* V *is said to be strongly statistically convergent to a point* p_0 *in* V *provided that*

$$\lim_n \frac{1}{n} |\{k \leq n \ : p_k \notin N_{p_0}(t)\}| = 0;$$

i.e., $\delta(\{k \in \mathbb{N} : p_k \notin N_{p_0}(t)\}) = 0$. *In this case,* p_0 *is called the strong statistical limit of the sequence* $p = (p_k)$ *and we write* $S - \lim_k p_k = p_0$.

Definition 1.9 *Let* $(V, \vartheta, \tau, \tau^*)$ *be a* PN*-space and* $p = (p_k)$ *be any sequence in* V. *If* $\left(p_{k(j)}\right)$ *be a subsequence of* (p_k) *and* $K = \{k(j) : j \in \mathbb{N}\}$*, then we denote* $\left(p_{k(j)}\right)$ *by* $(p)_K$. *If* $\lim_n \frac{1}{n} |\{k(j) : j \in \mathbb{N}\}| = 0$*, then we say that* $\left(p_{k(j)}\right)$ *is a thin subsequence of* (p_k). *On the other hand,* K *is non-thin provided that* $\limsup_n \frac{1}{n} |\{k(j) : j \in \mathbb{N}\}| > 0$.

Definition 1.10 *Let* $(V, \vartheta, \tau, \tau^*)$ *be a* PN*-space and* $p = (p_k)$ *be any sequence in* V. *Then an element* $q \in V$ *is a strong statistical limit point of* (p_k) *provided that there exists a non-thin subsequence of* (p_k) *that strongly converges to* q. *We denote the set of all strong statistical limit points of* (p_k) *by* $\Lambda(S, p)$.

Definition 1.11 *Let* $(V, \vartheta, \tau, \tau^*)$ *be a* PN*-space and* $p = (p_k)$ *be any sequence in* V. *Then an element* $r \in V$ *is a strong statistical cluster point of* (p_k) *provided that for every* $t > 0$*, we have* $\limsup_n \frac{1}{n} |\{k \in \mathbb{N} : p_k \in N_r(t)\}| > 0$. *We denote the set of all strong statistical cluster points of* (p_k) *by* $\Gamma(S, p)$.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} is denoted by q_r.

Definition 1.12 *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. A sequence $p = (p_k)$ in V is said to be strongly lacunary statistically convergent to a point p_0 in V if*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : p_k \notin N_{p_0}(t)\}| = 0 \ .$$

In this case, p_0 is called the strong lacunary statistical limit of the sequence $p = (p_k)$ and we write $S_\theta - \lim_k p_k = p_0$.

We now consider the quite natural definitions of strong lacunary statistical limit and strong lacunary statistical cluster points of sequences on a PN-space.

2 Main Results

Let $\theta = (k_r)$ be a lacunary sequence. For a PN-space $(V, \vartheta, \tau, \tau^*)$, let $p = (p_k)$ be a sequence in V. Let $(p_{k(j)})$ be a subsequence of p and $K = \{k(j) : j \in \mathbb{N}\}$, then we denote $(p_{k(j)})$ by $(p)_K$. If

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0;$$

then $(p)_K$ is called θ-thin subsequence. On the other hand $(p)_K$ is a θ-nonthin subsequence of p provided that

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

Definition 2.1 *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. An element $\mu \in V$ is called a strong lacunary statistical limit point (briefly strong $S_\theta-$limit point) of a sequence $p = (p_k)$ in V provided that there is a θ-nonthin subsequence of p that is strongly convergent to μ.*

Let $\Lambda(S_\theta, p)$ denotes the set of all strong S_θ-limit points of the sequence $p = (p_k)$.

Definition 2.2 *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. A point $\gamma \in V$ is said to be a strong lacunary statistical cluster point (briefly strong $S_\theta-$cluster point) of a sequence $p = (p_k)$ in V provided that for all $t > 0$,*

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : p_k \in N_\gamma(t)\}| > 0.$$

Let $\Gamma(S_\theta, p)$ denotes the set of all strong S_θ-cluster points of the sequence $p = (p_k)$.

Theorem 2.1 *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. For any sequence $p = (p_k)$ in V, $\Lambda(S_\theta, p) \subseteq \Gamma(S_\theta, p)$.*

Proof. For $\mu \in \Lambda((S_\theta, p)$, there is a θ-nonthin subsequence $(p_{k(j)})$ of p that strongly converges to μ. Since $(p_{k(j)})$ is a θ-nonthin subsequence so we have

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : p_k \in N_\mu(t)\}| > 0. \tag{1}$$

Now for every $t > 0$, the containment $\{k \in I_r : p_k \in N_\mu(t)\} \supseteq \{k(j) \in I_r : p_{k(j)} \in N_\mu(t)\}$ gives

$$\{k \in I_r : p_k \in N_\mu(t)\} \supseteq \{k(j)\, I_r : j \in \mathbb{N}\} - \left\{k(j) \in I_r : p_{k(j)} \notin N_\mu(t)\right\};$$

which immediately implies

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : p_k \in N_\mu(t)\}| \geq \limsup_{r\to\infty} \frac{1}{h_r} |\{k(j)\, I_r : j \in \mathbb{N}\}|$$
$$-\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k(j) \in I_r : p_{k(j)} \notin N_\mu(t)\right\}\right|. \tag{2}$$

Further, the strong convergence of $(p_{k(j)})$ to μ gives for $t > 0$, the set $\left\{k(j) \in I_r : (p_{k(j)} \notin N_\mu(t)\right\}$ is finite for which we have

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\left\{k(j) \in I_r : (p_{k(j)} \notin N_\mu(t)\right\}\right| = 0\ . \tag{3}$$

Using (1) and (3) in (2), we get

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : p_k \in N_\mu(t)\}| \geq d > 0.$$

This shows that $\mu \in \Gamma(S_\theta, p)$ and therefore we have the containment $\Lambda(S_\theta, p) \subseteq \Gamma(S_\theta, p)$. ■

Theorem 2.2 *Let $\theta = (k_r)$ be a lacunary sequence and $(V, \vartheta, \tau, \tau^*)$ be a PN-space. For any sequence $p = (p_k)$ in V, $\Gamma(S_\theta, p) \subseteq L(p)$, where $L(p)$ denotes the set of all strong limit points of $p = (p_k)$.*

Proof. Assume that $\gamma \in \Gamma(S_\theta, p)$, then for all $t > 0$, we have

$$\limsup_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : p_k \in N_\gamma(t)\}| > 0. \tag{4}$$

For $t > 0$, if we denote $K = \{k \in I_r : p_k \in N_\gamma(t)\}$, then the set $K = \{k_1 < k_2 < \cdots\}$ is an infinite set as otherwise i.e. if K is finite set then left side of (4) becomes zero and we obtain a contradiction. This shows that we have a subsequence $(p)_K$ of the sequence $p = (p_k)$ that is strongly convergent to γ. Hence γ is a strong limit point of (p_k) and therefore we have the containment$\Gamma(S_\theta, p) \subseteq L(p)$. ■

Theorem 2.3 *For any lacunary sequence $\theta = (k_r)$ and any sequence $p = (p_k)$ in a PN-space $(V, \vartheta, \tau, \tau^*)$, $\Gamma(S_\theta, p)$ is a closed set.*

Proof. To prove the theorem it is sufficient to prove that $cl(\Gamma(S_\theta, p)) \subseteq \Gamma(S_\theta, p)$ where $cl(A)$ denotes the strong closure of any set A. Let $\mu \in cl(\Gamma(S_\theta, p))$, then for any $t > 0$, $\Gamma(S_\theta, p)$ contains some point $\gamma \in N_\mu(t)$. Choose t' such that $N_\gamma(t') \subseteq N_\mu(t)$. Since $\gamma \in \Gamma(S_\theta, p)$, therefore

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : p_k \in N_\gamma(t')\}\right| > 0;$$

which immediately gives

$$\limsup_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : p_k \in N_\mu(t)\}\right| > 0.$$

This shows that $\mu \in \Gamma(S_\theta, p)$ and therefore we have $cl(\Gamma(S_\theta, p)) \subseteq \Gamma(S_\theta, p)$. ■

Theorem 2.4 *Let $\theta = (k_r)$ be a lacunary sequence. For any sequence $p = (p_k)$ in a PN-space $(V, \vartheta, \tau, \tau^*)$, if $S_\theta - \lim_k p_k = p_0$, then $\Lambda(S_\theta, p) = \Gamma(S_\theta, p) = p_0$.*

Proof. We first show that $\Lambda(S_\theta, p) = \{p_0\}$. Let $t > 0$ and assume $\Lambda(S_\theta, p) = \{p_0, q_0\}$ such that $p_0 \neq_0$. By definition there exist two θ-nonthin subsequences $(p_{k(i)})$ and $(p_{l(j)})$ of the sequence $p = (p_k)$ which are respectively strongly convergent to p_0 and q_0. Since $(p_{l(j)})$ strongly converges to q_0 , therefore for any $t > 0$, there is a positive integer m such that p_k is in $N_{q_0}(t)$ whenever $k \geq m$. This shows that for any $t > 0$ we have

$$\lim_r \frac{1}{h_r} \left|\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\}\right| = 0 \ . \tag{5}$$

Moreover, for any $t > 0$ one can write

$$\{l(j) \in I_r : j \in \mathbb{N}\} = \{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\} \cup \{l(j) \in I_r : p_{l(j)} \notin N_{q_0}(t)\};$$

which implies

$$\limsup_r \frac{1}{h_r} \left|\{l(j) \in I_r : j \in \mathbb{N}\}\right| = \limsup_r \frac{1}{h_r} \left|\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\}\right|$$

$$+\limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\right\}\right|. \quad (6)$$

Since $(l(j))$ is θ-nonthin subsequence so we have together with (5),

$$\limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\right\}\right| > 0. \quad (7)$$

Also using the fact $S_\theta - \lim_k p_k = p_0$, we have

$$\lim_r \frac{1}{h_r} \left|\{k \in I_r : p_k \notin N_{p_0}(t)\}\right| = 0, \quad (8)$$

which gives for any $t > 0$

$$\limsup_r \frac{1}{h_r} \left|\{k \in I_r : p_k \in N_{p_0}(t)\}\right| > 0. \quad (9)$$

Also for $p_0 \neq q_0$, $\left\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\right\} \cap \{k \in I_r : p_k \in N_{p_0}(t)\} = \emptyset$. So we have,

$$\left\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\right\} \subseteq \{k \in I_r : p_k \in N_{p_0}(t)\},$$

which immediately with use of (8)

$$\limsup_r \frac{1}{h_r} \left|\left\{l(j) \in I_r : p_{l(j)} \in N_{q_0}(t)\right\}\right| \leq \limsup_r \frac{1}{h_r} \left|\{k \in I_r : p_k \notin N_{p_0}(t)\}\right| = 0;$$

which contradict (7). Hence $\Lambda(S_\theta, p) = \{p_0\}$. Similarly, we can show that $\Gamma(S_\theta, p) = \{p_0\}$. ■

Theorem 2.5 *Let $\theta = (k_r)$ be a lacunary sequence. If $p = (p_k)$ and $q = (q_k)$ are two sequences in $(V, \vartheta, \tau, \tau^*)$ such that $\lim_r \frac{1}{h_r} \left|\{k \in I_r : p_k \neq q_k\}\right| = 0$, then $\Lambda(S_\theta, p) = \Lambda(S_\theta, q)$ and $\Gamma(S_\theta, p) = \Gamma(S_\theta, q)$.*

Proof. Assume $\gamma \in \Lambda(S_\theta, p)$, then there exists a θ-nonthin subsequence $(p)_K$ of the sequence $p = (p_k)$ that converges to γ.
Since, $\lim_r \frac{1}{h_r} \left|\{k \in I_r : k \in K, p_k \neq q_k\}\right| = 0$, it follows that

$$\limsup_r \frac{1}{h_r} \left|\{k \in I_r : k \in K, p_k = q_k\}\right| > 0 \quad (10)$$

Therefore, there exists a θ-nonthin subsequence $(q)_K$ of the sequence $q = (q_k)$ that converges to γ. This shows that $\gamma \in \Lambda(S_\theta, q)$ and therefore $\Lambda(S_\theta, p) \subseteq \Lambda(S_\theta, q)$. By symmetry we have $\Lambda(S_\theta, q) \subseteq \Lambda(S_\theta, p)$. Hence we have $\Lambda(S_\theta, p) = \Lambda(S_\theta, q)$. Similarly we can prove $\Gamma(S_\theta, p) = \Gamma(S_\theta, q)$. ■

Theorem 2.6 *Let $\theta=(k_r)$ be a lacunary sequence and $p=(p_k)$ be a sequence in $(V,\vartheta,\tau,\tau^*)$, then we have*
(i) If $\liminf_r q_r > 1$ then $\Lambda(S_\theta,p)\subseteq\Lambda(S,p)$;
(ii) If $\limsup_r q_r < \infty$ then $\Lambda(S,p)\subseteq\Lambda(S_\theta,p)$ and
(iii) If $1<\liminf_r q_r\leq\limsup_r q_r<\infty$ then $\Lambda(S,p)=\Lambda(S_\theta,p)$.

Proof. (i) Let $\liminf_r q_r > 1$, then there exists a $\delta>0$ such that $q_r>1+\delta$ for sufficiently large r which implies that $\frac{k_r}{h_r}\leq\frac{\delta+1}{\delta}$. Let $\mu\in\Lambda(S_\theta,p)$, then by definition, there exists a set $K=\{k(j):j\in\mathbb{N}\}$ such that $\lim_{j\to\infty}p_{k(j)}=\mu$ and

$$\limsup_{r\to\infty}\frac{1}{h_r}\left|\{k(j)\in I_r:j\in\mathbb{N}\}\right|\ >0 \tag{11}$$

Since,

$$\begin{aligned}\frac{1}{k_r}\left|\{k(j)\leq k_r:j\in\mathbb{N}\}\right| &\geq \frac{1}{k_r}\left|\{k(j)\in I_r:j\in\mathbb{N}\}\right|\\ &=\left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\left|\{k(j)\in I_r:j\in\mathbb{N}\}\right|\\ &\geq\left(\frac{\delta}{\delta+1}\right)\frac{1}{h_r}\left|\{k(j)\in I_r:j\in\mathbb{N}\}\right|;\end{aligned}$$

it follows by (11) that

$$\limsup_{r\to\infty}\frac{1}{k_r}\left|\{k(j)\leq k_r:j\in\mathbb{N}\}\right|>0.$$

Since $\left(p_{k(j)}\right)$ is already strongly convergent to μ, it follows that $\mu\in\Lambda(S,p)$. Hence we have $\Lambda(S_\theta,p)\subseteq\Lambda(S,p)$.

(ii) If $\limsup_r q_r<\infty$, then there exists a real number H such that $q_r<H$ for all r. Without loss of generality, we can assume $H>1$. Now for all r,

$$\frac{h_r}{k_{r-1}}=\frac{k_r-k_{r-1}}{k_{r-1}}=q_r-1\leq H-1.$$

Now, Let $\mu\in\Lambda(S,p)$, then by definition there is a set $K=\{k(j):j\in\mathbb{N}\}$ with $\delta(K)>0$ and $\lim_{j\to\infty}p_{k(j)}=\mu$. Let $N_r=\left|\{k\in I_r:k\in K\}\right|=\left|K\cap I_r\right|$

and $t_r = \frac{N_r}{h_r}$. For any integer n satisfying $k_{r-1} < n \leq k_r$, we can write

$$\begin{aligned}\frac{1}{n}\left|\{k \leq n : k \in K\}\right| &\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : k \in K\}\right| \\ &= \frac{1}{k_{r-1}}\{N_1 + N_2 + N_3 + \cdots + N_r\} \\ &= \frac{1}{k_{r-1}}\{t_1 h_1 + t_2 h_2 + t_3 h_3 + \cdots + t_r h_r\} \\ &= \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\ &\leq \frac{1}{\sum_{i=1}^{r-1} h_i}\sum_{i=1}^{r-1} h_i t_i + (H-1)t_r.\end{aligned}$$

Suppose $t_r \to 0$ as $r \to \infty$. Since θ is a lacunary sequence and the first part on the right side of above expression is a regular weighted mean transform of the sequence $t = (t_r)$, therefore it too tends to zero as $r \to \infty$. Since $n \to \infty$ as $r \to \infty$, it follows that $\delta(K) = 0$ which is a contradiction as $\delta(K) \neq 0$. Thus we have $\lim_{r\to\infty} t_r \neq 0$ and therefore by definition $\delta_\theta(K) \neq 0$. This shows that $\mu \in \Lambda(S_\theta, p)$. Hence $\Lambda(S, p) \subseteq \Lambda(S_\theta, p)$.

(iii) This is an immediate consequence of (i) and (ii).

Theorem 2.7 *Let $\theta = (k_r)$ be a lacunary sequence and $p = (p_k)$ be a sequence in $(V, \vartheta, \tau, \tau^*)$, then we have,*

(i) If $\liminf_r q_r > 1$ then $\Gamma(S_\theta, p) \subseteq \Gamma(S, p)$;

(ii) If $\limsup_r q_r < \infty$ then $\Gamma(S, p) \subseteq \Gamma(S_\theta, p)$ and

(iii) If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ then $\Gamma(S, p) = \Gamma(S_\theta, p)$.

Proof for the theorem, goes on the similar lines as for Theorem 2.6, so is omitted here.

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