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A-Quasi Normal Operators in Semi Hilbertian Spaces

S. Panayappan[1] and N. Sivamani[2]

[1]Department of Mathematics, Government Arts College,
Coimbatore- 641018, Tamilnadu, India
E-mail: panayappan@gmail.com
[2]Department of Mathematics, Tamilnadu College of Engineering,
Coimbatore- 641659, Tamilnadu, India
E-mail: sivamanitce@gmail.com

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Abstract

In this paper we introduce the concept of A-quasinormal operators acting on semi Hilbertian spaces H *with inner product* $\langle , \rangle_A$. *The object of this paper is to study conditions on* T *which imply A-quasi normality. If* S *and* T *are A-quasi normal operators, we shall obtain conditions under which their sum and product are A-quasi normal.*

Keywords: A*-adjoint,* A*-Normal, Semi inner product, and Moore-Penrose inverse and quasinormal.*

1 Introduction

Throughout this paper H denotes a complex Hilbert space with inner product $\langle .,. \rangle$ and the norm $\| . \|$. $L(H)$ stands the Banach algebra of all bounded linear operators on $H. I = I_H$ being the identity operator and if $V \subset H$ is a closed subspace, P_V is the orthogonal projection onto V.

$L(H)^+$ is the cone of positive operators,

i.e. $L(H)^+ = \{A \in L(H): \langle Ax, x\rangle \geq 0, \forall x \in H\}$.

Any positive operator $A \in L(H)^+$ defines a positive semi-definite sesquilinear form

$$\langle .,. \rangle_A : H \times H \to C, \langle x, y\rangle_A = \langle Ax, y\rangle.$$

By $\| . \|_A$ we denote the semi norm induced by $\langle .,. \rangle_A$ i.e. $\| x \|_A = \langle x, x\rangle_A^{\frac{1}{2}}$. Note that $\|x\|_A = 0$ if and only if $x \in N(A)$. Then $\| . \|_A$ is a norm on H if and only if A is an injective operator, and the semi - normed space $(L(H), \| . \|_A)$ is complete if and only if $R(A)$ is closed. Moreover $\langle .,. \rangle_A$ induces a semi norm on the subspace $\{T \in L(H) \,|\, \exists c > 0, \|Tx\|_A \leq c\|x\|_A, \forall x \in H\}$. For this subspace of operators it holds

$$\|T\|_A = \sup_{x \in \overline{R(A)}} \frac{\|Tx\|_A}{\|x\|_A} < \infty$$

Moreover $\|T\|_A = \sup \{ |\langle Tx, y\rangle_A|; x, y \in H \text{ and } \|x\|_A \leq 1, \|y\|_A \leq 1 \}$.
For $x, y \in H$, we say that x and y are A-orthogonal if $\langle x, y\rangle_A = 0$.
The following theorem due to Douglas will be used (for its proof refer [5].)

Theorem 1.1 Let $T, S \in L(H)$. *The following conditions are equivalent.*

(i) $R(S) \subset R(T)$.

(ii) *There exists a positive number* λ *such that* $SS^* \leq \lambda TT^*$.

(iii) *There exists* $W \in L(H)$ *such that* $TW = S$.

From now on, A *denotes a positive operator on* H *(i.e.* $A \in L(H)^+$*).*

Definition 1.2 *Let* $T \in L(H)$*, an operator* $W \in L(H)$ *is called an* A*-adjoint of* T *if* $\langle Tu, v\rangle_A = \langle u, Wv\rangle_A$ *for every* $u, v \in H$*, or equivalently* $AW = T^*A$, T *is called* A*- selfadjoint if* $AT = T^*A$ *and* T *is called* A*-positive if* AT *is positive.*

By Douglas Theorem, an operator $T \in L(H)$ admits an A-adjoint if and only if $R(T^*A) \subset R(A)$ and if W is an A-adjoint of T and $AZ = 0$ for some $Z \in L(H)$ then

$W+Z$ is also an A-adjoint of T. Hence neither the existence nor the uniqueness of an A-adjoint operator is guaranteed. In fact an operator $T \in L(H)$ may admit none, one or many A-adjoints.
From now on, $L_A(H)$ denotes the set of all $T \in L(H)$ which admit an A-adjoint,

i.e. $L_A(H) = \{T \in L(H) : R(T^* A) \subset R(A)\}$

$L_A(H)$ is a subalgebra of $L(H)$ which is neither closed nor dense in $L(H)$.
On the other hand the set of all A-bounded operators in $L(H)$ (i.e. with respect the semi norm $\| \cdot \|_A$ is

$$L_{A^{\frac{1}{2}}}(H) = \left\{T \in L(H) : T^* R(A^{\frac{1}{2}}) \subset R(A^{\frac{1}{2}})\right\} = \left\{T \in L(H) : R(A^{\frac{1}{2}} T^* A^{\frac{1}{2}}) \subset R(A)\right\}$$

Note that $L_A(H) \subset L_{A^{\frac{1}{2}}}(H)$, which shows that if T admits an A-adjoint then it is A-bounded.
If $T \in L(H)$ with $R(T^* A) \subset R(A)$, then T, admits an A-adjoint operator, Moreover there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^* A$, i.e. $T^\# = A^+ T^* A$, where A^+ is the Moore-Penrose inverse of T. The A-adjoint operator $T^\#$ verifies

$$AT^\# = T^* A,\ R(T^\#) \subseteq \overline{R(A)} \text{ and } N(T^\#) = N(T^* A).$$

In the next we give some important properties of $T^\#$ without proof (refer [3], [4] and [5]).

Theorem 1.3 Let $T \in L_A(H)$. *Then*

(1) If $AT = TA$ then $T^\# = PT^*$.
(2) $T^\# T$ and $TT^\#$ are A-self adjoint and A-positive.
(3) $\|T\|_A^2 = \|T^\#\|_A^2 = \|T^\# T\| = \|TT^\#\|$
(4) $\|S\|_A = \|T^\#\|_A$ for every $S \in L(H)$ which is an A-adjoint of T.
(5) If $S \in L_A(H)$ then $ST \in L_A(H)$, $(ST)^\# = T^\# S^\#$ and $\|TS\|_A = \|ST\|_A$.
(6) $T^\# \in L_A(H), (T^\#)^\# = PTP$ and $((T^\#)^\#)^\# = T^\#$.

Definition 1.4 *An operator $T \in L_A(H)$ is called A-normal if $T^\# T = TT^\#$ (for more details refer [1]).*

2 A- Quasinormal Operators

Definition 2.1 *An operator* $T \in L_A(H)$ *is called* A*-quasinormal if* T *commutes with* $T^{\#}T$ *i.e.* $T(T^{\#}T) = (T^{\#}T)T$.

Let $T = U + V \in L_A(H)$ *where* $U = \frac{T+T^{\#}}{2}$ *and* $V = \frac{T-T^{\#}}{2}$*. We shall write* $B^2 = TT^{\#}$ *and* $C^2 = T^{\#}T$ *where* B *and* C *are non-negative definite. We give necessary and sufficient conditions for an operator to be* A*-quasinormal [2] and [6].*

Theorem 2.2 T *is* A*-quasinormal with* $N(A)$ *is invariant subspace for* T *if and only if* C *commutes with* U *and* V.

Proof. Since $N(A)$ is invariant subspace for T we observe that $PT = TP$ and $T^{\#}P = PT^{\#}$.
Let T be A-quasinormal then

$$T(T^{\#}T) = (T^{\#}T)T$$
$$T^{\#}T^{\#\#}T^{\#} = T^{\#}T^{\#}T^{\#\#}$$
$$T^{\#}PTPT^{\#} = T^{\#}T^{\#}PTP$$
$$PT^{\#}PTT^{\#} = T^{\#}PT^{\#}PT$$
$$T^{\#}TT^{\#} = T^{\#}T^{\#}T$$

Hence $T^{\#}TT^{\#} = T^{\#^2}T.$

Now it is easy to see that $C^2U = UC^2$. Since C is non-negative definite, it follows that $CU = UC$. Similarly $CV = VC$.
Conversely, let $CU = UC$ and $CV = VC$. Then $C^2U = UC^2$ and $C^2V = VC^2$.
Hence $C^2T = TC^2$. Therefore $T^{\#}T^{2} = TT^{\#}T$.
In the following theorem we give conditions under which an operator T is A-quasi normal.

Theorem 2.3 *If* T *is an operator such that (i)* B *commutes with* U *and* V *(ii)* $C^2T = TB^2$. *Then* T *is* A*-quasinormal.*

Proof. Since $BU = UB$ and $BV = VB$ we have $B^2U = UB^2$ and $B^2V = VB^2$

Then $B^2T + B^2T^{\#} = TB^2 + T^{\#}B^2$

$$B^2T - B^2T^{\#} = TB^2 - T^{\#}B^2$$

This gives $B^2T = TB^2 = C^2T$. Hence T is A-quasinormal.

Theorem 2.4 *Let T be A-quasi normal, $C^2T = TB^2$ and $N(A)$ be an invariant subspace for T. Then B commutes with U and V.*

Proof. Since $C^2T = TB^2$ we have $T^{\#}T^2 = T^2T^{\#}$. Hence $T^{\#^2}T = TT^{\#^2}$.
Since T is A-quasi normal we have

$$B^2U = \frac{TT^{\#}T + TT^{\#^2}}{2} = \frac{T^{\#}T^2 + T^{\#^2}T}{2} = \frac{T^2T^{\#} + T^{\#}TT^{\#}}{2} = \frac{T + T^{\#}}{2}TT^{\#} = UB^2.$$

Hence $BU = UB$. Similarly $BV = VB$.

Theorem 2.5 *Let S and T be two A-quasinormal operators. Then their product ST is A-quasinormal if the following conditions are satisfied (i) $ST = TS$ (ii) $ST^{\#} = T^{\#}S$.*

Proof.
$$\begin{aligned}
&(ST)(ST)^{\#}(ST)\\
&= (ST)(T^{\#}S^{\#})(ST)\\
&= (ST)(S^{\#}T^{\#})(ST)\\
&= S(TS^{\#})(T^{\#}S)T\\
&= SS^{\#}(TS)T^{\#}T\\
&= SS^{\#}(ST)T^{\#}T\\
&= (SS^{\#}S)(TT^{\#}T)\\
&= (S^{\#}S^2)(T^{\#}T^2)\\
&= S^{\#}(S^2T^{\#})T^2\\
&= S^{\#}(T^{\#}S^2)T^2\\
&= (T^{\#}S^{\#})(S^2T^2)\\
&= (ST)^{\#}(ST)^2
\end{aligned}$$

Hence ST is A-quasinormal.

Theorem 2.6 *Let S and T be two A-quasinormal operators such that $ST = TS = S^{\#}T = T^{\#}S = 0$. Then $S + T$ is A-quasinormal.*

Proof.
$$\begin{aligned}
&(S+T)(S+T)^{\#}(S+T)\\
&= (S+T)(S^{\#}+T^{\#})(S+T)\\
&= (S+T)(S^{\#}S + S^{\#}T + T^{\#}S + T^{\#}T)\\
&= (S+T)(S^{\#}S + T^{\#}T)\\
&= SS^{\#}S + ST^{\#}T + TS^{\#} + TT^{\#}T\\
&= S^{\#}S^2 + T^{\#}T^2
\end{aligned}$$

$$=(S+T)^{\#}(S+T)^2$$

Hence $S+T$ is A-quasi normal.

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