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Frames, Riesz Bases and Double Infinite Matrices

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Abstract. *In this paper we have used double infinite matrix* $A = (a_{iljk})$ *of real numbers to define the A-frame. Some results on Riesz basis and A-frame also have been studied. This Work is motivated from the work of Moricz and Rhoades [7].*
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1 Introduction

Let $U(F)$ and $V(F)$ be finite dimensional vector spaces over the field F of dimension n. The elements $(y_1, \ldots, y_n) \in V$ and $(e_1, e_2, \cdots, e_n)$ is an ordered basis in U. Then there exists a unique linear transformation such that

$$Te_i = y_i, \quad i = 1, \cdots, n. \tag{1.1}$$

Let us extend the transformation T to linear transformation of vectors from the basis such that

$$T\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i y_i.$$

It is clear from (1.1) that T is completely defined because any element in U can be expressed as a linear combination of basis vectors uniquely. Also, if U is $n-$dimensional and V is $m-$dimensional then the class of all linear transformations from $U \to V$ be $nm-$dimensional.

Let an ordered bases in U and V be $\{e_j\}_{j=1}^n$ and $\{e_i\}_{i=1}^m$ respectively. Then the set of all linearly independent $[a_{ij}]\,.(i = 1, \cdots, m, j = 1, \cdots, n)$ i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \cdots & \\ \vdots & & & \vdots \\ a_{m1} & & & a_{mn} \end{pmatrix}_{m\times n}$$

be characterized by the mappings

$$a_{ij}e_k = \delta_{jk}e_i \quad i = 1, \cdots, m, k, j = 1, \cdots, n.$$

Now we have the following definitions

Definition 1.1 Let $A = (a_{iljk}), (i, l, j, k = 1, 2, \cdots)$, be a double non-negative infinite matrix of real or complex numbers. Let (X, Y) denote the class of all such matrices A such that the series $A(x_{il}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{iljk}x_{jk}$ converges for all double sequences $x_{jk} \in X$ and the sequence $\{A(x_{il})\}$ will be called $A-$means or $A-$transform of x_{il}. Also $Ax = \lim_{i,l\to\infty} A(x_{il})$, whenever it exists.

Definition 1.2. A double matrix $A = (a_{iljk})$ is said to be regular if the matrix transformation $A : X \to Y$ is defined on a convergent sequence to a convergent sequence and limit is preserved i.e., $\lim_{i,l\to\infty} A(x_{il}) = \lim_{i,l\to\infty} x_{il}$.

Definition 1.3. [7] A double matrix $A = (a_{iljk})$ is said to be regular if the following conditions holds.

(I) $\lim_{i,l\to\infty} \sum_{j,k=0}^{\infty} a_{iljk} = 1$,

(II) $\lim_{i,l\to\infty} \sum_{j=0}^{\infty} |a_{iljk}| = 0, \quad (k = 0, 1, \cdots)$,

(III) $\lim_{i,l\to\infty} \sum_{j=0}^{\infty} |a_{iljk}| = 0, \quad (j = 0, 1, \cdots)$,

(IV) $\|A\| = \sup_{i,l>0} \sum_{j,k=0}^{\infty} |a_{il}| < \infty$.

2 Frames

The theory for frames and bases has developed very fast over the last 15 years. The concept of frames were introduced by Duffin and Schaeffer [5]in the context of non-harmonic Fourier series. A sequence in a Hilbert space H is a frame if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|^2 \leq \sum_n |< x, x_n >|^2 \leq C_2\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Any numbers C_1, C_2 for which (2.1) is valid are called frames bounds. They are not unique if we can choose $C_1 = C_2$, the frame is called tight and is said to be exact if it ceases to be a frame by removing any of its elements. The theory of frames are discussed in variety of sources, including [1,3,4,6,8]. The purpose of the present paper is to define $A-$frame for an infinite double non-negative regular matrix and to study some results on $A-$frame and Riesz basis.

Let H be a separable Hilbert space with inner product $< .,. >$ and norm $\|.\| =< .,. >^{1/2}$. In the sequel z, and z^+ denote the set of integers and strictly positive integers respectively.

Definition 2.1. A family of elements $\{x_n, n \in z^+\} \subseteq H$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum |< f, x_n >|^2 \leq B\|f\|^2, \forall f \in H. \tag{2.2}$$

It is given [1] that $\{x_n, n \in z^+\}$ is a Bessel sequence with bound M if and only if, for every finite sequence of scalors $\{c_k\}$;

$$\|\sum_k c_k x_k\|^2 \leq M \sum_k |c_k|^2 . \tag{2.3}$$

Chui and Shi's [2] remarked that $\{x_n, n \in z^+\}$ is a Bessel sequence with bound M if and only if (2.3) is satisfied for every sequence $\{c_k\} \in l^2$.

In the consequence of above discussion we have the following lemma.

Lemma 2.1. $\{x_n, n \in z^+\}$ is a Bessel sequence if and only if

$$T : \{c_n\} \rightarrow \sum_n c_n x_n$$

is well defined operator from l^2 into H. In that case T is automatically bounded, and the adjoint of T is given by

$$T^* : H \rightarrow l^2, \quad T^* f = \{< f, x_n >\}.$$

An important consequence of above lemma 2.1 that if $\{x_n\}$ is a Bessel sequence, then $\sum_n c_n x_n$ converges unconditionally for all $\{c_n\} \in l^2$. When

$\{x_n, n \in z^+\} \subset H$ is a frame, the operator T and T^* are well defined, so we define the frame operator

$$S : H \to H, \quad Sf = TT^* f = \sum_n < f, x_n > x_n.$$

Two sequences $\{x_n, n \in z^+\}$ and $\{y_n, n \in z^+\}$ in H are called biorthogonal if $< x_n, y_n > \delta_{m,n}$, where $\delta_{m,n}$ is the Kronecker delta.

To prove that S is bounded, positive and surjective we have the following theorem from [1].

Theorem A. Let $\{x_n, n \in z^+\} \subset H$

(a) The following are equivalent

(i) $\{x_n, n \in z^+\}$ is a frame for H with frame bounds C_1 and C_2.

(ii) $S : H \to H$ is a topological isomerphism with norm bounds $\|S\| \leq C_2$ and $\|S\| \leq C_1^{-1}$.

(b) In case of either condition in part (a), we obtain that

$$C_1 I \leq S \leq C_2 I \quad C_2^{-1} I \leq S^{-1} \leq C_1^{-1} I,$$

$\{S^{-1}x_n\}$ is a frame for H with frame bounds C_2^{-1} and C_1^{-1} and for all $x \in H$,

$$f = SS^{-1} f = \sum_n < x, S^{-1} x_n > x_n, \tag{2.4}$$

and

$$f = \sum_n < x, x_n > S^{-1} x_n. \tag{2.5}$$

If $\{x_n, n \in z^+\}$ is a frame, S is called frame operator, $\{S^{-1}x_n\}$ is called dual frame of $\{x_n\}$, (2.4) is the frame decomposition of x and (2.5) is the dual frame decomposition of x. I is the identity map, $S \leq C_2\ I$ means that $< (C_2 I - S)x, x > \geq 0$ for each $x \in H$.
We also have

Theorem B.[1]. Let $\{x_n, n \in z^+\} \subset H$ be a frame for H with frame bounds C_1 and C_2. Then for each sequence $\{C_n\} \in l^2$ such that $x = \sum_n C_n x_n$ converges in H and $\|x\|^2 \leq C_2 \|C\|_{l^2}^2$ and for any arbitrary vector v there exists a moment sequence $\{y_n, n \in z^+\}$ such that $v = \sum_{n=1}^{\infty} x_n y_n$ and $C_2^{-1}\|v\|^2 \leq \sum_{n=1}^{\infty} |y_n|^2 \leq C_2 \|v\|^2$.

Theorem C.[1]. A sequence $\{x_n, n \in z^+\}$ in a Hilbert space H is an exact frame for H if and only if it is bounded unconditional basis for H.

3 Main Results

Theorem 3.1. Let $A = (a_{iljk})$ be a double non-negative regular infinite matrix. Then for any $f \in L^2(R)$ the frame condition for $A-$transform of (a_{iljk}) is

$$C_1\|f\|^2 \leq \sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 \leq C_2\|f\|^2, \tag{3.1}$$

where $A(\phi_{i,l}) = \sum_{j,k=0}^{\infty} a_{iljk}\phi_{j,k}$, $\{\phi_{i,l}\}$ is a sequence of vectors and $0 < C_1 \leq C_2 < \infty$ are frame bounds.

$$\begin{aligned}\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 &= \sum_{i,l\in z} \int_{-\infty}^{\infty} |f(x)|^2 \left|\overline{A(\phi_{i,l})}\right|^2 dx \\ &\leq \|f\|^2 \sum_{i,l\in z} |A(\phi_{i,l})|^2 \\ &= \|f\|^2\|A\|^2 \sum_{i,l\in z} |\phi_{i,l}|^2\end{aligned}$$

Since A is regular matrix and by the definition of $A(\phi_{i,l})$, we get

$$\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 \leq C_2\|f\|^2. \tag{3.2}$$

Now for any $f \in L^2(R)$, let

$$\widetilde{f} = \left[\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2\right]^{-1/2} f,$$

or

$$< \widetilde{f}, A(\phi_{i,l}) >= \left[\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2\right]^{-1/2} < f, A(\phi_{i,l}) >$$

then

$$\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 \leq 1.$$

Hence, for positive constant α, we get

$$\|\widetilde{f}\|^2\|\phi_{i,l}\|^2 \leq \alpha,$$

or

$$\left[\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2\right]^{-1} |< f, A(\phi_{i,l}) >|^2 \leq \alpha.$$

Since A is regular, it gives

$$\sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 \, \|f\|^2 \leq \alpha 1.$$

Thus,

$$C_1\|f\|^2 \leq \sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 . \tag{3.3}$$

Combining (3.2) and (3.3) the proof of theorem is immediate.

Theorem 3.2. $\{A(\phi_{i,l})\}$ is a frame for any $f \in L^2(R)$ if and only if the mapping

$$T : \{\beta_{i,l}\} \to \sum_{i,l\in z} \beta_{i,l} A(\phi_{i,l})$$

is a well defined mapping from l^2 into $L^2(R)$. Here $\beta_{i,l} =< f, A(\phi_{i,l}) >$ is $A-$moment sequence of $f \in L(R)$ relative to the frame.

Proof. First we shall prove that if $\{A(\phi_{i,l})\}$ is $A-$frame and $\{\beta_{i,l}\} \in l^2$, then $\sum_{i,l\in z} \beta_{i,l} A(\phi_{i,l})$ converges, and

$$\| \sum_{i,l\in z} \beta_{i,l} A(\phi_{i,l})\|^2 \leq C_2 \| \sum_{i,l\in z} |\beta_{i,l}|^2 . \tag{3.4}$$

To prove this let us assume

$$f_{j,k} = \sum_{i,l=1}^{j,k} \beta_{i,l} A(\phi_{i,l})$$

then for any $j, k \geq j_0, k_0$, using Schwartz inequality with the frame condition (3.1) we obtain

$$\|f_{j,k} - f_{j_0,k_0}\|^2 = \{ \sum_{i,l=j_0+1,k_0+1}^{j,k} |\beta_{i,l}|^2\}^{1/2}\{C_2\|f_{j,k} - f_{j_0,k_0}\|^2\}^{1/2}.$$

Which gives

$$\|f_{j,k} - f_{j_0,k_0}\|^2 \leq C_2 \sum_{i,l=j_0+1,k_0+1}^{j,k} |\beta_{i,l}|^2 .$$

Now we assume that $\{A(\phi_{i,l})\}$ is a frame. Since $\{A(\phi_{i,l})\}$ is a Bessel sequence, T is a bounded operator from l^2 into $L^2(R)$ by (3.4). Now for any $f \in L^2(R)$ we define a linear transformation S by the relation

$$Sf = \sum_{i,l\in z} < f, A(\phi_{i,l}) > A(\phi_{i,l}).$$

The transformation is self adjoint and it gives with (3.1) that

$$C_1\|f\|^2 \leq < Sf, f > \leq C_2\|f\|^2.$$

This conclude that S is positive, bounded and surjective. Thus $S = TT^*$ is surjective. Hence T is surjective.

Now suppose that T is a well defined operator from l^2 onto $L^2(R)$. By (3.4) $\{A(\phi_{i,l})\}$ satisfies the upper frame condition. Now consider that T be any bounded operator from a Hilbert space H^1 into a Hilbert space H. Then the set $C_T = H^1 \ominus N(T)$ i.e., the orthogonal complement of null space of T in H^1 is well defined, T is injective on C_T and ran T^* is dense in C_T. We denote T^+ the inverse map from ran T to C_T i.e., $T^+ : H \to C_T$. By writing $T^+f = \{(T^+f)_{i,l}\}$ for $f \in H$, we get

$$f = TT^+f = \sum_{i,l\in z} (T^+f)_{i,l} A(\phi_{i,l}).$$

We have

$$\begin{aligned} \|f\|^4 = | < f, f > |^2 &= \left| < \sum_{i,l\in z} (T^+f)_{i,l} A(\phi_{i,l}), f > \right|^2 \\ &\leq \sum_{i,l\in z} \left|(T^+f)_{i,l}\right|^2 \sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 \\ &\leq \|T^+\|^2\|f\|^2 \sum_{i,l\in z} |< f, A(\phi_{i,l}) >|^2 . \end{aligned}$$

Thus, we obtain

$$\sum |< f, A(\phi_{i,l}) >|^2 \geq \frac{1}{\|T^+\|^2}\|f\|^2, \forall f \in H.$$

Taking $H = L^2(R)$. The proof is completed in view of Theorem 3.1.

Theorem 3.3 Let any sequence of numbers $\{\beta_{i,l}\} \in l^2$ is a moment sequence of any function $f \in L^2(R)$ with respect to $\{A(\phi_{i,l})\}$. If $\{A(\phi_{i,l})\}$ is an exact $A-$frame then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{i,l\in z} |\beta_{i,l}|^2 \leq \| \sum_{i,l\in z} \beta_{i,l} A(\phi_{i,l})\|^2 \leq C_2 \sum_{i,l\in z} |\beta_{i,l}|^2$$

Proof. Since $\{A(\phi_{i,l})\}$ is an exact $A-$frame therefore $\{S^{-1}A(\phi_{i,l})\}$ is a biorthogonal sequence. By Theorem 3.2 we conclude that for a given sequence $\{\beta_{i,l}\} \in l^2$ and for any $f \in L^2(R)$, the series $f = \sum_{i,l\in z} \beta_{i,l} A(\phi_{i,l})$ has a finite norm. The proof is completed with (3.1) by using the fact that $\{A(\phi_{i,l})\}$ is bounded.

References

1. J.J. Benedettoo and D.F. Walnut, *Gabor Frames for L^2 and Related Spaces, in $Wavelets; Mathematics and Applications$,*(Benedettoo J.J. and Frazier, M.W. eds) pp. 97-162, CRS Press, Boca Raton, (1994).

2. C.K. Chui and Shi Xianliang, Bessel's sequences and affine frames, *Applied and Comput. Harm. Anal.*, 1(1993),29-49.

3. I. Daubechies, The wavelet transform, time frequency localization and signal analysis, *IEEE Trans. Inform. Theory*, 36(5)(1990), 961-1005.

4. I. Daubechies, Ten Lectures on Wavelets, *SIAM Publications Phildelphia*, (1992).

5. R.J. Duffin and A.C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.*, 72(1952),341-366.

6. C.E. Heil and D.F. Walnut, Continuous and discrete wavelet transforms, *SIAM Review*, 31(1989),629-666.

7. F. Mosicz and B.E. Rhoacles, Comparison theorems for double summablility methods, *Publ. Math. Debrecen*, 36(1989), no. 1-4, 207-220.

8. R.M. Young, An Introduction to Non-Harmonic Fourier Series, *Academic Press*, New York, (1980).