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Some Results on Soft S-Act

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Abstract

For a semigroup S we define some operations on the set of soft S-acts. Also by defining the soft congruence we introduce the notion of bi-act. The purpose of this paper is to investigate certain results on the soft bi-acts. Some analoges of algebraic properties like the isomorphisms theorems will be appeared as well.
Keywords: *Soft S-act, Soft homomorphism, Soft congruence.*

1 Introduction

Molodtsov [11] introduce the notation of soft set in 1999 to overcome uncertainties which cannot dealt with by classical methods as economics, engineering, environmental science, medical science and social science. In 2007 This notion generalized to the soft groups by Aktas [2]. Then the normal soft groups studied by Liu [9]. The interesting generalization of soft sets done by Acar [1] in 2010, where he introduced the notions of soft semirings, soft ideal and idealistic soft semirings. One may see the investigated properties on soft modules in [13]. Atagm and Sezgin studied in [3] the soft substructure of ring and modules. In [12, 4] Ali defined the soft S-act, soft relations on soft sets and then proved some results.

In this paper our main purpose is to define the notion of congruence on the set of soft S-acts and bi-acts. Consequently, we will be able to prove isomorphism theorems.

2 Preliminaries

Let$X$ be an initial universe of objects and E be a set of parameters in relation to the objects in X. Also let $P(X)$ denote the power set of X.

Definition 2.1. *[2] A pair (F, A) is called a soft set over X, where $A \subseteq E$ and $F : A \to P(X)$ is a set valued mapping.*

Definition 2.2. *[4] Let (σ, A) be a soft set over $X \times X$. Then (σ, A) is called a soft binary relation over X. In fact, (σ, A) is a parameterized collection of binary relations over X. That is we have a binary relation $\sigma(a)$ on X for each parameter $a \in A$. We denote the collection of all soft binary relations over X by $\xi_{Br}(X)$.*

Definition 2.3. *[4] Let (σ, A) and (ρ, B) be two soft binary relations over X. Then their composition is defined as the soft set $(\delta, C) = (\sigma, A) \circ (\rho, B)$ where C=$A \times B$ and $\delta(\alpha, \beta) = \sigma(\alpha) \circ \rho(\beta)$ for all $(\alpha, \beta) \in C$.*
Note that $\sigma(\alpha) \circ \rho(\beta)$ denotes the ordinary composition of binary relations on X. Specifically, we have:

$$\sigma(\alpha) \circ \rho(\beta) = \{(x, y) \in X \times X : \exists z \in X, (x, z) \in \sigma(\alpha) \text{ and } (z, y) \in \rho(\beta)\}.$$

Definition 2.4. *[4] Let (σ, A) and (ρ, B) be two soft binary relations over X such that $A \cap B \neq \emptyset$. Then their restricted composition is defined as the soft set $(\delta, C) = (\sigma, A) \bullet (\rho, B)$, where C=$A \cap B$ and $\delta(\gamma) = \sigma(\gamma) \circ \rho(\gamma)$ for all $\gamma \in C$.*

Definition 2.5. *[4] A soft binary relation (σ, A) over a set X is called a soft reflexive relation over X if $\sigma(\alpha)$is a reflexive relation on X for all $\alpha \in A$ with $\sigma(\alpha) \neq \emptyset$.*

if (σ, A) is a soft binary relation over a set X, then the converse soft binary relation of (σ, A) is also a soft binary relation over X, denoted by $(\sigma, A)^{-1}$. concretely, $(\sigma, A)^{-1} = (\rho, A)$ is a oft set over X, where $\rho(\alpha) = \sigma^{-1}(\alpha)$ and $\sigma^{-1}(\alpha) = \{(x, y) \in X \times X : (y, x) \in \sigma(\alpha)\}$, i.e. the converse of $\sigma(\alpha)$ for all $\alpha \in A$.

Definition 2.6. *[4] A soft binary relation (σ, A) over a set X is called a soft symmetric relation over X if $(\sigma, A)^{-1} = (\sigma^{-1}, A)$.*

Definition 2.7. *[4] Let (σ, A) be a soft binary relation over a set X. Then $\sigma(\alpha)$ is a soft transitive relation over X if $(\sigma, A) \bullet (\sigma, A) \subseteq (\sigma, A)$*

Definition 2.8. *[4] A soft binary relation (σ, A) over a set X is called soft equivalence relation over X if it is, soft reflexive, soft symmetric and soft transitive.*

Corollary 2.9. *[4] A soft binary relation (σ, A) over a set X is soft equivalence relation over X if and only if $\sigma(\alpha) \neq \emptyset$ is a equivalence relation on X for all $\alpha \in A$.*

It is well known that each equivalence relation on a set partition the set into disjoint equivalence classes and a partition of the set provides us an equivalence relation on the set. Therefore, a soft equivalence relation over X provides us a parameterized collection of partitions of X. Let $[x]_{\sigma(\alpha)}$ denote the equivalence class containing x $\in$ X determined by $\sigma(\alpha)$ for $\alpha \in A$. Then it is clear that $y \in [x]_{\sigma(\alpha)}$ if and only if $(x, y) \in \sigma(\alpha)$.

Definition 2.10. *[4] Let $(\sigma, A)$and (ρ, B) be two soft binary relations over a set X. Then*
(1) $(\sigma, A) \subseteq (\rho, B) \Leftrightarrow A \subseteq B, \sigma(\alpha) \subseteq \rho(\alpha)$ *for all* $\alpha \in A$.
(2) $(\sigma, A) \cap_R (\rho, B) = (\delta, C)$,*where* $C = A \cap B \neq \emptyset$ *and* $\delta(\gamma) = \sigma(\gamma) \cap \rho(\gamma)$ *for all* $\gamma \in C$.
(3) $(\sigma, A) \cup_R (\rho, B) = (\delta, C)$,*where* $C = A \cap B \neq \emptyset$ *and* $\delta(\gamma) = \sigma(\gamma) \cup \rho(\gamma)$ *for all* $\gamma \in C$.
(4) $(\sigma, A) \wedge (\rho, B) = (\delta, C)$,*where* $C = A \times B$ *and* $\delta(\alpha, \gamma) = \sigma(\alpha) \cap \rho(\gamma)$ *for all* $(\alpha, \gamma) \in C$.
(5) $(\sigma, A) \vee (\rho, B) = (\delta, C)$,*where* $C = A \times B$ *and* $\delta(\alpha, \gamma) = \sigma(\alpha) \cup \rho(\gamma)$ *for all* $(\alpha, \gamma) \in C$.
(6) $(\sigma, A) \cup_\varepsilon (\rho, B) = (\delta, C)$,*where* $C = A \cup B$ *and for all* $\alpha \in C$,

$$\delta(\alpha)=\begin{cases} \sigma(\alpha) \ \textit{if}\ \alpha \in A \setminus B \\ \rho(\alpha) \ \textit{if}\ \alpha \in B \setminus A \\ \sigma(\alpha) \cup (\alpha) \ \textit{if}\ \alpha \in A \cap B \end{cases}$$

(7) $(\sigma, A) \cap_\varepsilon (\rho, B) = (\delta, C)$,*where* $C = A \cup B$ *and for all* $\alpha \in C$

$$\delta(\alpha)=\begin{cases} \sigma(\alpha) \ \textit{if}\ \alpha \in A \setminus B \\ \rho(\alpha) \ \textit{if}\ \alpha \in B \setminus A \\ \sigma(\alpha) \cap \rho(\alpha) \ \textit{if}\ \alpha \in A \cap B \end{cases}$$

3 Soft S-Acts

In this section first we bring the definition of soft S-acts and then we defined some operations on them.

Definition 3.1. *[12] Let X be right S-act. Then $(F, A)$is called a soft right S-act over X if $F(\alpha) \neq \emptyset$ is a subact of X for all $\alpha \in A$.*

In the sequel, by an S-act we shall mean a right S-act, and by a soft S-act we mean a soft right S-act, unless otherwise stated.

Definition 3.2. *A soft S-act (F, A) is said to be null soft S-act over X, and denoted by $\aleph$ if for all $\alpha \in A$, $F(\alpha) = \emptyset$.*

Definition 3.3. *A soft S-act (F, A) is said to be an absolute soft S-act over X, and denoted by Λ if for all $\alpha \in A$, $F(\alpha) = X$.*

Example 3.4. *Let $S = \{1, a, b, c, d, e\}$ be a monoid with the following table*

.	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	b	c	c	d
b	b	b	b	c	c	c
c	c	c	c	c	c	c
d	d	c	c	c	c	d
e	e	d	c	c	d	e

Then S is an S-act over itself. Now consider a soft set (F, S) over S such that $F(1) = S$, $F(a) = \{a, b, c, d\}$, $F(b) = \{b, c\}$, $F(c) = \{c\}$, $F(d) = \{c, d\}$, $F(e) = \{c, d, e\}$. Then (F, S) is a soft S-act over S. The tabular representation of this soft S-act is as following

	1	a	b	c	d	e
1	1	1	1	1	1	1
a	0	1	1	1	1	0
b	0	0	1	1	0	0
c	0	0	0	1	0	0
d	0	0	0	1	1	0
e	0	0	0	1	1	1

Definition 3.5. *If (F, A) and (G, B) are two soft S-acts over X when $A \cap B = \emptyset$ then $(F, A) \cup (G, B)$ is defined as soft set (H, C)over X, where $C = A \cup B$ and for all $c \in C$,*

$$H(c) = \begin{cases} F(c)\ ,\ \text{if } c \in A \\ G(c)\ ,\ \text{if } c \in B \end{cases}$$

Definition 3.6. *If (F, A) and (G, B) are two soft S-acts over X then cartesian product of them is defined as soft set over X such that, $(H, A \times B)$ where $H(\alpha, \beta) = F(\alpha) \times G(\beta)$ for all $\alpha \in A$ and $\beta \in B$.*

Definition 3.7. *If S is a group and (F, A) be soft S-act over X then the complement of (F, A) is defined as soft set (F^c, A) over X where $F^c(\alpha) = X - F(\alpha)$ for all $\alpha \in A$.*

Definition 3.8. *The extended intersection of two soft S-acts (F, A) and (G, B) is defined as soft set $(H, C) = (F, A) \cap_{\varepsilon} (G, B)$ over X, where $C = A \cup B$ and*

$$H(c) = \begin{cases} F(c) \ , \text{ if } c \in A \setminus B \\ G(c) \ , \text{ if } c \in B \setminus A \\ F(c) \cap G(c) \ , \text{ if } c \in A \cap B \end{cases}$$

Definition 3.9. *Let (F, A) and (G, B) be two soft S-acts over X and Y respectively. Then $(F, A) \times (G, B)$ is defined as soft set (H, C) over $X \times Y$ where, $C = A \times B$ and $H(\alpha, \beta) = F(\alpha) \times G(\beta)$.*

Proposition 3.10. *Let (F, A) and (G, B) be two soft S-acts over X. Then $(F, A) \cap_R (G, B)$, $(F, A) \cup_R (G, B)$, $(F, A) \cup (G, B)$, $(F, A) \wedge (G, B)$, $(F, A) \times (G, B)$, $(F, A) \cup_{\varepsilon} (G, B)$, $(F, A) \vee (G, B)$, $(F, A) \cap_{\varepsilon} (G, B)$ are soft S-acts over X.*

Proposition 3.11. *Let (F, A) and (G, B) be two soft S-acts over X and Y respectively. Then $(F, A) \times (G, B)$ is soft S-act over $X \times Y$.*

Definition 3.12. *Let(F, A) and (G, B)be two soft S-acts over X ,then (G, B) is a soft subact of (F, A) if*
a) $B \subseteq A$
b) $G(b)$ is a subact of $F(b)$, for all $b \in B$.

Definition 3.13. *We say that (F, A) is soft equal to (G, B), and write $(F, A) = (G, B)$, if (F, A) is a soft subact of (G, B) and (G, B) is a soft subact of (F, A).*

Proposition 3.14. *If (F, A) is a soft S-act over X and (G, B),(H, C) are two soft subact of (F, A). Then $(G, B) \cap_R (H, C)$, $(G, B) \cup_R (H, C)$, $(G, B) \cup (H, C)$, $(G, B) \wedge (H, C)$, $(G, B) \times (H, C)$,$(G, B) \cup_{\varepsilon} (H, C)$, $(G, B) \vee (H, C)$, $(G, B) \cap_{\varepsilon} (H, C)$ are soft subact of (F, A).*

Proposition 3.15. *Let (F_1, A_1) and (G_1, B_1) be two soft subact of soft S-acts (F, A) and (G, B) over S-act X and Y respectively, then $(F_1, A_1) \times (G_1, B_1)$ is soft subact of $(F, A) \times (G, B)$ over $X \times Y$.*

Definition 3.16. *Let (F, A) and (G, B) be two soft S-acts over X and Y, respectively, and $f : X \to Y$ and $g : A \to B$ be two functions. We say that (f, g) is a soft homomorphism and (F, A) is homomorphic to (G, B) if the following hold:*
1) f is an S-homomorphism from X to Y.
2) g is a mapping from A onto B.
3) $f(F(a)) = G(g(a))$ for all $a \in A$.

If f is an isomorphism from X to Y and g is a one to one mapping from A to B, then (f, g) is called soft isomorphism.

If $(f, g) : (F, A) \to (G, B)$ and $(h, k) : (G, B) \to (H, C)$ are soft homomorphisms, then the composition of (f, g) and (h, k) is defined as $(h, k) \circ (f, g) = (p, q)$ where p=$h \circ f$ and q=$k \circ g$.
The composition of two soft homomorphism is soft homomorphism which we call that soft composition.

Let $f : \mathcal{A} \to \mathcal{B}$ be a S-homomorphism from S-act $\mathcal{A}$ to S-act $\mathcal{B}$ and (F, A) be a soft set over $\mathcal{A}$. Then $(f(F), A)$ is soft set over $\mathcal{B}$ where $f(F)$ is defined by $f(F)(a) = f(F(a))$ for all $a \in \mathcal{A}$.

Proposition 3.17. *If (F, A) and (G, B) be two soft S-acts over X, (G, B) be soft subact of (F, A), and f is homomorphism from X to Y,then $(f(F), A)$ and $(f(G), B)$are both soft S-act over Y and $(f(G), B)$ is soft sub-act of $(f(F), A)$.*

Definition 3.18. *If (F, A) is a soft S-act over X, we define $i_{(F,A)}$ a soft homomorphism (f, g) from (F, A) to itself, where $f = i_X$ and $g = i_A$ and $f(F(a)) = F(a)$ for all $a \in A$.*

Definition 3.19. *Let (F, A) and (G, B) be two soft S-acts over X, and (f, g) be a soft homomorphism from (F, A) to (G, B). The inverse of (f, g) is a unique soft homomorphism $(f^\star, g^\star)$ if and only if $f^\star = f^{-1}$ and $g^\star = g^{-1}$*
So $(f, g) \circ (f^\star, g^\star) = (f \circ f^\star, g \circ g^\star) = (i_X, i_B) = id_{(G,B)}$ and $(f^\star, g^\star) \circ (f, g) = id_{(F,A)}$.

Proposition 3.20. *Let (G, B) be a soft subact of a soft S-act (F, A) over S-act X. Then $(f(G), g(B))$ is soft subact of (H, C) where (H, C) is sot S-act over S-act Y and (f, g) is a soft homomorphism from (F, A) to (H, C).*

Proof. Since $G(b)$ is subact of $F(b)$ for all $b \in B$. Therefore $G(b) \subseteq F(b)$. This implies that $f(G(b)) \subseteq f(F(b)) = H(g(b))$. Therefore $f(G(b))$ is a subact of Y. As g is function so $g(B) \subseteq C$. Hence $(f(G), g(B)$ is a soft subact of (H, C).

4 Congruence Relations over Soft S-Acts

In this section we want to define soft congruence relation over soft S-act.

Definition 4.1. *A soft relation (σ, A) over an S-act X is called a soft S-compatible relation if $\sigma(\alpha)$ is an S- compatible relation over X for all $\alpha \in A$.*

Definition 4.2. *A soft equivalence relation (σ, A) over an S-act X is a called a soft congruence relation if $\sigma(\alpha)$ is an congruence relation over X for all $\alpha \in A$.*

Lemma 4.3. *A soft equivalence relation (σ, A) over an S-act X is soft congruence relation if and only if $\sigma(\alpha)$ is S-compatible for all $\alpha \in A$.*

Proposition 4.4. *Let (σ, A) and (ρ, B) be two soft equivalence relations over an S-act X. Then we have*
1) $(\sigma, A) \cap_R (\rho, B)$ is a soft congruence over X when $A \cap B \neq \emptyset$.
2) $(\sigma, A) \cup_\varepsilon (\rho, B)$ is a soft congruence over X when $A \cap B = \emptyset$.
3) $(\sigma, A) \wedge (\rho, B)$ is a soft congruence over X.
4) $(\sigma, A) \cap_\varepsilon (\rho, B)$ is a soft congruence over X.

Proposition 4.5. *Let (σ, A) be a soft congruence relation over an S-act X and $X/\sigma(\alpha) = \{[x]_{\sigma(\alpha)} : x \in X\}$ where $\alpha \in A$. Then for any $\alpha \in A$, $X/\sigma(\alpha)$ is an S-act under the binary operation induce by X, that is $[x]_{\sigma(\alpha)}s = [xs]_{\sigma(\alpha)}$ for all $x \in X$ and $s \in S$. Moreover, X is homomorphic to $X/\sigma(\alpha)$ for each $\alpha \in A$ by the natural map $\sqcap_\alpha : X \to X/\sigma(\alpha)$ given by $f_\alpha(x) = [x]_{\sigma(\alpha)}$.*

We can denote $X/(\sigma, A)$ as the collection of all S-acts which are homomorphic images of X.
Let σ be a congruence on X and (F, A) be a soft S-act over X. Denote the soft set $(F, A)/\sigma$ by (K, A) where $K(\alpha) = \{[a]_\sigma : a \in F(\alpha)\}$ for all $\alpha \in A$. Since $F(\alpha)$ is a subact of X, $K(\alpha)$ is a subact of X/σ for all $\alpha \in A$. Hence, $(F, A)/\sigma$ is a soft subact over X/σ. Now let $i_A : A \to A$ be the identity mapping over A and σ^τ be the natural homomorphism given by $\sigma^\tau(a) = [a]_\sigma$ for all a∈A. Then (σ^τ, i_A) is soft homomorphism from (F, A) to $(F, A)/\sigma$, which is called natural soft homomorphism.
If (F, A) is a soft S-act over X and (σ, A) is soft binary relation over X such that $\sigma(\alpha) \neq \emptyset$ is congruence on $F(\alpha) \neq \emptyset$ for all $\alpha \in A$, then we say that (σ, A) is a soft congruence relation on (F, A).

Definition 4.6. *Let (F, A) and (G, B) be two soft S-act over X and Y, respectively. Let (f, g) be a soft homomorphism from (F, A) to (G, B) we define ker(f, g) as a soft binary relation (σ, A) over X where*

$$\sigma(\alpha) = \{(a, b) \in F(\alpha) \times F(\alpha) : f(a) = f(b)\} \text{ for all } \alpha \in A.$$

Proposition 4.7. *Let (F, A) and (G, B) be two soft S-act over X and Y, respectively. If (f, g) is a soft homomorphism from (F, A) to (G, B) then ker(f, g) is a soft congruence relation on (F, A).*

Proof. Since kerf is clearly an equivalence relation over X it is easy to see that $\sigma(\alpha)$ is an equivalence relation on the subact $F(\alpha)$ for all $\alpha \in A$.

Now let $(a, b) \in \sigma(\alpha)$. So $f(a) = f(b)$,whence we have
$f(as) = f(a)s = f(b)s = f(bs)$.
This shows that $(as, bs) \in \sigma(\alpha)$, and $\sigma(\alpha)$ is a congruence on $F(\alpha)$ for all $\alpha \in A$. Hence $\ker(f, g)$ is a soft congruence relation on (F, A).

5 Soft Congruence on Biacts

Definition 5.1. *Let X be a $T - S$ biact over monoids T and S. A soft relation (σ, A) over $T - S$- biact X is called a soft $T - S$-compatible if for all $t \in T$, $a, b \in X$ and $s \in S$ we have*
$(a, b) \in \sigma(\alpha) \Rightarrow (tas, tbs) \in \sigma(\alpha))$ for all $\alpha \in A$.
A soft equivalence relation (σ, A) over $T - S$- biact X, which is a soft $T - S$-compatible is called soft $T - S$-congruence over X.

If $|S|$=1,we have a definition of a soft T-compatible relation (soft T-congruence), and if $|T|$=1,we have a definition of a soft S-compatible relation (soft S-congruence).

Proposition 5.2. *A binary relation (σ, A) over $T - S$-biact X is called a soft $T - S$-compatible (congruence) if and only if for all $\alpha \in A$, $\sigma(\alpha)$ be a $T - S$- compatible relation (congruence).*

Lemma 5.3. *A soft relation (σ, A) over a $T - S$-biact X is a soft $T - S$-congruence if and only if (σ, A) is both a soft T-congruence and soft S-congruence.*

Proposition 5.4. *If $(\sigma, A)$and (ρ, A) be two soft $T - S$-congruences over $T - S$-biact X then,*
1)$(\sigma, A) \cap_R (\rho, A)$ is a soft $T - S$-congruence over X.
2)$(\sigma, A) \cap_\varepsilon (\rho, A)$ is a soft $T - S$-congruence over X.
3)$(\sigma, A) \wedge (\rho, A)$ is a soft $T - S$-congruence over X.

Definition 5.5. *Let (σ, A) be soft $T - S$-compatible over $T - S$-biact X. The relation $(\sigma, A)^c = \{(\sigma^c(\alpha)) : forall\ \alpha \in A\}$ where*

$$\sigma^c(\alpha) = \{(ta_1s, ta_2s) \in A \times A | t \in T, (a_1, a_2) \in \sigma(\alpha), s \in S\}$$

is called a soft $T - S$- compatible closure of $\sigma(\alpha)$ and $(\sigma, A)^c = \{\sigma(\alpha)^c | c \in C and \alpha \in A\}$ is soft $T - S$-compatible closure of (σ, A)

The unique smallest soft $T - S$-congruence on a $T - S$-biact X, containing (σ, A) which is denote by $(\sigma, A)^\#$ is called the soft congruence closure of (σ, A). In fact,

$$(\sigma, A)^{\#} = \bigcap_{R_{i\in I,(\sigma,A)\subseteq(\sigma_i,A_i)}} (\sigma_i, A_i).$$

Proposition 5.6. *Let $(\sigma, A)$and (ρ, A) be two binary relations on a $T-S$-biact X. Then*
1)$(\sigma, A) \subseteq (\sigma, A)^c$.
2)$((\sigma, A)^c)^{-1} \subseteq ((\sigma, A)^{-1})^c$.
3)$(\sigma, A) \subseteq (\rho, B) \Rightarrow (\sigma, A)^c \subseteq (\rho, B)^c$.
4)$((\sigma, A)^c)^c = (\sigma, A)$.
5)$((\sigma, A) \cup_\varepsilon (\rho, B))^c = (\sigma, A)^c \cup_\varepsilon (\rho, B)^c$
6)$(\sigma, A) = (\sigma, A)^c$ if and only if (σ, A) is a $T-S$-compatible on $T-S$- biact X.

Proof. (1)-(3) are obvious.
(4) By (1) $(\sigma, A)^c \subseteq ((\sigma, A)^c)^c$. Let for all $\alpha \in A$, $(a_1, a_2) \in ((\sigma^c(\alpha)))^c$. Then $a_1 = t'a'_1s'$, $a_2 = t'a'_2s'$ for some $t' \in T$, $s' \in S$ and $(a'_1, a'_2) \in (\sigma^c(\alpha))$. Therefore, $a'_1 = ta_1s$, $a'_2 = ta_2s$ for some $t \in T$, $s \in S$ and $(a_1, a_2) \in (\sigma(\alpha))$. Hence

$$a_1 = (t't)a_1(ss'), a_2 = (t't)a_2(ss'), t', t \in T, s, s' \in S$$

i.e, $(a_1, a_2) \in (\sigma^c(\alpha))$, for all $c \in C$. Therefor $((\sigma, A)^c)^c \subseteq (\sigma, A)$, and then $((\sigma, A)^c)^c = (\sigma, A)$.
(5) Let $(\sigma, A) \cup_\varepsilon (\rho, B)) = (H, C)$ where $C = A \cup B$ and H is a function from C to $P(X \times X)$ defined by

$$H(c)=\begin{cases} \sigma(c) \text{ , if } c \in A \setminus B \\ \rho(c) \text{ , if } c \in B \setminus A \\ \sigma(c) \cup \rho(c) \text{ , if } c \in A \cap B, \end{cases}$$

for all $c \in C$. Therefore $(\sigma, A) \cup_\varepsilon (\rho, B))^c = (H, C)^c$, where

$$H^c(c)=\begin{cases} \sigma^c(c) \text{ , if } c \in A \setminus B \\ \rho^c(c) \text{ , if } c \in B \setminus A \\ (\sigma(c) \cup \rho(c))^c \text{ , if } c \in A \cap B. \end{cases}$$

Now, let $(\sigma, A)^c \cup_\varepsilon (\rho, B)^c = (K, C)$, where $C = A \cup B$ and K is a function from C to $P(X \times X)$ defined by

$$K(c)=\begin{cases} (\sigma^c(c)) \text{ , if } c \in A \setminus B \\ (\rho^c(c)) \text{ , if } c \in B \setminus A \\ (\sigma^c(c)) \cup (\rho^c(c)) \text{ , if } c \in A \cap B. \end{cases}$$

So, if $c \in A \setminus B$ or $B \setminus A$, then $K(c) = H^c(c)$. If $c \in A \cap B$, then $(\sigma^c(c)) \subseteq (\sigma(c) \cup \rho(c))^c$ and $(\rho^c(c)) \subseteq (\sigma(c) \cup \rho(c))^c$. Consequently,

$$(\sigma^c(c)) \cup (\rho^c(c)) \subseteq (\sigma(c) \cup \rho(c))^c.$$

Conversely, suppose that $(a_1', a_2') \in (\sigma(c) \cup \rho(c))^c$, for all $c \in C$. Then $a_1' = ta_1s, a_2' = ta_2s$ for some $t \in T, s \in S$ and $(a, b) \in (\sigma(c) \cup \rho(c))$. That is either $(a_1, a_2) \in (\sigma(c))$ or $(a_1, a_2) \in (\rho(c))$, and hence either $(a_1', a_2') \in (\sigma^c(c))$ or $(a_1', a_2') \in (\rho^c(c))$. Thus, $(a_1', a_2') \in (\sigma^c(c)) \cup (\rho^c(c))$, for all $c \in C$.
(6) Let $(\sigma, A) = (\sigma, A)^c$. Then,

$$(a_1, a_2) \in \sigma(\alpha) \Rightarrow (ta_1s, ta_2s) \in (\sigma(\alpha))^c = (\sigma(\alpha)),$$

for all $\alpha \in A$,$t \in T, s \in S$ *and* $a_1, a_2 \in A$. Thus, (σ, A) is $T - S$-compatible. Conversely, if (σ, A) is $T - S$-compatible relation and $(a_1', a_2') \in (\sigma(\alpha))^c$ for all $\alpha \in A$, then $a_1' = ta_1s, a_2' = ta_2s$ for some $t \in T, s \in S, (a_1, a_2) \in \sigma(\alpha)$. Therefor,$(a_1', a_2') = (ta_1s, ta_2s) \in \sigma(\alpha)$ that is $(\sigma(\alpha))^c \subseteq \sigma(\alpha)$.

Proposition 5.7. *If (σ, A) is a soft $T - S$-compatible over a $T - S$-biact X, then $(\sigma, A)^n = (\sigma, A) \bullet (\sigma, A) \bullet (\sigma, A) \bullet ... \bullet (\sigma, A)$ (n factors), where $\sigma^n(\alpha) = \sigma(\alpha) \circ \sigma(\alpha) ... \circ \sigma(\alpha)$ (n factors) is also $T - S$-compatible for all $\alpha \in A$.*

Proof. Let for all $\alpha \in A$, $(a_1, a_2) \in (\sigma^n(\alpha))$. Then there exist $b_1, b_2, ..., b_{n-1} \in X$ such that

$$(a_1, b_1), (b_1, b_2), ..., (b_{n-1}, a_2) \in (\sigma(\alpha))$$

since $(\sigma(\alpha))$ is $T - S$-compatible then,

$$(ta_1s, tb_1s), (tb_1s, tb_2s), ..., (tb_{n-1}s, ta_2s) \in (\sigma(\alpha))$$

for all $t \in T$, $s \in S$. So $(ta_1s, ta_2s) \in (\sigma^n(\alpha))$. This yields that $(\sigma, A)^n$ is $T - S$-compatible.

For next proposition, consider soft binary relation $(\sigma, A)^\infty = (\sigma^\infty, A) = \bigcup_{n=1}^{\infty} (\sigma, A)^n$ which is called the soft transitive closure of (σ, A) and is defined in [4].

Proposition 5.8. *let (σ, A) be soft binary relation on $T - S$ biact $\mathcal{A}$. Then* $(\sigma, A)^\sharp = ((\sigma, A)^c)^e = ((\sigma, A)^c \cup_\varepsilon ((\sigma, A)^{-1})^c \cup_\varepsilon (\sigma^{1_\mathcal{A}}, A)^c)^\infty$
where $(\sigma^{1_\mathcal{A}}, A)$ is identity relation over $\mathcal{A}$ such that

$$\sigma^{1_\mathcal{A}}(a) = \begin{cases} 1_\mathcal{A} & , if \sigma(\alpha) \neq \emptyset \\ \emptyset & , otherwise. \end{cases}$$

Proof. By considering [4] we know that $(\sigma, A)^e = ((\sigma, A) \cup_\varepsilon ((\sigma, A)^{-1}) \cup_\varepsilon (\sigma^{1_\mathcal{A}}, A))^\infty$, Therefor, it is clear that $(\sigma, A) \subseteq (\sigma, A)^c \subseteq ((\sigma, A)^c)^e$.
Now we show that $((\sigma, A)^c)^e$ is a soft $T-S$-congruence on $\mathcal{A}$. Note that $((\sigma, A)^c)^e = (\nu, A)^\infty$, where $(\nu, A) = (\sigma, A)^c \cup_\varepsilon ((\sigma, A)^{-1})^c \cup_\varepsilon (\sigma^{1_\mathcal{A}}, A)^c$.
Suppose that for all $\alpha \in A$, $(a_1, a_2) \in (\sigma^c(\alpha))^e$. Then $(a_1, a_2) \in \nu^n(\alpha)$ for all $n \in N$. By the proposition 5.6, we get:

$$(\nu, A) = (\sigma, A)^c \cup_\varepsilon ((\sigma, A)^{-1})^c \cup_\varepsilon (\sigma^{1_\mathcal{A}}, A)^c =$$
$$((\sigma, A) \cup_\varepsilon ((\sigma, A)^{-1} \cup_\varepsilon (\sigma^{1_\mathcal{A}}, A))^c = (\nu, A)^c$$

Hence (ν, A) is a soft $T-S$-compatible and thus $(\nu, A)^n$ is also soft $T-S$-compatible. Therefor for all $\alpha \in A$ we have $(ta_1s, ta_2s) \in \nu^n(\alpha) \subseteq (\sigma^c(\alpha))^e$ for all $t \in T$, $s \in S$. Thus $((\sigma, A)^c)^e$ is soft $T-S$-congruence on $\mathcal{A}$.
Let now (ρ, A) be a soft congruence on $\mathcal{A}$ containing (σ, A). Then $(\sigma, A)^c \subseteq (\rho, A)^c = (\rho, A)$, and therefor $((\sigma, A)^c)^e \subseteq (\rho, A)$. Thus proof is complete.

Definition 5.9. *let (σ, A) be soft equivalence relation on $T-S$-biact X. Then $(\sigma, A)^b) = \{\sigma^b(\alpha) : \alpha \in A\}$ where,*

$$\sigma^b(\alpha) = \{(a_1, a_2) \in X \times X : (ta_1s, ta_2s) \in (\sigma(\alpha)) forallt \in T, s \in S\}$$

Proposition 5.10. *$(\sigma, A)^b$ is the largest soft $T-S$-congruence on $T-S$-biact X contained in(σ, A).*

Proof. It is clear that (σ, A) is soft equivalence relation,because of $\sigma^b(\alpha)$ is a soft equivalence relation and $\sigma^b(\alpha) \subseteq \sigma(\alpha)$ for all $\alpha \in A$,so $(\sigma, A)^b \subseteq (\sigma, A)$.If $(a_1, a_2) \in \sigma^b(\alpha)$ and $t' \in T, s' \in S$ then we have

$$(t(t'a_1s')s, t(t'a_2s')s) = ((tt')a_1(s's), (tt')a_2(s's) \in (\sigma(\alpha)$$

for all $t \in T, s \in S$ and there for $(t'a_1s', t'a_2s') \in (\sigma^b(\alpha)$. Thus $\sigma^b(\alpha)$ is largest $T-S$-congruence for all $\alpha \in A$ contained in $\sigma(\alpha)$, so $(\sigma, A)^b)$ is the largest soft $T-S$-congruence on $T-S$- biact X contained in (σ, A).

6 Homomorphism Theorem

In this section we want to prove homomorphism theorem for soft S-acts.

Theorem 6.1. *(Homomorphism Theorem) Let (F, A) and (G, B) be two soft S-acts over X and Y, respectively. Let $(f, g) : (F,A) \to (G,B)$ be a soft homomorphism and ρ be a congruence on X such that $\rho \subseteq kerf$. Then there exists a unique soft homomorphism $(h, g) : (F, A)/\rho \to (G, B)$ such that $h(F(\alpha)/\rho) = G(g(\alpha))$ for all $\alpha \in A$, and the following diagram commutes where $(\rho^\tau, i_A) : (F, A) \to (F, A)/\rho$ is the natural soft homomorphism.*

$$\begin{tikzcd}
(F,A) \arrow[r, "{(f,g)}"] \arrow[d, "{(\rho^{\tau},i_A)}"'] & (G,B) \arrow[dl, "{(h,g)}"] \\
(F,A)/\rho &
\end{tikzcd}$$

If $\rho = kerf$, g is injective, and f is surjective then, (h,g) is a soft isomorphism.

Proof. Since (f,g) is a soft homomorphism from (F,A) to (G,B), it follows that $f : X \to Y$ is a soft S-act homomorphism, and $g : A \to B$ is a surjective map such that $fF(a) = G(g(a)$ for all $a \in A$. Let h:$X/\rho \to Y$ be a map defined by $h([x]_\rho) = f(x)$ where x∈X. Note that
$[a]_\rho = [b]_\rho \Rightarrow (a,b) \in \rho \subseteq kerf \Rightarrow f(a) = f(b)$
So h is a well-defined map. Moreover, $h : X/\rho \to Y$ is an S-act homomorphism. In fact, let x∈X and s∈S, clearly
$h([x]_\rho s) = h([xs]_\rho) = f(xs) = f(x)s = h([x]_\rho)s.$
The soft S-act $(F,A)/\rho$ is given as (K,A) where $K(\alpha) = F(\alpha)/\rho$ for all $\alpha \in A$. then we have
$h(K(\alpha)) = h(F(\alpha)/\rho)$ $=\{h([a]_\rho) : a \in F(\alpha)\}$ $=\{f(a) : a \in F(\alpha)\}$ $=f(F(\alpha))$ $=G(g(\alpha))$.
Therefore, we conclude that (h,g) is a soft homomorphism from $(F,A)/\rho$ to (G,B).
Finally it is easy to see that diagram is commutative and the soft homomorphism (f,g) is unique by definition of h.

Remark 6.2. *By definition $ker(f,g) \subseteq kerf$ and $ker(f,g)$ is soft congruence on X ,so by above hypothesis of theorem there is a unique soft homomorphism from $(F,A)/ker(f,g)$ to (G,B).*

Theorem 6.3. *(Second isomorphism theorem) Let(F,A), (G,B) be two soft S-acts over X and Y respectively. Let (σ,A), (ρ,A) be two soft congruences over (F,A) and (G,B) respectively. Consider two soft epimorphism $(f_1, id_A) : (F,A)_X \to \frac{(F,A)_X}{(\sigma,A)_X}$ and $(f_2, id_A) : (G,A)_Y \to \frac{(G,A)_Y}{(\rho,A)_Y}$. Then $(f_1 \times f_2, id_A \times id_A) : (F,A)_X \times (G,A)_Y \to \frac{(F,A)_X}{(\sigma,A)_X} \times \frac{(G,A)_Y}{(\rho,A)_Y}$ is soft epimorphism and*

$$\frac{(F,A)_X \times (G,A)_Y}{ker(f_1 \times f_2)} \cong \frac{(F,A)_X}{(\sigma,A)_X} \times \frac{(G,A)_Y}{(\rho,A)_Y}$$

Proof. First it is easy to see that $(f_1 \times f_2, id_A \times id_A)$ is soft epimorphism from $(F,A)_X \times (G,A)_Y$ to $\frac{(F,A)_X}{(\sigma,A)_X} \times \frac{(G,A)_Y}{(\rho,A)_Y}$. Now by using $ker(f_1 \times f_2) = kerf_1 \times kerf_2$ and first isomorphism theorem proof is complete.

Theorem 6.4. *(Third isomorphism theorem) Let (F, A) be soft S-act over X, and (σ, A), (ρ, A) be two soft congruences over (F, A) such that $(\sigma, A) \subseteq (\rho, A)$, then $\frac{(\rho,A)}{(\sigma,A)}$ is soft congruence over $\frac{(F,A)_X}{(\sigma,A)_X}$ and*

$$\frac{(F,A)/(\sigma,A)}{(\rho,A)/(\sigma,A)} \cong (F, A)/(\rho, A).$$

Proof. The proof is straightforward.

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