Gen. Math. Notes, Vol. 28, No. 1, May 2015, pp. 18-39
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
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The $2t$-Pebbling Property on the Jahangir Graph $J_{2,m}$

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(Received: 3-4-15 / Accepted: 9-5-15)

Abstract

The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. A graph G satisfies the $2t$-pebbling property if $2t$ pebbles can be moved to a specified vertex when the total starting number of pebbles is $2f_t(G) - q + 1$ where q is the number of vertices with at least one pebble. In this paper, we are going to show that the graph $J_{2,m}$ ($m \geq 3$) satisfies the $2t$-pebbling property.

Keywords: *Graph pebbling, Jahangir grpah, 2t-pebbling property.*

1 Introduction

An n-dimensional cube Q_n, or n-cube for short, consists of 2^n vertices labelled by $(0, 1)$-tuples of length n. Two vertices are adjacent if their labels are different in exactly one entry. Saks and Lagarias (see [1]) propose the following question: suppose 2^n pebbles are arbitrarily placed on the vertices of an n-cube. Does there exist a method that allows us to make a sequence of moves, each move taking two pebbles off one vertex and placing one pebble on an adjacent vertex, in such a way that we can end up with a pebble on any desired vertex? This question is answered in the affirmative in [1].

A configuration C of pebbles on a graph $G = (V, E)$ can be thought of as a function $C : V(G) \to N \cup \{0\}$. The value $C(v)$ equals the number of pebbles placed at vertex v, and the size of the configuration is the number $|C| = \sum_{v \in V(G)} C(v)$ of pebbles placed in total on G. Suppose C is a configuration of pebbles on a graph G. A pebbling move (step) consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, the target vertex, if we can apply pebbling moves repeatedly (if necessary) so that in the resulting configuration the vertex v has at least one pebble.

Definition 1.1 *([8]) The t-pebbling number of a graph G, $f_t(G)$, is the least n such that, for any configuration of n pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the pebbling number of G.*

Fact 1.2 *([12]) For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.*

Fact 1.3 *([12]) The pebbling number of a graph G satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

Saks and Lagarias question then reduces to asking whether $f(Q_n) \leq n$, where Q_n is the n-cube. Chung [1] answered this question in the affirmative, by proving a stronger result.

Theorem 1.4 *([1]) In an n-cube with a specified vertex v, the following are true:*

- *If 2^n pebbles are assigned to vertices of the n-cube, one pebble can be moved to v.*
- *Let q be the number of vertices that are assigned an odd number of pebbles. If there are all together more than $2^{n+1} - q$ pebbles, then two pebbles can be moved to v.*

Definition 1.5 *([3]) Given the t-pebbling of G, let p be the number of pebbles on G, let q be the number of vertices with at least one pebble. We say that G satisfies the $2t$-pebbling property if it is possible to move $2t$ pebbles to any specified target vertex of G starting from every configuration in which $p \geq 2f_t(G) - q + 1$ or equivalently $p + q > 2f_t(G)$ for all t.*

If q stands for the number of vertices with an odd number of pebbles, we call the property, the odd $2t$-pebbling property.

Definition 1.6 *([3]) We say a graph satisfies the odd $2t$-pebbling property for all t. If, for any arrangement of pebbles with at least $2f_t(G)-r+1$ pebbles, where r is the number of vertices in the arrangement with an odd number of pebbles, it is possible to put $2t$ pebbles on any target vertex using pebbling moves.*

It is easy to see that a graph which satisfies the $2t$-pebbling property also satisfies the odd $2t$-pebbling property for all t.
With regard to t-pebbling number of graphs, we find the following theorems:

Theorem 1.7 *([9]) Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

Theorem 1.8 *([2]) Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ be a cycle of length $n-1$. Then the t-pebbling number of the wheel graph W_n is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

Theorem 1.9 *([5]) For $G = K^*_{s_1,s_2,\cdots,s_r}$,*

$$f_t(G) = \begin{cases} 2t+n-2, & \text{if } 2t \leq n - s_1 \\ 4t + s_1 - 2, & \text{if } 2t \geq n - s_1 \end{cases}.$$

Theorem 1.10 *([9]) Let $K_{1,n}$ be an n-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.*

Theorem 1.11 *([9]) Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+1}-(-1)^{k+2}}{3} + (t-1)2^k$.*

Theorem 1.12 *([9]) Let P_n be a path on n vertices. Then $f_t(P_n) = t(2^{n-1})$.*

Theorem 1.13 *([9]) Let Q_n be the n-cube. Then $f_t(Q_n) = t(2^n)$.*

With regard to the $2t$-pebbling property of graphs, we find the following theorems:

Theorem 1.14 *([12]) All diameter two graphs satisfy the two-pebbling property.*

Theorem 1.15 *([3]) All paths satisfy the $2t$-pebbling property for all t.*

Theorem 1.16 *([3]) All even cycles satisfy the $2t$-pebbling property for all t.*

Theorem 1.17 *([3]) The n-cube Q_n satisfies the $2t$-pebbling property for all t.*

Theorem 1.18 *([4]) Let K_n be a complete graph on n vertices. Then K_n satisfies the $2t$-pebbling property for all t.*

Theorem 1.19 *([5]) The star graph $K_{1,n}$, where $n > 1$ satisfies the $2t$-pebbling property.*

Theorem 1.20 *([5]) Any complete r-partite graph satisfies the $2t$-pebbling property.*

In Section 2, we state the pebbling results of the Jahangir graph $J_{2,m}$ and then we prove that $J_{2,m}$ satisfies the $2t$-pebbling property in Section 3 and Section 4.

2 Jahangir Graph Definition and its Known Pebbling Results

Definition 2.1 *([11]): Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm+1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.*

Labeling for $J_{2,m}$ ($m \geq 3$): Let v_{2m+1} be the label of the center vertex and $v_1, v_2, \cdots, v_{2m}$ be the label of the vertices that are incident clockwise on cycle C_{2m} so that $deg(v_1) = 3$.

The t-pebbling number of Jahangir graph $J_{2,m}$ ($m \geq 3$)is as follows:

Theorem 2.2 *([6, 8]) For the Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.*

Theorem 2.3 *([6, 8]) For the Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.*

Theorem 2.4 *([6, 8]) For the Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.*

Theorem 2.5 *([6, 7, 8]) For the Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$ where $m \geq 6$.*

Notation 2.6 *Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the set $A \subseteq V(G)$. We define the sets $S_1 = \{v_1, v_3, \cdots, v_{2m-1}\}$ and $S_2 = \{v_2, v_4, \cdots, v_{2m}\}$ from the labeling of $J_{2,m}$.*

Remark 2.7 *Consider a graph G with n vertices and $2f(G) - q + 1$ pebbles on it and we choose a target vertex v from G. If $p(v) = 1$, then the number of pebbles remained in G is $2f(G) - q \geq f(G)$, since $f(G) \geq n$ and $q \leq n$, and hence we can move the second pebble to v. Let us assume that $p(v) = 0$. We let $p(u) \geq 2$ where $uv \in E(G)$. We move one pebble to v from u. Then the graph G has at least $2f(G) - q + 1 - 2 \geq f(G)$, since $f(G) \geq n$ and $q \leq n-1$, and hence we can move the second pebble to v. So, we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.*

3 The 2-Pebbling Property of the Jahangir Graph $J_{2,m}$

Theorem 3.1 *The graph $J_{2,3}$ satisfies the 2-pebbling property.*

Proof: The graph $J_{2,3}$ has at least $2f(J_{2,3})-q+1 \geq 17-q \geq 10$ pebbles on it.

Case 1: Let v_7 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_7) = 0$ and $p(v_i) \leq 1$ for all $v_iv_7 \in E(J_{2,3})$. Thus, one of the non-adjacent vertices of v_7 has at least $\lceil \frac{17-q-3}{3} \rceil \geq \lceil \frac{8}{3} \rceil \geq 3$. Without loss of generality, we let $p(v_2) \geq 3$. If $p(v_1) = 1$ or $p(v_3) = 1$ then we can move one pebble to v_7 using at most three pebbles through v_2 and v_1 or v_2 and v_3. Then, the graph $J_{2,3}$ has at least $17-q-3 \geq 8$, since $q \leq 6$ and hence we are done by Theorem 2.2. Assume that $p(v_1) = 0$ and $p(v_3) = 0$. Since $q \leq 4$, we have $17-q-1 \geq 12$, and hence one of the non-adjacent vertices of v_7, say v_2, has at least four pebbles. So, we move one pebble to v_7 from v_2 at a cost of four pebbles and then the remaining number of pebbles on $J_{2,3}$ are $17-q-4 \geq 9$, since $q \leq 4$ and hence we are done by Theorem 2.2.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$, $p(v_2) \leq 1$, $p(v_6) \leq 1$ and $p(v_7) \leq 1$. We assume that $p(v_3) \geq 2$. If $p(v_2) = 1$ or $p(v_7) = 1$ then we move one pebble to v_1 using at most three pebbles. Then the number of pebbles remained on $J_{2,3}$ is $17-q-3 \geq 8$, since $q \leq 6$ and hence we are done by Theorem 2.2. Let $p(v_2) = 0$ and $p(v_7) = 0$. So, $17-q \geq 13$. If $p(v_3) \geq 4$ then we move one pebble to v_1 using at most four pebbles and then $17-q-4 \geq 9$ pebbles have remained in $J_{2,3}$ and hence we are done. So, we assume that $p(v_3) \leq 3$. Similarly, we assume $p(v_5) \leq 3$. Let $p(v_3) = 2$ or 3. If $p(v_5) \geq 1$ then we can move two pebbles to v_7 at a cost of at most five pebbles, since $p(v_4) \geq 4$ and hence one pebble is moved to v_1. Then, the remaining number of pebbles on $J_{2,3}$ are $17-q-5 \geq 8$ and hence we are done by Theorem 2.2. Let $p(v_5) = 0$. Since $p(v_3) \geq 2$ and $p(v_4) \geq 4$, we can move one pebble to v_1 at a cost of at most six pebbles. Then, $17-q-6 \geq 8$, since $q \leq 3$ and hence we are done by Theorem 2.2. So, we assume that $p(v_3) \leq 1$. Similarly, we assume $p(v_5) \leq 1$. Thus, $p(v_4) \geq 6$. Let $p(v_3) = 1$.

If $p(v_2) = 0$ and $p(v_7) = 0$ then we move three pebbles to v_3 from v_4 and hence one pebble is moved to v_1. Thus, $p(v_4) - 6 \geq 5$. If $p(v_6) = 1$ then we can move one pebble to v_6 from v_4 and hence another one pebble can be moved to v_1. Let $p(v_5) = 1$ and $p(v_6) = 0$. We can move two pebbles to v_7

since $p(v_4) \geq 12$ and hence one another pebble is moved to v_1. Now, we let $p(v_5) = p(v_6) = 0$. Clearly, $p(v_4) = 15$ and hence we can move two pebbles to v_1 through v_3.

If $p(v_2) = 0$ and $p(v_7) = 1$ then we can move one pebble to v_1 using at most four pebbles through v_3 and v_7, since $p(v_4) \geq 8$. Thus $17 - q - 4 \geq 8$ $(q \leq 5)$ and hence we are done by Theorem 2.2.

If $p(v_2) = 1$ and $p(v_7) = 1$ then we move three pebbles to v_3 from v_4 since $p(v_4) \geq 6$ and hence we can move one pebble each from v_2 and v_7 to v_1.

So we assume that $p(v_3) = 0$. Similarly, $p(v_5) = 0$. Since $p(v_4) \geq 10$, if $p(v_2) = p(v_7) = 1$ or $p(v_2) = p(v_6) = 1$ or $p(v_6) = p(v_7) = 1$ then clearly we can move two pebbles to v_1. Next we let $p(v_2) = 1$. Clearly, $p(v_6) = p(v_7) = 0$. Thus, $p(v_4) = 14$ and hence we can move two pebbles to v_1 by moving three pebbles to v_2 from v_4. Assume $p(v_2) = 0$. Similarly, we assume $p(v_6) = p(v_7) = 0$. Then $p(v_4) = 16$ and hence we can move two pebbles to v_1 easily.

Case 3: Let v_2 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_2) = 0$, $p(v_1) \leq 1$, and $p(v_3) \leq 1$. Let $p(v_4) \geq 4$. If $p(v_3) = 1$ then we can move one pebble to v_2 using at most three pebbles. Thus the graph $J_{2,3}$ has at least $17 - q - 3 \geq 8$ (since $q \leq 6$) pebbles and hence we are done. Let $p(v_3) = 0$. Thus we move one pebble to v_2 using four pebbles from v_4 then the remaining number of pebbles on $J_{2,3}$ is $17 - q - 4 \geq 8$ and hence we are done. So we assume that $p(v_4) \leq 3$. Similarly, we assume that $p(v_6) \leq 3$ and $p(v_7) \leq 3$.Let $p(v_4) = 2$ or 3. If $p(v_3) = 1$ or $p(v_7) \geq 2$ then clearly we can move two pebbles to v_2. Assume $p(v_3) = 0$ and $p(v_7) \leq 1$. Let $p(v_6) = 2$ or 3. If $p(v_1) = 1$ then we are done. If not, then $17 - q \geq 13$ implies that $p(v_5) \geq 6$. If $p(v_7) = 1$ then we can move one pebble to v_2 at a cost of at most five pebbles and hence we are done, since $17 - q - 5 \geq 8$. If not, then $17 - q \geq 14$ implies that $p(v_5) \geq 8$. We can move one pebble to v_2 at a cost of at most six pebbles and then the remaining number of pebbles on $J_{2,3}$ is at least $17 - q - 6 \geq 8$ and hence we are done. Assume $p(v_6) \leq 1$. If $p(v_1) = p(v_6) = 1$ or $p(v_1) = p(v_7) = 1$ then we can move one pebble to v_2 at a cost of four pebbles, since $p(v_5) \geq 6$. If not, then $17 - q \geq 13$. We can move one pebble to v_3 using three pebbles from v_5, if $p(v_7) = 1$. Then we move another one pebble to v_3 from v_4 and hence one pebble is moved to v_2 at a cost of at most five pebbles. Then we have at least $17 - q - 5 \geq 8$ pebbles and hence we are done. If $p(v_7) = 0$ then $17 - q \geq 14$. Clearly, we can move two pebbles to v_3 using at most six pebbles from v_4 and v_5 and then $J_{2,3}$ has at least eight pebbles remained on it and hence we are done. Assume $p(v_4) \leq 1$. Similarly, $p(v_6) \leq 1$. Clearly, $p(v_5) \geq 6$. Let $p(v_1) = 1$. We move one pebble to v_2 using four pebbles from v_5 and one pebble from v_1.

If $p(v_3) = p(v_4) = 1$ or $p(v_3) = p(v_7) = 1$ then we can move another one pebble to v_2, since $p(v_5) - 4 \geq 2$ and hence we are done.

If not, then $p(v_5) - 4 \geq 4$. If $p(v_7) = p(v_4) = 1$ or $p(v_7) = p(v_6) = 1$ then we can move one pebble to v_2 and hence we are done. Otherwise, $p(v_5) - 4 \geq 6$. If $p(v_7) = 1$ or $p(v_6) = 1$ or $p(v_4) = 1$ then also we can move one pebble to v_2. Assume $p(v_7) = p(v_6) = p(v_4) = 0$. Thus $p(v_5) - 4 \geq 8$ and hence we can move one pebble to v_2.

So, we assume that $p(v_1) = 0$. Similarly, $p(v_3) = 0$. Clearly $p(v_5) \geq 10$. Let $v_6 = 1$. We move three pebbles to v_6 from v_5 and hence one pebble is moved to v_2 from v_6.
If $p(v_7) = 1$ and $p(v_4) = 1$ then we can move another one pebble to v_2, since $p(v_5) - 6 \geq 4$.

If $p(v_7) = 1$ and $p(v_4) = 0$ then we can move another one pebble to v_2, since $p(v_5) - 6 \geq 6$.

If $p(v_7) = 0$ and $p(v_4) = 0$ then we can move another one pebble to v_2, since $p(v_5) - 6 \geq 8$.

So we assume that $p(v_6) = 0$. Similarly, $p(v_4) = 0$. Let $p(v_7) = 1$. Thus $p(v_5) = 14$ and so we can move seven pebbles to v_7 and hence we are done. Otherwise, $p(v_5) = 16$ and hence we can move two pebbles to v_2.

Theorem 3.2 *The graph $J_{2,4}$ satisfies the 2-pebbling property.*

Proof: The graph $J_{2,4}$ has at least $2f(J_{2,4}) - q + 1 \geq 33 - q \geq 24$ pebbles on it.

Case 1: Let v_9 be the target vertex.

Clearly, $p(v_9) = 0$, and $p(v_i) \leq 1$ for all $v_i v_9 \in E(J_{2,4})$ (by Remark 2.7). Thus one of the non-adjacent vertices of v_9 has at least $\lceil \frac{33-q-4}{4} \rceil \geq \lceil \frac{21}{4} \rceil \geq 6$. Without loss of generality, we let $p(v_2) \geq 6$. Since $p(v_2) \geq 6$, we move one pebble to v_9 from v_2 at a cost of four pebbles and then the remaining number of pebbles on $J_{2,4}$ are $33 - q - 4 \geq 21$, since $q \leq 8$ and hence we are done by Theorem 2.3.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $v_iv_1 \in E(J_{2,4})$. Let $p(v_2) = 1$. If $p(v_3) + p(v_4) \geq 4$ then we can move one pebble to v_2 and hence we move one pebble to v_1 at a cost of at most five pebbles. Then the graph $J_{2,4}$ has at least $33 - q - 5 \geq 20$ and hence we can move one more pebble to v_1, by Theorem 2.3. We assume $p(v_3)+p(v_4) \leq 3$ such that we cannot move a pebble to v_2. Also, we may assume that, $p(v_5)+p(v_9) \leq 3$ such that one pebble cannot be moved to v_1. Thus $p(v_6) + p(v_7) \geq 33 - q - 8 \geq 17$ and hence we can move two pebbles to v_1.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. Let $p(v_3) = 1$. Clearly, $p(v_4) \leq 1$ and $p(v_9) \leq 1$. If $p(v_5) \geq 4$ or $p(v_7) \geq 4$ or $p(v_5) \geq 2$ and $p(v_7) \geq 2$ then we can move one pebble to v_3 and then one pebble is moved to v_2 at a cost of five pebbles. Then the remaining number of pebbles on $J_{2,4}$ are $33 - q - 5 \geq 20$ and hence we can move another one pebble to v_2, by Theorem 2.3. Assume $p(v_5) + p(v_7) \leq 4$ such that we cannot move one pebble to v_3. Thus $p(v_6) \geq 33 - q - 9 \geq 15$. We move one pebble to v_3 using eight pebbles from v_6 and hence we move one pebble to v_2. Then We have $33 - q - 9 \geq 16$ pebbles remain on $J_{2,4}$ and hence we can move another one pebble to v_2 by Theorem 2.3. Assume $p(v_3) = 0$. Similarly, we assume $p(v_1) = 0$. Let $p(v_8) = 2$ or 3 (Since, $p(v_8) \leq 3$). If $p(v_9) \geq 2$ then we move one pebble to v_2 through v_1 at a cost of four pebbles and hence we have $33 - q - 4 \geq 23$ and we are done. Assume $p(v_9) \leq 1$. We have $p(v_5) + p(v_6) + p(v_7) \geq 20$, since $q \leq 6$ and $p(v_4) \leq 3$. So we can move one pebble to v_1 using at most eight pebbles from the vertices v_5, v_6 and v_7. Then we move one pebble to v_1 from v_8 and hence we move one pebble to v_2 at a cost of at most ten pebbles. Thus the remaining number of pebbles on $J_{2,4}$ is $33 - q - 10 \geq 17$ and hence we can move another one pebble to v_2 by Theorem 2.3. Assume $p(v_8) \leq 1$. In a similar way, we may assume that $p(v_4) \leq 1$ and $p(v_9) \leq 1$. Thus, $p(v_5) + p(v_6) + p(v_7) \geq 24$. Let $p(v_7) \geq 2$. If $p(v_9) = 1$ or $p(v_8) = 1$ then we move one pebble to v_1 and then we can move one more pebble to v_1 using at most eight pebbles from the vertices v_5, v_6 and v_7 and hence one pebble is moved to v_2 at a cost of at most eleven pebbles. Thus, the graph $J_{2,4}$ has at least $33 - q - 11 \geq 16$ pebbles and hence we are done by Theorem 2.3. Assume $p(v_8) = p(v_9) = 0$. If $p(v_5) + p(v_6) \geq 4$ with $p(v_5) \geq 1$ then we can move one pebble to v_3 through v_9. Then we move another one pebble to v_3 using at most eight pebbles from the vertices v_5, v_6 and v_7. Thus we move one pebble to v_2 at a cost of at most thirteen pebbles and hence we have $33 - q - 13 \geq 16$ pebbles remain on $J_{2,4}$ and we are done by Theorem 2.3. Assume $p(v_5) = 0$. Since $p(v_6) + p(v_7) \geq 29$, we can move one pebble to v_2 through v_9 and v_1 at a cost of at most fourteen pebbles. Then the number of pebbles remaining on $J_{2,4}$ is $33 - q - 14 \geq 16$ and hence we are done by Theorem 2.3. Assume $p(v_7) \leq 1$. Similarly, we assume $p(v_5) \leq 1$. That is, $p(v_6) \geq 22$. Let $p(v_4) = 1$ then we move one pebble to v_4 from v_6 and hence one pebble is moved to v_3 at a cost of five pebbles. If $p(v_9) = 1$ then we move one pebble to v_9 from v_6 and so we move one pebble to v_3. So we move one pebble to v_2 at a cost of at most ten pebbles and then the graph $J_{2,4}$ has at least $33 - q - 10 \geq 17$ and we are done by Theorem 2.3. Assume $p(v_9) = 0$. If $p(v_5) = 1$ or $p(v_7) = 1$ then we move three pebbles to v_5 or v_7 and then one more pebble is moved to v_3 and so v_2 at a cost of at most twelve pebbles. Thus $33 - q - 12 \geq 16$ and hence we are done. Let $p(v_5) = p(v_7) = 0$. Then we can move two pebbles to v_3 using the pebbles at v_4 and v_6 and hence one pebble is moved to v_2 at a cost of at most thirteen pebbles. Thus the graph $J_{2,4}$ has at least $33 - q - 13 \geq 17$ and hence we are done. So, we assume $p(v_4) = 0$. Similarly, we may assume that $p(v_8) = 0$ and $p(v_9) = 0$. We have $p(v_5) + p(v_6) + p(v_7) \geq 30$. Clearly, we can move eight pebbles to v_9 from these pebbled vertices and hence two pebbles can be moved to v_2.

Theorem 3.3 *The graph $J_{2,5}$ satisfies the 2-pebbling property.*

Proof: The graph $J_{2,5}$ has at least $2f(J_{2,5}) - q + 1 \geq 37 - q \geq 26$ pebbles on it.

Case 1: Let v_{11} be the target vertex.

Clearly, $p(v_{11}) = 0$, and $p(v_i) \leq 1$ for all $v_i v_{11} \in E(J_{2,5})$ (by Remark 2.7). Thus one of the non-adjacent vertices of v_{11} has at least $\lceil \frac{37-q-5}{5} \rceil \geq \lceil \frac{22}{5} \rceil \geq 5$. Without loss of generality, we let $p(v_2) \geq 5$. Since $p(v_2) \geq 5$, we move one pebble to v_{11} from v_2 at a cost of four pebbles and then the remaining number of pebbles on $J_{2,5}$ is $37 - q - 4 \geq 23$, since $q \leq 8$ and hence we are done by Theorem 2.4.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $v_i v_1 \in E(J_{2,5})$. If $p(v_3) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1. Then the graph $J_{2,5}$ has at least $37 - q - 4 \geq 23$ pebbles and hence we are done by Theorem 2.4. So, we assume that $p(v_i) \leq 3$, for all $v_i v_{11} \in E(J_{2,5})$ and at most one adjacent vertex only, of v_{11} can contain more than two pebbles (Otherwise, we can move one pebble to v_1 through v_{11} and hence we can do easily). Thus, $p(v_4) + p(v_6) + p(v_8) \geq 18$. Clearly, we can move one pebble to v_1 at a cost of at most eight pebbles from the vertices v_4, v_6 and v_8 and then the number of pebbles remained on $J_{2,5}$ is at least $37 - q - 8 \geq 21$ and hence we are done by Theorem 2.4.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. We may assume that $p(v_1) = p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{10}) \leq 1$ and $p(v_{11}) \leq 1$. Let $p(v_5) \geq 4$. If $p(v_7) \geq 4$ or $p(v_9) \geq 4$ or $p(v_7) \geq 2$ and $p(v_9) \geq 2$ then we can move one pebble to v_1 through v_{11} and v_3 at a cost of at most eight pebbles. Thus we have $37 - q - 8 \geq 21$ pebbles remained on $J_{2,5}$ and hence we are done by Theorem 2.4. Assume $p(v_7) \leq 3$ and $p(v_9) \leq 3$ and $p(v_7) + p(v_9) \leq 4$ such that two pebbles cannot be moved to v_{11}. Let $p(v_7) \geq 2$. If $p(v_{11}) = 1$ or $p(v_5) \geq 6$ then we can move one pebble to v_2 and hence we can do easily. Assume $p(v_{11}) = 0$ and $p(v_5) = 4$ or 5. This implies that $p(v_6) + p(v_8) \geq 19$ and hence we can move one pebble to v_2 from the vertices v_5, v_7, and v_6 or v_8. Then the remaining number of pebbles on $J_{2,5}$ is at least $37 - q - 10 \geq 20$ and hence we are done by Theorem 2.4. Assume $p(v_7) \leq 1$ and $p(v_9) \leq 1$. Since $p(v_6) + p(v_8) \geq 18$, we can move one pebble to v_3 using eight pebbles from the vertices v_6 and v_8. If $p(v_3) = 1$ or $p(v_{11}) = 1$ then we move another one pebble to v_3 at a cost of three pebbles. Thus we move a pebble to v_2 at a total cost eleven pebbles and then the remaining number of pebbles on $J_{2,5}$ is at least $37 - q - 11 \geq 18$ and hence we are done by Theorem 2.4. Assume $p(v_3) = p(v_{11}) = 0$. Again we can move one pebble to v_2 at a cost of at most twelve pebbles from the vertices v_5, v_6 and v_8. Then the graph $J_{2,5}$ has at least $37 - q - 12 \geq 19$ and hence we are done by Theorem 2.4. So, we assume $p(v_5) \leq 3$. In a similar way, we may assume that $p(v_9) \leq 3$ and $p(v_7) \leq 3$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: If $p(v_{11}) = 1$ then we can move one pebble to v_2 using at most seven pebbles and hence we are done since $37 - q - 7 \geq 22$ and by Theorem 2.4. Assume $p(v_{11}) = 0$. This implies that $p(v_6) + p(v_8) \geq 19$ and hence we can move one pebble to v_{11} from v_6 or v_8 at a cost of four pebbles. Thus we can move one pebble to v_2 at a total cost of ten pebbles then the remaining number of pebbles on $J_{2,5}$ is at least $37 - q - 10 \geq 20$ and hence we are done by Theorem 2.4.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{11}) = 1$ and then $J_{2,5}$ has at least $37 - q - 11 \geq 18$ and hence we are done. If $p(v_{11}) = 0$, then we can move one pebble to v_2 using the pebbles at v_6, v_8 and the two vertices of $S_1 - \{v_1, v_3\}$. Then we have $37 - q - 12 \geq 18$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{11}) = 1$ and then $J_{2,5}$ has at least $37 - q - 11 \geq 18$ and hence we are done. Let $p(v_{11}) = 0$ and also let v_5 be the vertex with $p(v_5) \geq 2$. If $p(v_4) = 1$ then we move one pebble to v_3 and then we can move one more pebble to v_3 using the pebbles at v_6, v_8, since $p(v_6) + p(v_8) \geq 23$. Thus we move one pebble to v_2 from v_3, and then we have $37 - q - 11 \geq 18$ and hence we are done. Assume $p(v_4) = 0$ and thus $p(v_6) + p(v_8) \geq 25$. If $p(v_7) = 1$ then we can move three pebbles to v_{11} at a cost of at most thirteen pebbles from the vertices v_6, v_7 and v_8. Assume $p(v_7) = 0$ then $p(v_6) + p(v_8) \geq 25$ and hence we can move four pebbles from the vertices v_5, v_6 and v_8 at a cost of fourteen pebbles. Then $J_{2,5}$ has at least $37 - q - 14 \geq 18$ and hence we are done. In a similar way, we can move two pebbles to v_2 if $p(v_9) \geq 2$ and $p(v_7) \geq 2$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles each: Clearly, $p(v_6) + p(v_8) \geq 23$. Let $p(v_6) + p(v_8) = 23$. Without loss of generality, we let $p(v_6) \geq 12$. If $p(v_8) \geq 2$ then we move one pebble to v_3 through v_7 and v_{11}. Using two pebbles from the vertex v_6, we move one more pebble to v_3 and hence one pebble is moved to v_2. Then $p(v_6) + p(v_8) = 19$ and so we can move eight pebbles to v_7 and hence we can move one more pebble to v_2. Assume $p(v_8) \leq 1$ This implies that we have $p(v_6) \geq 22$, so we move one pebble to v_7 and then we move five pebbles to v_4. Thus v_3 receives four pebbles and hence we are done.

Assume $p(v_6) + p(v_8) \geq 24$. Without loss of generality, we let $p(v_8) \geq 12$. Let $p(v_{10}) = 1$. If $p(v_9) = 1$ then we move one pebble to v_{10} from v_8 then we move one pebble to v_1. And then we move one more pebble to v_1 from the vertices v_6 and v_8 through v_7 and v_{11}. Thus we can move a pebble to v_2 at a total cost of twelve pebbles and so the graph $J_{2,5}$ has at least $37 - q - 12 \geq 18$ and hence we are done. Assume $p(v_9) = 0$. Clearly, we can move one pebble to v_2 using at most thirteen pebbles and hence the remaining number of pebbles on $J_{2,5}$ is at least $37 - q - 13 \geq 18$ and hence we are done. Assume $p(v_{10}) = 0$. In a similar way, we may assume that $p(v_4) = 0$.

If $p(v_{11}) = 1$ then we can move one pebble to v_2 at a cost of thirteen pebbles from the vertices v_6, v_8 and v_{11}. Then $J_{2,5}$ has at least $37 - q - 13 \geq 18$ and hence we are done. Assume $p(v_{11}) = 0$ and thus $p(v_6) + p(v_8) \geq 29$. Let $p(v_8) \geq 15$. If $p(v_7) = p(v_9) = 1$, then we move two pebbles to v_{11} using four pebbles from v_8. Clearly, we can move six pebbles from the vertices v_6 and v_8 through v_7 and hence we can move two pebbles to v_2. Assume $p(v_7) = 0$ or $p(v_9) = 0$ and thus we move one pebble to v_{11} from v_8 and then we can move seven pebbles to v_{11} through v_7 since $p(v_6) + p(v_8) - 2 \geq 31$. Assume $p(v_7) = p(v_9) = 0$ and thus $p(v_6) + p(v_8) \geq 33$. So, we can move 16 pebbles to v_7 from the vertices v_6 and v_8 and hence we are done.

Theorem 3.4 *The graph $J_{2,6}$ satisfies the 2-pebbling property.*

Proof: The graph $J_{2,6}$ has at least $2f(J_{2,6})-q+1 \geq 43-q \geq 30$ pebbles on it.

Case 1: Let v_{13} be the target vertex.

Clearly, $p(v_{13}) = 0$, and $p(v_i) \leq 1$ for all $v_iv_{13} \in E(J_{2,6})$ (by Remark 2.7). Thus one of the non-adjacent vertices of v_{13} has at least $\lceil \frac{43-q-6}{6} \rceil \geq \lceil \frac{25}{6} \rceil \geq 5$. Without loss of generality, we let $p(v_2) \geq 5$. Since $p(v_2) \geq 5$, we move one pebble to v_{13} from v_2 at a cost of four pebbles and then the remaining number of pebbles on $J_{2,6}$ is $43 - q - 4 \geq 27$, since $q \leq 12$ and hence we are done by Theorem 2.5.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $v_iv_1 \in E(J_{2,6})$. If $p(v_3) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1. Then the graph $J_{2,6}$ has at least $43 - q - 4 \geq 27$ pebbles and hence we are done by Theorem 2.5. So, we assume that $p(v_i) \leq 3$, for all $v_iv_{13} \in E(J_{2,6})$ and at most one adjacent vertex only, of v_{13} can contain more than two pebbles (Otherwise, we can move one pebble to v_1 through v_{13} and hence we can do easily). Thus, $p(S_2 - \{v_2, v_{12}\}) \geq 21$. Clearly, we can move one pebble to v_1 at a cost of at most eight pebbles from the vertices $S_2 - \{v_2, v_{12}\}$ and then the number of pebbles remained on $J_{2,6}$ is at least $43 - q - 8 \geq 23$ and hence we are done by Theorem 2.5.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. Also, we may assume that $p(v_1) = p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{10}) \leq 1$ and $p(v_{11}) \leq 1$. Let $p(v_5) \geq 4$. If a vertex of $S_1 - \{v_1, v_3, v_5\}$ has more than three pebbles or two vertices of $S_1 - \{v_1, v_3, v_5\}$ contains more than one pebble each then we can move one pebble to v_2 at a cost of eight pebbles. Thus the remaining number of pebbles on $J_{2,6}$ is at least $43 - q - 8 \geq 25$ and hence we are done by Theorem 2.5. So assume that $p(v_i) \leq 3$ where $v_i \in S_1 - \{v_1, v_3, v_5\}$ and at most one vertex only of $S_1 - \{v_1, v_3, v_5\}$ can contain two or three pebbles. Let $p(v_7) \geq 2$. If $p(v_{13}) = 1$ or $p(v_5) = 6$ or 7, then we can move one pebble to v_2 at a cost of at most eight pebbles and hence we are done since $43 - q - 8 \geq 25$. Assume $p(v_{13}) = 0$ and $p(v_5) = 4$ or 5. Clearly, $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 24$ and hence we can move one pebble to v_{13} from the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ and then we move another three pebbles to v_{13} from the vertices v_5 and v_7. Thus, we can move one pebble to v_2 from v_{13} and the remaining pebbles on $J_{2,6}$ is at least $43 - q - 10 \geq 24$ and hence we are done by Theorem 2.5. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5\}$. Clearly, $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 20$ and hence we can move one pebble to v_1 at a cost of at most eight pebbles and then we move one more pebble to v_1 from v_5. Thus we can move one pebble to v_2 from v_1 and then $J_{2,6}$ has at least $43 - q - 12 \geq 21$ and hence we are done. Assume $p(v_i) \leq 3$ for all $v_i \in p(S_1 - \{v_1, v_3\})$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move one pebble to v_2 through v_{13} and hence we are done since $43 - q - 8 \geq 25$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: If $p(v_{13}) = 1$ then we can move one pebble to v_2 using at most seven pebbles and hence we are done since $43 - q - 7 \geq 26$ and by Theorem 2.5. Assume $p(v_{13}) = 0$. This implies that $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 21$ and hence we can move one pebble to v_{13} from the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ at a cost of four pebbles. Thus we can move one pebble to v_2 at a total cost of ten pebbles, then the remaining number of pebbles on $J_{2,6}$ is at least $43 - q - 10 \geq 24$ and hence we are done by Theorem 2.5.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{13}) = 1$ and then $J_{2,6}$ has at least $43 - q - 11 \geq 22$ and hence we are done. If $p(v_{13}) = 0$, then we can move one pebble to v_2 using the pebbles at the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ and the two vertices of $S_1 - \{v_1, v_3\}$. Then we have $43 - q - 12 \geq 22$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{13}) = 1$ and then $J_{2,6}$ has at least $43 - q - 11 \geq 22$ and hence we are done. Let $p(v_{13}) = 0$ and also let v_5 be the vertex with $p(v_5) \geq 2$. If $p(v_4) = 1$ then we move one pebble to v_3 and then we can move one more pebble to v_3 using the pebbles at the vertices of $S_2 - \{v_2, v_4, v_{12}\}$, since $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 26$. Thus we move one pebble to v_2 from v_3, and then we have $43 - q - 11 \geq 22$ and hence we are done. Assume $p(v_4) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 28$. If $p(v_7) = 0$ or $p(v_9) = 0$ or $p(v_7) = p(v_9) = 0$ then we can move four pebbles from v_5 and the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ at a cost of at most fourteen pebbles. Then $J_{2,6}$ has at least $43 - q - 14 \geq 21$ and hence we are done. In a similar way, we can move two pebbles to v_2 if $p(v_i) \geq 2$ where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles each: Clearly, $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 27$. Let $p(S_2 - \{v_2, v_4, v_{12}\}) = 27$. Without loss of generality, we let $p(v_6) \geq 9$. If $p(v_8) \geq 2$ or $p(v_{10}) \geq 2$ then we move one pebble to v_3 through v_9 and v_{13}. Using two pebbles from the vertex v_6, we move one more pebble to v_3 and hence one pebble is moved to v_2. Then $p(S_2 - \{v_2, v_4, v_{12}\}) - 4 = 23$ and so we can move four pebbles to v_{13} and hence we can move one more pebble to v_2. Assume $p(v_8) \leq 1$ and $p(v_{10}) \leq 1$. This implies that we have $p(v_6) \geq 25$, so, from v_6, we move one pebble to v_7 and then we move five pebbles to v_4. Thus v_3 receives four pebbles and hence we are done.

Assume $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 28$. Without loss of generality, we let $p(v_{10}) \geq 10$. Let $p(v_{12}) = 1$. If $p(v_{11}) = 1$ then we move one pebble to v_{12} from v_{10} then we move one pebble to v_1. And then we move one more pebble to v_1 from the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ through v_7, v_9 and v_{11}. Thus we can move a pebble to v_2 at a total cost of twelve pebbles and so the graph $J_{2,6}$ has at least $43 - q - 12 \geq 21$ and hence we are done. Assume $p(v_{11}) = 0$. Clearly, we can move one pebble to v_2 using at most thirteen pebbles and hence the remaining number of pebbles on $J_{2,6}$ is at least $43 - q - 13 \geq 21$ and hence we are done. Assume $p(v_{12}) = 0$. In a similar way, we may assume that $p(v_4) = 0$.

If $p(v_{13}) = 1$ then we can move one pebble to v_2 at a cost of thirteen pebbles from the vertices of $S_2 - \{v_2, v_4, v_{12}\}$. Then $J_{2,6}$ has at least $43 - q - 13 \geq 21$ and hence we are done. Assume $p(v_{13}) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 32$. Let $p(v_{10}) \geq 11$. If $p(v_9) = p(v_{11}) = 1$, then we move two pebbles to v_{13} using four pebbles from v_{10}. Clearly, we can move six pebbles to v_{13} from the vertices of $S_2 - \{v_2, v_4, v_{12}\}$ through v_7, v_9 and hence we can move two pebbles to v_2. Assume $p(v_9) = 0$ or $p(v_{11}) = 0$ and thus we move one pebble to v_{13} from v_{10} and then we can move seven pebbles to v_{13} through v_7 and v_9 since $p(S_2 - \{v_2, v_4, v_{12}\}) - 2 \geq 34$. Assume $p(v_7) = p(v_9) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{12}\}) \geq 35$. So, we can move 8 pebbles to v_{13} from the vertices $S_2 - \{v_2, v_4, v_{12}\}$ and hence we are done.

Theorem 3.5 *The graph $J_{2,7}$ satisfies the 2-pebbling property.*

Proof: The graph $J_{2,7}$ has at least $2f(J_{2,7}) - q + 1 \geq 47 - q \geq 32$ pebbles on it.

Case 1: Let v_{15} be the target vertex.

Clearly, $p(v_{15}) = 0$, and $p(v_i) \leq 1$ for all $v_i v_{15} \in E(J_{2,7})$ (by Remark 2.7). Thus one of the non-adjacent vertices of v_{15} has at least $\lceil \frac{47-q-7}{7} \rceil \geq \lceil \frac{26}{7} \rceil \geq 4$. Without loss of generality, we let $p(v_2) \geq 4$. Since $p(v_2) \geq 4$, we move one pebble to v_{15} from v_2 at a cost of four pebbles and then the remaining number of pebbles on $J_{2,7}$ is $47 - q - 4 \geq 29$, since $q \leq 14$ and hence we are done by Theorem 2.5.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $v_i v_1 \in$

$E(J_{2,7})$. If $p(v_3) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1. Then the graph $J_{2,7}$ has at least $47 - q - 4 \geq 29$ pebbles and hence we are done by Theorem 2.5. So, we assume that $p(v_i) \leq 3$, for all $v_iv_{15} \in E(J_{2,7})$ and at most one adjacent vertex only, of v_{15} can contain more than two pebbles (Otherwise, we can move one pebble to v_1 through v_{15} and hence we can do easily). Thus, $p(S_2 - \{v_2, v_{14}\}) \geq 23$. Clearly, we can move one pebble to v_1 at a cost of at most eight pebbles from the vertices of $S_2 - \{v_2, v_{14}\}$ and then the number of pebbles remained on $J_{2,7}$ is at least $47 - q - 8 \geq 25$ and hence we are done by Theorem 2.5.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. We may assume that $p(v_1) = p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{14}) \leq 1$ and $p(v_{15}) \leq 1$. Let $p(v_5) \geq 4$. If a vertex of $S_1 - \{v_1, v_3, v_5\}$ has more than three pebbles or two vertices of $S_1 - \{v_1, v_3, v_5\}$ contains more than one pebble each then we can move one pebble to v_2 at a cost of eight pebbles. Thus the remaining number of pebbles on $J_{2,7}$ is at least $47 - q - 8 \geq 27$ and hence we are done by Theorem 2.5. So assume that $p(v_i) \leq 3$ where $v_i \in S_1 - \{v_1, v_3, v_5\}$ and at most one vertex only of $S_1 - \{v_1, v_3, v_5\}$ can contain two or three pebbles. Let $p(v_7) \geq 2$. If $p(v_{15}) = 1$ or $p(v_5) = 6$ or 7, then we can move one pebble to v_2 at a cost of at most eight pebbles and hence we are done since $47 - q - 8 \geq 27$. Assume $p(v_{15}) = 0$ and $p(v_5) = 4$ or 5. Clearly, $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 24$ and hence we can move one pebble to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and then we move another three pebbles to v_{15} from the vertices v_5 and v_7. Thus, we can move one pebble to v_2 from v_{15} and the remaining pebbles on $J_{2,7}$ is at least $47 - q - 10 \geq 26$ and hence we are done by Theorem 2.5. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5\}$. Clearly, $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 22$ and hence we can move one pebble to v_1 at a cost of at most eight pebbles and then we move one more pebble to v_1 from v_5. Thus we can move one pebble to v_2 from v_1 and then $J_{2,7}$ has at least $47 - q - 12 \geq 23$ and hence we are done. Assume $p(v_i) \leq 3$ for all $v_i \in p(S_1 - \{v_1, v_3\})$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move one pebble to v_2 through v_{15} and hence we are done since $47 - q - 8 \geq 27$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: If $p(v_{15}) = 1$ then we can move one pebble to v_2 using at most seven pebbles and hence we are done since $47 - q - 7 \geq 28$ and by Theorem 2.5. Assume $p(v_{15}) = 0$. This implies that $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 23$ and hence we can move one pebble to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ at a cost of four pebbles. Thus we can move one pebble to v_2 at a total cost of ten pebbles, then the remaining number of pebbles on $J_{2,7}$ is at least $47 - q - 10 \geq 26$ and hence we are done by Theorem 2.5.

Two vertices of $S_1-\{v_1,v_3\}$ have two or more pebbles each: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{15})=1$ and then $J_{2,7}$ has at least $47-q-11\geq 24$ and hence we are done. If $p(v_{15})=0$, then we can move one pebble to v_2 using the pebbles at the vertices of $S_2-\{v_2,v_4,v_{14}\}$ and the two vertices of $S_1-\{v_1,v_3\}$. Then we have $47-q-12\geq 23$ and hence we are done.

One vertex of $S_1-\{v_1,v_3\}$ has two or more pebbles: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{15})=1$ and then $J_{2,7}$ has at least $47-q-11\geq 24$ and hence we are done. Let $p(v_{15})=0$ and also let v_5 be the vertex with $p(v_5)\geq 2$. If $p(v_4)=1$ then we move one pebble to v_3 and then we can move one more pebble to v_3 using the pebbles at the vertices of $S_2-\{v_2,v_4,v_{14}\}$, since $p(S_2-\{v_2,v_4,v_{14}\})\geq 27$. Thus we move one pebble to v_2 from v_3, and then we have $47-q-11\geq 25$ and hence we are done. Assume $p(v_4)=0$ and thus $p(S_2-\{v_2,v_4,v_{14}\})\geq 28$. If $p(v_7)=0$ or $p(v_9)=0$ or $p(v_7)=p(v_9)=0$ then we can move four pebbles from v_5 and the vertices of $S_2-\{v_2,v_4,v_{14}\}$ at a cost of at most fourteen pebbles. Then $J_{2,7}$ has at least $47-q-14\geq 23$ and hence we are done. In a similar way, we can move two pebbles to v_2 if $p(v_i)\geq 2$ where $v_i\in S_1-\{v_1,v_3,v_5\}$.

No vertex of $S_1-\{v_1,v_3\}$ has two or more pebbles each: Clearly, $p(S_2-\{v_2,v_4,v_{14}\})\geq 27$. Let $p(S_2-\{v_2,v_4,v_{14}\})=27$. Without loss of generality, we let $p(v_6)\geq 7$. If a vertex of $S_2-\{v_2,v_4,v_{14}\}$ contains more than one pebble then we can move one pebble to v_3 through v_{15}. Using two pebbles from the vertex v_6, we move one more pebble to v_3 and hence one pebble is moved to v_2. Then $p(S_2-\{v_2,v_4,v_{14}\})-4=23$ and so we can move four pebbles to v_{13} and hence we can move one more pebble to v_2. Assume $p(v_i)\leq 1$ for all $v_i\in S_2-\{v_2,v_4,v_6,v_{14}\}$ This implies that we have $p(v_6)\geq 20$, so, from v_6, we move one pebble to v_7 and then we move nine pebbles to v_5. Thus v_3 receives four pebbles and hence we are done.

Assume $p(S_2-\{v_2,v_4,v_{14}\})\geq 28$. Without loss of generality, we let $p(v_{12})\geq 7$. Let $p(v_{14})=1$. If $p(v_{13})=1$ then we move one pebble to v_{14} from v_{12} then we move one pebble to v_1. And then we move one more pebble to v_1 from the vertices of $S_2-\{v_2,v_4,v_{14}\}$ through v_{15}. Thus we can move a pebble to v_2 at a total cost of twelve pebbles and so the graph $J_{2,7}$ has at least $47-q-12\geq 23$ and hence we are done. Assume $p(v_{13})=0$. Clearly, we can move one pebble to v_2 using at most thirteen pebbles and hence the remaining number of pebbles on $J_{2,7}$ is at least $47-q-13\geq 23$ and hence we are done. Assume $p(v_{14})=0$. In a similar way, we may assume that $p(v_4)=0$.

If $p(v_{15}) = 1$ then we can move one pebble to v_2 at a cost of thirteen pebbles from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$. Then $J_{2,7}$ has at least $47 - q - 13 \geq 24$ and hence we are done. Assume $p(v_{15}) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 33$. Let $p(v_{12}) \geq 9$. If $p(v_{11}) = p(v_{13}) = 1$, then we move two pebbles to v_{15} using four pebbles from v_{12}. Clearly, we can move six pebbles to v_{15} from the vertices of $S_2 - \{v_2, v_4, v_{14}\}$ and hence we can move two pebbles to v_2. Assume $p(v_{11}) = 0$ or $p(v_{13}) = 0$ and thus we move one pebble to v_{15} from v_{12} and then we can move seven pebbles to v_{15} since $p(S_2 - \{v_2, v_4, v_{14}\}) - 2 \geq 35$. Assume $p(v_{11}) = p(v_{13}) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{14}\}) \geq 37$. So, we can move 8 pebbles to v_{15} from the vertices $S_2 - \{v_2, v_4, v_{14}\}$ and hence we are done.

Theorem 3.6 *The graph $J_{2,m}$ satisfies the 2-pebbling property, where $m \geq 8$.*

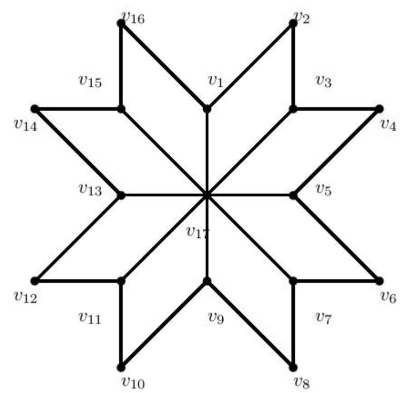

Figure 1: Jahangir graph $J_{2,8}$

Proof: The graph $J_{2,m}$ has at least $2f(J_{2,m}) - q + 1 \geq 4m + 21 - q \geq 2m + 20$ pebbles on it.

Case 1: Let v_{2m+1} be the target vertex.

Clearly, $p(v_{2m+1}) = 0$, and $p(v_i) \leq 1$ for all $v_i v_{2m+1} \in E(J_{2,m})$ (by Remark 2.7). Thus one of the non-adjacent vertices of v_{2m+1} has at least $\lceil \frac{4m+21-q-m}{m} \rceil \geq \lceil \frac{m+21}{m} \rceil \geq 2$. Without loss of generality, we let $p(v_2) \geq 2$. Since $p(v_2) \geq 2$, we can move one pebble to v_{2m+1} from v_2 at a cost of at most four pebbles and then the remaining number of pebbles on $J_{2,m}$ is $4m + 21 - q - 4 \geq 2m + 16$, since $q \leq 2m$ and hence we are done by Theorem 2.5.

Case 2: Let v_1 be the target vertex.

Clearly, by the Remark 2.7, we have $p(v_1) = 0$ and $p(v_i) \leq 1$ for all $v_i v_1 \in$

$E(J_{2,m})$. If $p(v_3) \geq 4$ or $p(v_3) \geq 2$ and $p(v_5) \geq 2$ then we can move one pebble to v_1. Then the graph $J_{2,m}$ has at least $4m + 21 - q - 4 \geq 2m + 16$ pebbles and hence we are done by Theorem 2.5. So, we assume that $p(v_i) \leq 3$, for all $v_i v_{2m+1} \in E(J_{2,m})$ and at most one adjacent vertex only, of v_{2m+1} can contain more than two pebbles (Otherwise, we can move one pebble to v_1 through v_{2m+1} and hence we can do easily). Thus, $p(S_2 - \{v_2, v_{2m}\}) \geq m + 16$. Clearly, we can move one pebble to v_1 at a cost of at most eight pebbles from the vertices of $S_2 - \{v_2, v_{2m}\}$ and then the number of pebbles remained on $J_{2,m}$ is at least $4m + 21 - q - 8 \geq 2m + 13$ and hence we are done by Theorem 2.5.

Case 3: Let v_2 be the target vertex.

Clearly, $p(v_2) = 0$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. We may assume that $p(v_1) = p(v_3) = 0$, $p(v_4) \leq 1$, $p(v_{2m}) \leq 1$ and $p(v_{2m+1}) \leq 1$. Let $p(v_5) \geq 4$. If a vertex of $S_1 - \{v_1, v_3, v_5\}$ has more than three pebbles or two vertices of $S_1 - \{v_1, v_3, v_5\}$ contains more than one pebble each then we can move one pebble to v_2 at a cost of eight pebbles. Thus the remaining number of pebbles on $J_{2,m}$ is at least $4m + 21 - q - 8 \geq 2m + 13$ and hence we are done by Theorem 2.5. So assume that $p(v_i) \leq 3$ where $v_i \in S_1 - \{v_1, v_3, v_5\}$ and at most one vertex only of $S_1 - \{v_1, v_3, v_5\}$ can contain two or three pebbles. Let $p(v_7) \geq 2$. If $p(v_{2m+1}) = 1$ or $p(v_5) = 6$ or 7, then we can move one pebble to v_2 at a cost of at most eight pebbles and hence we are done since $4m + 21 - q - 8 \geq 2m + 13$. Assume $p(v_{2m+1}) = 0$ and $p(v_5) = 4$ or 5. Clearly, $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 18$ and hence we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and then we move another three pebbles to v_{2m+1} from the vertices v_5 and v_7. Thus, we can move one pebble to v_2 from v_{2m+1} and the remaining pebbles on $J_{2,m}$ is at least $4m + 21 - q - 10 \geq 2m + 14$ and hence we are done by Theorem 2.5. Assume $p(v_i) \leq 1$ for all $v_i \in S_1 - \{v_1, v_3, v_5\}$. Clearly, $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 18$ and hence we can move one pebble to v_1 at a cost of at most eight pebbles and then we move one more pebble to v_1 from v_5. Thus we can move one pebble to v_2 from v_1 and then $J_{2,m}$ has at least $4m + 21 - q - 12 \geq 2m + 12$ and hence we are done. Assume $p(v_i) \leq 3$ for all $v_i \in p(S_1 - \{v_1, v_3\})$. If four vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each then clearly we can move one pebble to v_2 through v_{2m+1} and hence we are done since $4m + 21 - q - 8 \geq 2m + 16$.

Three vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: If $p(v_{2m+1}) = 1$ then we can move one pebble to v_2 using at most seven pebbles and hence we are done since $4m + 21 - q - 7 \geq 2m + 16$ and by Theorem 2.5. Assume $p(v_{2m+1}) = 0$. This implies that $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 18$ and hence we can move one pebble to v_{2m+1} from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ at a cost of at most four pebbles. Thus we can move one pebble to v_2 at a total cost of ten pebbles, then the remaining number of pebbles on $J_{2,m}$ is at least $4m + 21 - q - 10 \geq 2m + 14$ and hence we are done by Theorem 2.5.

Two vertices of $S_1 - \{v_1, v_3\}$ have two or more pebbles each: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{2m+1}) = 1$ and then $J_{2,m}$ has at least $4m+21-q-11 \geq 2m+13$ and hence we are done. If $p(v_{2m+1}) = 0$, then we can move one pebble to v_2 using the pebbles at the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ and the two vertices of $S_1 - \{v_1, v_3\}$. Then we have $4m + 21 - q - 12 \geq 2m + 12$ and hence we are done.

One vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles: Clearly, we can move one pebble to v_2 easily at a cost of eleven pebbles if $p(v_{2m+1}) = 1$ and then $J_{2,m}$ has at least $4m + 21 - q - 11 \geq 2m + 12$ and hence we are done. Let $p(v_{2m+1}) = 0$ and also let v_5 be the vertex with $p(v_5) \geq 2$. If $p(v_4) = 1$ then we move one pebble to v_3 and then we can move one more pebble to v_3 using the pebbles at the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$, since $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 18$. Thus we move one pebble to v_2 from v_3, and then we have $4m+21-q-11 \geq 2m+13$ and hence we are done. Assume $p(v_4) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 23$. If $p(v_7) = 0$ or $p(v_9) = 0$ or $p(v_7) = p(v_9) = 0$ then we can move four pebbles from v_5 and the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ at a cost of at most fourteen pebbles. Then $J_{2,m}$ has at least $4m + 21 - q - 14 \geq 2m + 11$ and hence we are done. In a similar way, we can move two pebbles to v_2 if $p(v_i) \geq 2$ where $v_i \in S_1 - \{v_1, v_3, v_5\}$.

No vertex of $S_1 - \{v_1, v_3\}$ has two or more pebbles each: Clearly, $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 23$. Let $p(S_2 - \{v_2, v_4, v_{2m}\}) = m + 23$. Without loss of generality, we let $p(v_6) \geq 1 + \lceil \frac{26}{m-3} \rceil$. If a vertex of $S_2 - \{v_2, v_4, v_{2m}\}$ contains more than one pebble then we can move one pebble to v_3 through v_{2m+1}. Using two pebbles from the vertex v_6, we move one more pebble to v_3 and hence one pebble is moved to v_2. Then the remaining number of pebbles on $J_{2,m}$ is at least $4m+21-q-8 \geq 2m+15$ and hence we can move one more pebble to v_2. Assume $p(v_i) \leq 1$ for all $v_i \in S_2 - \{v_2, v_4, v_6, v_{2m}\}$. This implies that we have $p(v_6) \geq 27$, so, from v_6, we move one pebble to v_7 and then we move nine pebbles to v_5. Thus v_3 receives four pebbles and hence we are done.

Assume $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 24$. Without loss of generality, we let $p(v_{2m-2}) \geq 1 + \lceil \frac{26}{m-3} \rceil$. Let $p(v_{2m}) = 1$. If $p(v_{2m-1}) = 1$ then we move one pebble to v_{2m} from v_{2m-2} then we move one pebble to v_1. And then we move one more pebble to v_1 from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$ through v_{2m+1}. Thus we can move a pebble to v_2 at a total cost of twelve pebbles and so the graph $J_{2,m}$ has at least $4m + 21 - q - 12 \geq 2m + 11$ and hence we are done. Assume $p(v_{2m-1}) = 0$. Clearly, we can move one pebble to v_2 using at most thirteen pebbles and hence the remaining number of pebbles on $J_{2,m}$ is at least $4m + 21 - q - 13 \geq 2m + 10$ and hence we are done. Assume $p(v_{2m}) = 0$. In a similar way, we may assume that $p(v_4) = 0$.

If $p(v_{2m+1}) = 1$ then we can move one pebble to v_2 at a cost of thirteen pebbles from the vertices of $S_2 - \{v_2, v_4, v_{2m}\}$. Then $J_{2,m}$ has at least $4m + 21 - q - 13 \geq 2m + 12$ and hence we are done. Assume $p(v_{2m+1}) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 24$. Let $p(v_{2m-2}) \geq 1 + \lceil \frac{27}{m-3} \rceil$. Assume $p(v_{11}) = 0$ or $p(v_{13}) = 0$ and thus we move one pebble to v_{2m+1} from v_{2m-2} and then we can move seven pebbles to v_{2m+1} since $p(S_2 - \{v_2, v_4, v_{2m}\}) - 2 \geq m + 22$. Assume $p(v_{11}) = p(v_{13}) = 0$ and thus $p(S_2 - \{v_2, v_4, v_{2m}\}) \geq m + 24$. So, we can move eight pebbles to v_{2m+1} from the vertices $S_2 - \{v_2, v_4, v_{2m}\}$ and hence we are done.

4 The $2t$-Pebbling Property of the Jahangir Graph $J_{2,m}$

In this section, we are going to prove that the Jahangir graph $J_{2,m}$ ($m \geq 3$) satisfies the $2t$-pebbling property. Clearly, the technique to prove this is Induction on t.

Theorem 4.1 *The graph $J_{2,3}$ satisfies the 2t-pebbling property.*

Proof: For $t = 1$, this theorem is true by Theorem 3.1. Assume the result is true for $t - 1 \geq 1$. Consider the graph $J_{2,3}$ with $2f_t(J_{2,3}) - q + 1$ pebbles on it. Clearly $2f_t(J_{2,3}) - q + 1 \geq 16t + 1 - q \geq 24$, since $q \leq 7$ and $t \geq 2$ and by Theorem 2.2. So, we can move two pebbles to the target vertex v_i of $J_{2,3}$ at a cost of at most sixteen pebbles by Theorem 2.2. Then the graph $J_{2,3}$ has at least $16t + 1 - q - 16 = 16(t - 1) + 1 - q$ and hence we can move the additional $2(t - 1)$ pebbles to v_i. Thus the graph $J_{2,3}$ satisfies the $2t$-pebbling property.

Theorem 4.2 *The graph $J_{2,4}$ satisfies the 2t-pebbling property.*

Proof: For $t = 1$, this theorem is true by Theorem 3.2. Assume the result is true for $t - 1 \geq 1$. Consider the graph $J_{2,4}$ with $2f_t(J_{2,4}) - q + 1$ pebbles on it. Clearly $2f_t(J_{2,4}) - q + 1 \geq 32t + 1 - q \geq 55$, since $q \leq 9$ and $t \geq 2$ and by Theorem 2.3. So, we can move two pebbles to the target vertex v_i of $J_{2,4}$ at a cost of at most 32 pebbles by Theorem 2.3. Then the graph $J_{2,4}$ has at least $32t + 1 - q - 32 = 32(t - 1) + 1 - q$ and hence we can move the additional $2(t - 1)$ pebbles to v_i. Thus the graph $J_{2,4}$ satisfies the $2t$-pebbling property.

Theorem 4.3 *The graph $J_{2,5}$ satisfies the 2t-pebbling property.*

Proof: For $t = 1$, this theorem is true by Theorem 3.3. Assume the result is true for $t-1 \geq 1$. Consider the graph $J_{2,5}$ with $2f_t(J_{2,5}) - q + 1$ pebbles on it. Clearly $2f_t(J_{2,5}) - q + 1 \geq 32t + 5 - q \geq 58$, since $q \leq 11$ and $t \geq 2$ and by Theorem 2.4. So, we can move two pebbles to the target vertex v_i of $J_{2,5}$ at a cost of at most 32 pebbles. Then the graph $J_{2,5}$ has at least $32t + 5 - q - 32 = 2(16(t-1)+2) + 1 - q$ pebbles and hence we can move the additional $2(t-1)$ pebbles to v_i. Thus the graph $J_{2,5}$ satisfies the $2t$-pebbling property.

Theorem 4.4 *The graph $J_{2,m}$ satisfies the $2t$-pebbling property, where $m \geq 6$.*

Proof: For $t = 1$, this theorem is true by Theorem 3.4, 3.5, and 3.6. Assume the result is true for $t-1 \geq 1$. Consider the graph $J_{2,m}$ with $2f_t(J_{2,m}) - q + 1$ pebbles on it. Clearly $2f_t(J_{2,m}) - q + 1 = 2\left[16(t-1) + f(J_{2,m})\right] - q + 1 \geq 33 + f(J_{2,m}) - q \geq 37$, since $q \leq 2m+1$ and $t \geq 2$ and by Theorem 2.5. So, we can move two pebbles to the target vertex v_i of $J_{2,m}$ at a cost of at most 32 pebbles. Then the graph $J_{2,m}$ has at least $2\left[16(t-1) + f(J_{2,m})\right] - q + 1 - 32 \geq 2\left[16(t-2) + f(J_{2,m})\right] - q + 1$ pebbles and hence we can move the additional $2(t-1)$ pebbles to v_i. Thus the graph $J_{2,m}$ satisfies the $2t$-pebbling property.

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