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On Some Absolute Summability Factors of Infinite Series

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Abstract

In this paper, a general theorem concerning $\varphi - \left|\bar{N}, p_n\right|_k$ *factors of infinite series has been proved. The presented result giving improvement as well as generalization of some known results.*

Keywords: *Absolute Summability, Infinite series*

1 Introduction

Let (φ_n) be a sequence of positive real numbers, let $\sum a_n$ be an infinite series with the sequence of partial sums (s_n). Let (t_n) denote the n-th (C, 1) means of the sequence (na_n). The series $\sum a_n$ is said to be summable $\left|C,1\right|_k$, $k \geq 1$, if (see [1])

(1.1) $$\sum_{n=1}^{\infty} \frac{1}{n} \left|t_n\right|^k < \infty.$$

and it is said to be summable $\varphi - \left|C,1\right|_k$, $k \geq 1$, if (see [5])

(1.2) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} \left|t_n\right|^k < \infty .$$

If we are taking $\varphi_n = n$, $\varphi - \left|C,1\right|_k$ reduces to $\left|C,1\right|_k$ summability.

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=1}^{n} p_v \to \infty \quad as \quad n \to \infty \quad (P_{-1} = P_{-1} = 0).$$

The sequence-to-sequence transformation

(1.3) $$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (u_n) of the Riesz mean or simply the $(\overline{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) $(see[2])$. The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$ if

(1.4) $$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty .$$

In the special case when $p_n = 1$ for all n, then $|R, p_n|_k$ summability is the same as $|C,1|_k$ summability. The series $\sum a_n$ is summable $\varphi - |N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |u_n - u_{n-1}|^k < \infty .$$

For $\varphi_n = n$, $\varphi - |R, p_n|_k$ summability is the same as $|R, p_n|_k$ summability .

Concerning $|C,1|_k$ summability, Mazhar [3] has proved the following

Theorem 1.1. If

(1.5) $$\lambda_m = o(1), \quad as \quad m \to \infty,$$

(1.6) $$\sum_{n=1}^{m} n \log n |\Delta^2 \lambda_n| = O(1), \quad as \quad m \to \infty,$$

(1.7) $$\sum_{v=1}^{m} \frac{|t_v|^k}{v} = O(\log m) \quad as \quad m \to \infty ,$$

then the series $\sum a_n \lambda_n$ *is summable* $|C,1|_k$, $k \geq 1$.

Ozarslan [4], on the other hand , generalized the previous result by giving the following

Theorem 1.2. *Let* (φ_n) *be a sequence of positive real numbers and the conditions (1.5)-(1.6) of Theorem (1.1) are satisfied . If*

(1.8) $$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(\log m) \quad as \quad m \to \infty,$$

(1.9) $$\sum_{n=v}^{\infty} \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_v^{k-1}}{v^k} \right),$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C,1|_k$, $k \geq 1$.

It should be mentioned that on taking $\varphi_n = n$ in Theorem (1.2), we get Theorem 1.1.

The aim of this paper is that to give three improvements to Theorem 1.2. Firstly by weakining the conditions and secondly by generalizing the result replacing log m by χ_m *, and thirdly by adding new parameter. In fact, we present the following*

2 Main Result

Theorem 2.1. *Let* $(\varphi_n),(\chi_n)$ *be sequences of positive real numbers such that* (χ_n) *is nondecreasing and the condition (1.5), is satisfied. If*

(2.1) $$np_n = O(P_n)\,,\ P_n = O(np_n)\,,\quad as\ n\to\infty,$$

(2.2) $$\beta_{n+1} = O(\beta_n)\,,$$

(2.3) $$\Delta\beta_n = O\left(n^{-1}\beta_n\right),\quad as\ n\to\infty,$$

(2.4) $$\sum_{n=1}^{\infty} n\chi_n\left|\Delta^2\lambda_n\right| = O(1)\,,$$

(2.5) $$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_v|^k|s_n|^k}{n^k\chi_n^{k-1}} = O(\chi_m)\,,\quad as\ m\to\infty,$$

(2.6) $$\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{v^kP_{n-1}} = O\left(\frac{\varphi_v^{k-1}}{v^{k-1}P_v}\right),$$

then the series $\sum a_n\lambda_n\beta_n$ is summable $\varphi-\left|\overline{N},p_n\right|_k,\ k\geq 1$.

Remark 1 *On putting* $p_n=1,\ \beta_n=1,\ \chi_n=\log n$, *we obtain an improvement to Theorem 1.2*
.

Remark 2 *It may be noted thet condition (2.5) is weaker than condition*

(2.5)′ $$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k} = O(\chi_m),$$

for we have $(2.5)'\Rightarrow(2.5)$, but not conversely. In fact if $(2.5)'$ holds, then

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k\chi_n^{k-1}} = O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k} = O(\chi_m),\quad (2.4)'$$

while, if (2.5) is satisfied, we have
since by the mean value theorem,

$$\Delta\chi_n^{k-1} = O(1)\ \chi_n^{k-2}|\Delta\chi_n|,$$

then, we have

$$\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k} = \sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k\chi_n^{k-1}}\chi_n^{k-1}$$

$$= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{\varphi_v^{k-1}|\beta_v|^k|s_v|^k}{v^k\chi_v^{k-1}}\right)\Delta\left(\chi_n^{k-1}\right) + \left(\sum_{n=1}^{m}\frac{\varphi_n^{k-1}|\beta_n|^k|s_n|^k}{n^k\chi_n^{k-1}}\right)\chi_m^{k-1}$$

$$= O(1)\sum_{n=1}^{m-1}\chi_n^{k-1}|\Delta\chi_n| + O(1)\ \chi_m^k$$

$$= O(1)\ \chi_m^{k-1}\sum_{v=1}^{m}|\Delta\chi_n| + O(1)\,\chi_m^k$$

$$= O(1)\ \chi_m^k + O(1)\,\chi_m^k$$

$$= O\left(\chi_m^k\right) \neq O\left(\chi_m\right),\quad \text{for } k>1.$$

3 Lemma

The following Lemma is needed

Lemma 3.1. *The conditions (1.5) and (2.4) implies*

(3.1) $$\sum_{n=1}^{\infty}\chi_n\left|\Delta\lambda_n\right| = O(1),$$

(3.2) $$n\chi_n\left|\Delta\lambda_n\right| = O(1), \quad as\ n\to\infty,$$

(3.3) $$\chi_n\left|\lambda_n\right| = O(1), \quad as\ n\to\infty.$$

Proof. By virtue of (1.5),

$$\begin{aligned}\sum_{n=1}^{\infty}\chi_n\left|\Delta\lambda_n\right| &= \sum_{n=1}^{\infty}\chi_n\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right| \\ &\le \sum_{n=1}^{\infty}\chi_n\sum_{v=n}^{\infty}\left|\Delta^2\lambda_v\right| \\ &= \sum_{v=1}^{\infty}\left|\Delta^2\lambda_v\right|\sum_{n=1}^{v}\chi_v \\ &= O(1)\sum_{v=1}^{\infty}v\chi_v\left|\Delta^2\lambda_v\right| \\ &= O(1).\end{aligned}$$

$$\begin{aligned}n\chi_n\left|\Delta\lambda_n\right| &= n\chi_n\left|\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right|\right| \\ &\le n\chi_n\sum_{v=n}^{\infty}\left|\Delta\left|\Delta\lambda_v\right|\right| \\ &\le n\chi_n\sum_{v=n}^{\infty}\left|\Delta^2\lambda_v\right| \\ &= O(1)\sum_{n=v}^{\infty}v\chi_v\left|\Delta^2\lambda_v\right| \\ &= O(1).\end{aligned}$$

$$\begin{aligned}\chi_n\left|\lambda_n\right| &= \chi_n\sum_{v=n}^{\infty}\Delta\left|\lambda_v\right| \\ &\le \chi_n\sum_{v=n}^{\infty}\left|\Delta\lambda_v\right| \\ &= O(1)\sum_{v=n}^{\infty}\chi_v\left|\Delta\lambda_v\right| \\ &= O(1), \quad \text{by the first part}.\end{aligned}$$

4 Proof of Theorem 2.1

Let T_n be the (N, p_n) mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n \beta_n$. By definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{r=1}^{v} a_r \lambda_r \beta_r = \frac{1}{P_n}\sum_{v=1}^{n}(P_n - P_{v-1}) a_v \lambda_v \beta_v ,$$

and hence

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v \lambda_v \beta_v , \qquad n \geq 1.$$

Using Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} s_v \Delta(P_{v-1}\lambda_v \beta_v) + \frac{p_n}{P_n}\lambda_n \beta_n s_n \\ &= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}\left(- p_v \lambda_v \beta_v s_v + P_v \lambda_v \Delta\beta_v s_v + P_v \beta_{v+1}\Delta\lambda_v s_v\right) + \frac{p_n}{P_n}\lambda_n \beta_n s_n \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4} . \end{aligned}$$

Since $\left|T_{n1} + T_{n2} + T_{n3} + T_{n4}\right|^k \leq 4^k \left(\left|T_{n1}\right|^k + \left|T_{n2}\right|^k + \left|T_{n3}\right|^k + \left|T_{n4}\right|\right)$, in order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}\left|T_{nr}\right|^k < \infty, \quad r = 1, 2, 3, 4.$$

Applying Holder's inequality, we have via Lemma 3.1

$$\begin{aligned} \sum_{n=2}^{m}\varphi_n^{k-1}\left|T_{n1}\right|^k &= \sum_{n=2}^{m}\varphi_n^{k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} p_v \lambda_v \beta_v s_v\right|^k \\ &= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}\, p_n^k}{P_n^k P_{n-1}^k}\left(\sum_{v=1}^{n-1} p_v \left|\lambda_v\right|\left|\beta_v\right|\left|s_v\right|\right)^k \\ &= O(1)\sum_{n=2}^{m}\frac{\varphi_n^{k-1}\, p_n^k}{P_n^k P_{n-1}}\sum_{v=1}^{n-1} p_v \left|\lambda_v\right|^k \left|\beta_v\right|^k \left|s_v\right|^k \left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1} \\ &= O(1)\sum_{v=1}^{m} p_v \left|\lambda_v\right|^k \left|\beta_v\right|^k \left|s_v\right|^k \sum_{n=v}^{m}\frac{\varphi_n^{k-1}\, p_n^k}{P_n^k P_{n-1}} \\ &= O(1)\sum_{v=1}^{m} p_v \left|\lambda_v\right|^k \left|\beta_v\right|^k \left|s_v\right|^k \sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^k\, P_{n-1}} \\ &= O(1)\sum_{v=1}^{m}\frac{p_v\, \varphi_v^{k-1}}{v^{k-1} P_v}\left|\lambda_v\right|^k \left|\beta_v\right|^k \left|s_v\right|^k \\ &= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}\left|\lambda_v\right|^{k-1}\left|\beta_v\right|^k \left|s_v\right|^k \left|\lambda_v\right| \\ &= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k\, \chi_v^{k-1}}\left|\beta_v\right|^k \left|s_v\right|^k \sum_{n=v}^{\infty}\Delta\left|\lambda_n\right| \\ &= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k\, \chi_v^{k-1}}\left|\beta_v\right|^k \left|s_v\right|^k \sum_{n=v}^{\infty}\left|\Delta\lambda_n\right| \end{aligned}$$

$$= O(1)\sum_{n=1}^{\infty}|\Delta\lambda_n|\sum_{v=1}^{n}\frac{\varphi_v^{k-1}}{v^k\chi_v^{k-1}}|\beta_v|^k|s_v|^k$$
$$= O(1)\sum_{n=1}^{\infty}\chi_n|\Delta\lambda_n|$$
$$= O(1).$$

$$\sum_{n=2}^{m}\varphi_n^{k-1}|T_{n2}|^k = \sum_{n=2}^{m}\varphi_n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_v\Delta\beta_v s_v\right|^k$$
$$= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v|\lambda_v||\Delta\beta_v||s_v|\right)^k$$
$$= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}v^{-1}P_v|\lambda_v||\beta_v||s_v|\right)^k$$
$$= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}}\sum_{v=1}^{n-1}v^{-k}p_v\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^k|\beta_v|^k|s_v|^k\left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|\beta_v|^k|s_v|^k\sum_{n=v}^{m}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|\beta_v|^k|s_v|^k\sum_{n-v}^{m}\frac{\varphi_n^{k-1}}{v^kP_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}p_v}{v^{k-1}P_v}|\lambda_v|^k|\beta_v|^k|s_v|^k$$
$$= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}|\lambda_v|^k|\beta_v|^k|s_v|^k$$
$= O(1)$, as in the case of T_{n1} .

$$\sum_{n=1}^{m+1}\varphi_n^{k-1}|T_{n3}|^k = O(1)\sum_{n=1}^{m+1}\varphi_n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\beta_{v+1}\Delta\lambda_v s_v\right|^k$$
$$= O(1)\sum_{n=1}^{m+1}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}^k}\left(\sum_{v=1}^{n-1}P_v|\beta_v||\Delta\lambda_v||s_v|\right)^k$$
$$= O(1)\sum_{n=1}^{m+1}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}^k}\sum_{v=1}^{n-1}\frac{P_v^k}{\chi_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\left(\sum_{v=1}^{n-1}\chi_v|\Delta\lambda_v|\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m}\frac{P_v^k}{\chi_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{\varphi_n^{k-1}p_n^k}{P_n^kP_{n-1}^k}$$
$$= O(1)\sum_{v=1}^{m}\frac{P_v^k}{\chi_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{\varphi_n^{k-1}}{v^kP_{n-1}^k}$$
$$= O(1)\sum_{v=1}^{m}\frac{P_v}{\chi_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k\sum_{n=v+1}^{m+1}\frac{\varphi_n^{k-1}}{v^kP_{n-1}}$$
$$= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^{k-1}\chi_v^{k-1}}|\beta_v|^k|\Delta\lambda_v||s_v|^k$$

$$
\begin{aligned}
&= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{\varphi_v^{k-1}|\beta_v|^k |s_v|^k}{v^k \chi_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \frac{\varphi_r^{k-1}|\beta_r|^k |s_r|^k}{r^k \chi_r^{k-1}} \\
&\qquad + O(1)\, m|\Delta\lambda_m| \sum_{v=1}^{m} \frac{\varphi_v^{k-1}|\beta_v|^k |s_v|^k}{v^k \chi_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta^2\lambda_v|\chi_v + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v|\chi_v + O(1)\, m|\Delta\lambda_m|\chi_m \\
&= O(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1} |T_{n4}|^k &= \sum_{n=1}^{m} \varphi_n^{k-1} \left| \frac{p_n}{P_n} \beta_n \lambda_n s_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |\beta_n|^k |s_n|^k |\lambda_n|^{k-1} |\lambda_n| \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1}|\beta_n|^k |s_n|^k}{n^k \chi_n^{k-1}} \sum_{v=n}^{\infty} |\Delta\lambda_v| \\
&= O(1) \sum_{v=1}^{m} |\Delta\lambda_v| \sum_{n=1}^{v} \frac{\varphi_n^{k-1}|\beta_n|^k |s_n|^k}{n^k \chi_n^{k-1}} \\
&= O(1) \sum_{v=1}^{m} \chi_v |\Delta\lambda_v| \\
&= O(1)\,.
\end{aligned}
$$

This completes the proof of the Theorem.

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